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ON DISCRETE INEQUALITIES OF GRÜSS TYPE

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Abstract. In the paper, we establish two discrete inequalities of Grüss type via inequalities of Watson-Greub-Rheiuboldt and Klamkin-Mclenaghan.

1. Introduction

In [6], Grüss proved the following integral inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right|\le\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma)\qquad(1.1)$$

provided that f and g are two integrable functions on $[a,b]$ and satisfy the condition

$$\phi\le f(x)\le\Phi \text{ and } \gamma\le g(x)\le\Gamma \text{ for } a.e. x\in[a,b].\qquad(1.2)$$

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

The discrete version of (1.1) states that (see for example [2]):

If $\overline{a}=(a_1,\dots a_n)$ and $\overline{b}=(b_1,\dots b_n)$ are two sequences of real numbers with

$$a\le a_i\le A<\infty \text{ and } b\le b_i\le B<\infty \text{ for all } i\in\{1,\dots,n\},$$

then

$$\begin{aligned}\left|C_n(\overline{a},\overline{b})\right|&\le\frac{1}{n}[\frac{1}{2}](1-\frac{1}{n}[\frac{1}{2}])(A-a)(B-b)\\&=\frac{1}{n^2}[\frac{n^2}{4}](A-a)(B-b)\\&\le\frac{1}{4}(A-a)(B-b)\end{aligned}\qquad(1.3)$$

where

$$C_n(\overline{a},\overline{b})=\frac{1}{n}\sum_{i=1}^n a_ib_i-\frac{1}{n}\sum_{i=1}^n a_i\cdot\frac{1}{n}\sum_{i=1}^n b_i.$$

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For other new results in the domain, see the papers [1, 3-8, 9-10] and the book [8].

Recently, Dragomir and Khan [5] proved the following two discrete inequalities of Grüss type:

Theorem A. *Let* $\overline{a} = (a_1, \ldots a_n)$ *and* $\overline{b} = (b_1, \ldots b_n)$ *be two n-tuples of positive real numbers with*

$$0 < a \le a_i \le A < \infty \text{ and } 0 < b \le b_i \le B < \infty \text{ for all } i \in \{1, \ldots, n\}.$$

Then we have the inequality

$$\left|C_n(\overline{a}, \overline{b})\right| \le \frac{1}{4} \frac{(A-a)(B-b)}{\sqrt{AaBb}} \cdot \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} b_i. \tag{1.4}$$

The constant $\frac{1}{4}$ *is the best possible.*

Theorem B. *Let* $\overline{a} = (a_1, \ldots a_n)$ *and* $\overline{b} = (b_1, \ldots b_n)$ *be defined as in Theorem A. Then we have the inequality*

$$\left|C_n(\overline{a}, \overline{b})\right| \le (\sqrt{A} - \sqrt{a})(\sqrt{B} - \sqrt{b}) \cdot \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} b_i. \tag{1.5}$$

The constant $c = 1$ *is the best possible.*

In this paper, we establish a weighted generalization of Theorem A and Theorem B, respectively.

2. Main Results

Theorem 1. *Let* $\overline{a} = (a_1, \ldots a_n)$ *and* $\overline{b} = (b_1, \ldots, b_n)$ *be defined as in Theorem A, and let* $\overline{p} = (p_1, \ldots, p_n)$ *be an n-tuples of nonnegative numbers with* $P_n = \sum_{i=1}^{n} p_i > 0$. *Then we have the inequality*

$$\left|C_n(\overline{p}, \overline{a}, \overline{b})\right| \le \frac{1}{4} \frac{(A-a)(B-b)}{\sqrt{AaBb}} \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \tag{2.1}$$

where

$$C_n(\overline{p}, \overline{a}, \overline{b}) = \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i$$

and the constant $\frac{1}{4}$ *is the best possible.*

Proof. By the Cauchy-Schwarz inequality, we have

$$\left|C_n(\overline{p}, \overline{a}, \overline{b})\right| = \left| \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \right|$$

$$
\begin{aligned}
&= \left|\frac{1}{2P_n^2}\sum_{i,j=1}^{n} p_i p_j (a_i - a_j)(b_i - b_j)\right| \\
&\le \frac{1}{2P_n^2}\sum_{i,j=1}^{n} p_i p_j \,|(a_i - a_j)|\,|(b_i - b_j)| \\
&\le \frac{1}{2P_n^2}[\sum_{i,j=1}^{n} p_i p_j (a_i - a_j)^2]^{\frac{1}{2}}[\sum_{i,j=1}^{n} p_i p_j (b_i - b_j)^2]^{\frac{1}{2}} \\
&= \frac{1}{2P_n^2}[2P_n\sum_{i=1}^{n} p_i a_i^2 - 2(\sum_{i=1}^{n} p_i a_i)^2]^{\frac{1}{2}}[2P_n\sum_{i=1}^{n} p_i b_i^2 - 2(\sum_{i=1}^{n} p_i b_i)^2]^{\frac{1}{2}} \\
&= [\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i^2 - (\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i)^2]^{\frac{1}{2}}[\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i^2 - (\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i)^2]^{\frac{1}{2}}. \qquad (2.2)
\end{aligned}
$$

Using the Waston-Greub-Rheiuboldt inequality [8, p.122]

$$
\sum_{i=1}^{n} w_i z_i^2 \cdot \sum_{i=1}^{n} w_i u_i^2 \le \frac{(M_1M_2 + m_1m_2)^2}{4m_1m_2M_1M_2} \cdot (\sum_{i=1}^{n} w_i z_i u_i)^2, \qquad (2.3)
$$

provided $0 \le w_i < \infty$, $0 < m_1 \le z_i \le M_1 < \infty$ and $0 < m_2 \le u_i \le M_2 < \infty$ $(i = 1, \ldots, n)$, we have

$$
(\sum_{i=1}^{n} p_i a_i^2) \cdot P_n \le \frac{(A+a)^2}{4aA} \cdot (\sum_{i=1}^{n} p_i a_i)^2
$$

giving

$$
\frac{(\sum_{i=1}^{n} p_i a_i^2) \cdot P_n - (\sum_{i=1}^{n} p_i a_i)^2}{(\sum_{i=1}^{n} p_i a_i)^2} \le \frac{(A+a)^2 - 4aA}{4aA} = \frac{(A-a)^2}{4aA},
$$

thas is,

$$
\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i^2 - \frac{1}{P_n}(\sum_{i=1}^{n} p_i a_i)^2 \le \frac{(A-a)^2}{4aA} \cdot (\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i)^2. \qquad (2.4)
$$

Similarly, we have

$$
\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i^2 - \frac{1}{P_n}(\sum_{i=1}^{n} p_i b_i)^2 \le \frac{(B-b)^2}{4bB} \cdot (\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i)^2. \qquad (2.5)
$$

Using (2.2), (2.4) and (2.5), we otain the inequality (2.1). As note in the proof of Theorem in [5], we obtain that the constant $\frac{1}{4}$ is the best possible. This completes the proof.

Remark 1. Choose $p_i = \frac{1}{n}$ $(1, \ldots, n)$ in Theorem 1. Then the inequality (2.1) reduces to (1.4).

Theorem 2. *Let* $\overline{a} = (a_1, \ldots, a_n)$, $\overline{b} = (b_1, \ldots, b_n)$ *and* $\overline{p} = (p_1, \ldots, p_n)$ *be defined as in Theorem 1. Then we have*

$$\left|C_n(\overline{p}, \overline{a}, \overline{b})\right| \leq (\sqrt{A} - \sqrt{a})(\sqrt{B} - \sqrt{b})(\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i)^{\frac{1}{2}}(\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i)^{\frac{1}{2}}. \tag{2.6}$$

The constant $c = 1$ *is the best possible.*

Proof. Using the Klamkin-Mclenaghan inequality [7, p.125]

$$\sum_{i=1}^{n} w_i z_i^2 \cdot \sum_{i=1}^{n} w_i u_i^2 - (\sum_{i=1}^{n} w_i z_i u_i)^2 \leq (\sqrt{M} - \sqrt{m})^2 \sum_{i=1}^{n} w_i z_i u_i \cdot \sum_{i=1}^{n} w_i u_i^2, \tag{2.7}$$

provided $0 \leq w_i < \infty$, $z_i > 0$, $u_i > 0$ and $0 < m \leq \frac{z_i}{u_i} \leq M < \infty$ $(i = 1, \ldots, n)$, we have

$$(\sum_{i=1}^{n} p_i a_i^2) \cdot P_n - (\sum_{i=1}^{n} p_i a_i)^2 \leq (\sqrt{A} - \sqrt{a})^2 (\sum_{i=1}^{n} p_i a_i) \cdot P_n$$

that is,

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i^2 - (\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i)^2 \leq (\sqrt{A} - \sqrt{a})^2 (\frac{1}{P_n}\sum_{i=1}^{n} p_i a_i). \tag{2.8}$$

Similarly, we have

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i^2 - (\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i)^2 \leq (\sqrt{B} - \sqrt{b})^2 (\frac{1}{P_n}\sum_{i=1}^{n} p_i b_i). \tag{2.9}$$

Using (2.2), (2,8) and (2.9), we obtain the inequality. As note in proof Theorem 1 in [5], we obtain that the constant $c = 1$ is the best possible. This completes the proof.

Remark 2. Choose $p_i = \frac{1}{n}$ $(1, \ldots, n)$ in Theorem 2. Then the inequality (2.6) reduces to (1.5).

References

[1] D. Andrica and C. Badea, *Grüss' inequality for positive linear functionals*, Periodica Math. Hungarica **19**(1988), 155-167.

[2] M. Biernacki, H. Pidek and C. Ryll-Nardzewski, *Sur une inégalité entre des intégrales definies*, Ann. Univ. Mariae Curie-Skolodowska **A4** (1950), 1-4.

[3] S. S. Dragomir, *Integral Grüss type inequality for mappings with values in Hilbert spaces and its applications*, J. Korean Math. Soc. **38** (2001), 1261-1273.

[4] S. S. Dragomir, *A generalization Grüss's inequality in inner product spaces and applications*, J. Math. Anal. Appl. **237** (1999), 74-82.

[5] S. S. Dragomir and L. Khan, *Two discrete inequalities of Grüss type via Pólya-Szegö and Shisha results for real numbers* , Tamkang J. of Math. accepted.

[6] A. M. Fink, *A treatise on Grüss' inequality*, Analyic and Geometric Inequalities and Applications, T. M. Rassias and H. M. Srivastava, Editors Kluwer, 1999, 93-114.

[7] G. Grüss, *Über das Maximum des absoluten Betrages Von* $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, Math. Z. **39** (1935), 215-226.

[8] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic publishers, Dordrecht, 1993.

[9] J. E. Pečarić, *On an inequality of G. Grüss*, Mat. Vesnik **7** (20) (35) (1983), 59-64.

[10] J. E. Pečarić and D. Andrica, *On some Grüss type inequalities*, It. Sem. Func. Equat. Approx. Conv., Cluj-Napoca (1986), 221-214.

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