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NEW INEQUALITIES OF OSTROWSKI TYPE FOR TWICE DIFFERENTIABLE MAPPINGS

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Abstract. In this paper, using integral identities for twice differentiable mappings, we establish new inequalities which in the special cases yield certain Ostrowski type inequalities recently established in the literature.

1. Introduction

In [1] Cerone, Dragomir and Roumeliotis proved the following identity:

$$\int_a^b f(t)dt = (b-a)f(x) - (b-a)\left(x - \frac{a+b}{2}\right) f'(x) + \int_a^b k(x,t) f''(t)dt, \tag{1.1}$$

for $x \in [a,b]$, where $f : [a,b] \to R$ is a twice differentiable mapping on (a,b) and $k(x,t) : [a,b]^2 \to R$ is given by

$$k(x,t) = \begin{cases} \frac{(t-a)^2}{2} & \text{if } t \in [a,x] \\ \frac{(t-b)^2}{2} & \text{if } t \in (x,b] \end{cases}. \tag{1.2}$$

In another paper [2] Dragomir and Barnett have proved the following identity:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) + \frac{1}{(b-a)^2}\int_a^b \int_a^b p(x,t)p(t,s) f''(s)dsdt, \tag{1.3}$$

for $x \in [a,b]$, where $f : [a,b] \to R$ is continuous on $[a,b]$ and twice differentiable on (a,b) and $p(x,t) : [a,b]^2 \to R$ is given by

$$p(x,t) = \begin{cases} t-a & \text{if } t \in [a,x] \\ t-b & \text{if } t \in (x,b] \end{cases}. \tag{1.4}$$

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In the same papers [1] and [2] based on the identities (1.1) and (1.3), the authors have respectively established the following Ostrowski type inequalities:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(x - \frac{a+b}{2}\right) f'(x) \right| \leq E(x)\|f''\|_\infty, \tag{1.5}$$

for $x \in [a, b]$, where

$$E(x) = \frac{1}{24}(b-a)^2 + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2, \tag{1.6}$$

$f : [a, b] \to R$ is a twice differentiable mapping on (a, b) and $f'' : (a, b) \to R$ is bounded i.e. $\|f''\|_\infty = \sup_{t\in(a,b)} |f''(t)| < \infty$, and

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right| \leq L(x)\|f''\|_\infty, \tag{1.7}$$

for $x \in [a, b]$, where

$$L(x) = \frac{1}{2}\left\{ \left[\frac{(x - \frac{a+b}{2})^2}{(b-a)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (b-a)^2, \tag{1.8}$$

$f : [a, b] \to R$ is continuous on $[a, b]$ and twice differentiable on (a, b), whose second derivative $f'' : (a, b) \to R$ is bounded on (a, b).

For similar results, see the book [4] by Mitrinović, Pečarić and Fink and the recent papers [5, 6], where further references are given. The main purpose of the present paper is to establish new Ostrowski type inequalities involving a pair of twice differentiable mappings. The analysis used in the proofs is quite elementary and our results in the special cases recapture the inequalities (1.5) and (1.7).

2. Statement of Results

Our main results are given in the following theorems.

Theorem 1. *Let f, $g : [a, b] \to R$ be twice differentiable mappings on (a, b) and f'', $g'' : (a, b) \to R$ are bounded i.e. $\|f''\|_\infty = \sup_{t\in(a,b)} |f''(t)| < \infty$, $\|g''\|_\infty = \sup_{t\in(a,b)} |g''(t)| < \infty$. Then*

$$\Bigg| 2\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right) - \left[f(x) - \left(x - \frac{a+b}{2}\right)f'(x)\right]\left(\frac{1}{b-a}\int_a^b g(t)dt\right)$$
$$- \left[g(x) - \left(x - \frac{a+b}{2}\right)g'(x)\right]\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\Bigg|$$

$$\leq E(x)\left[\|f''\|_\infty\left(\frac{1}{b-a}\int_a^b |g(t)|dt\right)+\|g''\|_\infty\left(\frac{1}{b-a}\int_a^b |f(t)|dt\right)\right], \tag{2.1}$$

$$\left|\left(\frac{1}{b-a}\int_a^b f(t)dt\right)g(x)+\left(\frac{1}{b-a}\int_a^b g(t)dt\right)f(x)+\left(x-\frac{a+b}{2}\right)(fg)'(x)-2f(x)g(x)\right|$$
$$\leq E(x)[\|f''\|_\infty|g(x)|+\|g''\|_\infty|f(x)|], \tag{2.2}$$

for $x \in [a,b]$, where $E(x)$ is given by (1.6).

Theorem 2. *Let f, $g : [a,b] \to R$ are continuous on $[a,b]$ and twice differentiable on (a,b), whose second derivatives f'', $g'' : (a,b) \to R$ are bounded on (a,b) i.e. $\|f''\|_\infty = \sup_{t\in(a,b)}|f''(t)| < \infty$, $\|g''\|_\infty = \sup_{t\in(a,b)}|g''(t)| < \infty$. Then*

$$\left|f(x)\left(\frac{1}{b-a}\int_a^b g(t)dt\right)+g(x)\left(\frac{1}{b-a}\int_a^b f(t)dt\right)-2\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right)\right.$$
$$-\left[\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right)\right.$$
$$\left.\left.+\frac{g(b)-g(a)}{b-a}\left(x-\frac{a+b}{2}\right)\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\right]\right|$$
$$\leq L(x)\left[\|f''\|_\infty\left(\frac{1}{b-a}\int_a^b |g(t)|dt\right)+\|g''\|_\infty\left(\frac{1}{b-a}\int_a^b |f(t)|dt\right)\right], \tag{2.3}$$

$$\left|2f(x)g(x)-\left\{\left[\frac{1}{b-a}\int_a^b f(t)dt+\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right]g(x)\right.\right.$$
$$\left.\left.+\left[\frac{1}{b-a}\int_a^b g(t)dt+\frac{g(b)-g(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right]f(x)\right\}\right|$$
$$\leq L(x)[\|f''\|_\infty|g(x)|+\|g''\|_\infty|f(x)|], \tag{2.4}$$

for $x \in [a,b]$, where $L(x)$ is given by (1.8).

Remark 1. It is easy to observe that, by taking $g(x) = 1$ and hence $g'(x) = 0$, $g''(x) = 0$ in Theorems 1 and 2, we recapture respectively the inequalities (1.5) and (1.7) given in [1] and [2].

3. Proof of Theorem 1

From the hypotheses, we have the following identities (see [1, pp.34-35]):

$$\frac{1}{b-a}\int_a^b f(t)dt = \left[f(x)-\left(x-\frac{a+b}{2}\right)f'(x)\right]+\frac{1}{b-a}\int_a^b k(x,t)f''(t)dt, \tag{3.1}$$

$$\frac{1}{b-a}\int_a^b g(t)dt = \left[g(x) - \left(x - \frac{a+b}{2}\right)g'(x)\right] + \frac{1}{b-a}\int_a^b k(x,t)g''(t)dt, \tag{3.2}$$

for $x \in [a,b]$, where $k(x,t)$ is given by (1.2). Multiplying both (3.1) and (3.2) by $\frac{1}{b-a}\int_a^b g(t)dt$ and $\frac{1}{b-a}\int_a^b f(t)dt$ respectively, we get

$$\begin{aligned}
&2\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right)\\
&= \left[f(x) - \left(x - \frac{a+b}{2}\right)f'(x)\right]\left(\frac{1}{b-a}\int_a^b g(t)dt\right)\\
&\quad + \left[g(x) - \left(x - \frac{a+b}{2}\right)g'(x)\right]\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\\
&\quad + \left(\frac{1}{b-a}\int_a^b k(x,t)f''(t)dt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right)\\
&\quad + \left(\frac{1}{b-a}\int_a^b k(x,t)g''(t)dt\right)\left(\frac{1}{b-a}\int_a^b f(t)dt\right),
\end{aligned} \tag{3.3}$$

for $x \in [a,b]$. From (3.3) and using the properties of modulus we have

$$\begin{aligned}
&\left|2\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right) - \left[f(x) - \left(x - \frac{a+b}{2}\right)f'(x)\right]\left(\frac{1}{b-a}\int_a^b g(t)dt\right)\right.\\
&\quad \left. - \left[g(x) - \left(x - \frac{a+b}{2}\right)g'(x)\right]\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\right|\\
&\le \left[\|f''\|_\infty\left(\frac{1}{b-a}\int_a^b |g(t)|dt\right) + \|g''\|_\infty\left(\frac{1}{b-a}\int_a^b |f(t)|dt\right)\right]\left(\frac{1}{b-a}\int_a^b |k(x,t)|dt\right).
\end{aligned} \tag{3.4}$$

In fact, from (1.2) and using the elementary calculations as in [1, p.35] we obtain

$$\frac{1}{b-a}\int_a^b |k(x,t)|dt = E(x), \quad x \in [a,b]. \tag{3.5}$$

Using (3.5) in (3.4) we get the required inequality in (2.1).

Rewriting (3.1) and (3.2) as

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \left(x - \frac{a+b}{2}\right)f'(x) - \frac{1}{b-a}\int_a^b k(x,t)f''(t)dt, \tag{3.1$'$}$$

$$g(x) = \frac{1}{b-a}\int_a^b g(t)dt + \left(x - \frac{a+b}{2}\right)g'(x) - \frac{1}{b-a}\int_a^b k(x,t)g''(t)dt, \tag{3.2$'$}$$

Multiplying both sides of $(3.1)'$ and $(3.2)'$ by $g(x)$ and $f(x)$ respectively and adding, we get

$$2f(x)g(x) = \left(\frac{1}{b-a}\int_a^b f(t)dt\right)g(x) + \left(\frac{1}{b-a}\int_a^b g(t)dt\right)f(x) + \left(x - \frac{a+b}{2}\right)(fg)'(x)$$

$$-\left(\frac{1}{b-a}\int_a^b k(x,t)f''(t)dt\right)g(x)-\left(\frac{1}{b-a}\int_a^b k(x,t)g''(t)dt\right)f(x). \quad (3.6)$$

Rewriting (3.6) and using the properties of modulus and (3.5) we get the desired inequality in (2.2).

4. Proof of Theorem 2

From the hypotheses, we have the following identities (see [2, pp.70-71]):

$$\begin{aligned} f(x) &= \frac{1}{b-a}\int_a^b f(t)dt + \frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right) \\ &+\frac{1}{(b-a)^2}\int_a^b\int_a^b p(x,t)p(t,s)f''(s)dsdt, \end{aligned} \quad (4.1)$$

$$\begin{aligned} g(x) &= \frac{1}{b-a}\int_a^b g(t)dt + \frac{g(b)-g(a)}{b-a}\left(x-\frac{a+b}{2}\right) \\ &+\frac{1}{(b-a)^2}\int_a^b\int_a^b p(x,t)p(t,s)g''(s)dsdt, \end{aligned} \quad (4.2)$$

for $x \in [a,b]$, where $p(x,t)$ is given by (1.4). Multiplying both sides of (4.1) and (4.2) by $\frac{1}{b-a}\int_a^b g(t)dt$ and $\frac{1}{b-a}\int_a^b f(t)dt$ respectively and adding, we get

$$\begin{aligned} & f(x)\left(\frac{1}{b-a}\int_a^b g(t)dt\right)+g(x)\left(\frac{1}{b-a}\int_a^b f(t)dt\right) \\ &= 2\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right) \\ &+\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right) \\ &+\frac{g(b)-g(a)}{b-a}\left(x-\frac{a+b}{2}\right)\left(\frac{1}{b-a}\int_a^b f(t)dt\right) \\ &+\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b p(x,t)p(t,s)f''(s)dsdt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right) \\ &+\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b p(x,t)p(t,s)g''(s)dsdt\right)\left(\frac{1}{b-a}\int_a^b f(t)dt\right), \end{aligned} \quad (4.3)$$

for $x \in [a,b]$. From (4.3) and using the properties of modulus we have

$$\Bigg| f(x)\left(\frac{1}{b-a}\int_a^b g(t)dt\right)+g(x)\left(\frac{1}{b-a}\int_a^b f(t)dt\right)-2\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right)$$
$$-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right)$$

$$-\frac{g(b)-g(a)}{b-a}\left(x-\frac{a+b}{2}\right)\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\Bigg|$$
$$\leq\left[\|f''\|_\infty\left(\frac{1}{b-a}\int_a^b |g(t)|dt\right)+\|g''\|_\infty\left(\frac{1}{b-a}\int_a^b |f(t)|dt\right)\right]$$
$$\times\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b |p(x,t)|\,|p(t,s)|dsdt\right). \tag{4.4}$$

In [2, pp.71-74] by using (1.4) and simple algebraic manipulations the authors have obtain

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b |p(x,t)|\,|p(t,s)|dsdt = L(x), \tag{4.5}$$

for $x\in[a,b]$. Using (4.5) in (4.4), we get the inequality in (2.3).

To prove the inequality (2.4), we multiply both sides of (4.1) and (4.2) by $g(x)$ and $f(x)$ respectively and add them to get

$$\begin{aligned}2f(x)g(x) &= \left[\frac{1}{b-a}\int_a^b f(t)dt+\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right]g(x)\\ &+\left[\frac{1}{b-a}\int_a^b g(t)dt+\frac{g(b)-g(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right]f(x)\\ &+\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b p(x,t)p(t,s)f''(s)dsdt\right)g(x)\\ &+\left(\frac{1}{(b-a)^2}\int_a^b\int_a^b p(x,t)p(t,s)g''(s)dsdt\right)f(x).\end{aligned} \tag{4.6}$$

Rewriting (4.6) and using the properties of modulus and (4.5) we get the required inequality in (2.4).

5. Further Inequalities

In this section, we establish further inequalities of the type given in Theorems 1 and 2 based on the identity proved by Dragomir and Sofo in [3].

Theorem 3. *Let f, $g:[a,b]\to R$ be mappings whose first derivatives are absolutely continuous on $[a,b]$ and assume that the second derivatives f'', $g''\in L_\infty[a,b]$. Then*

$$\Bigg|2\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right)$$
$$-\left\{\frac{1}{2}\left[f(x)+\frac{f(a)+f(b)}{2}\right]-\left(x-\frac{a+b}{2}\right)f'(x)\right\}\left(\frac{1}{b-a}\int_a^b g(t)dt\right)$$

$$-\left\{\frac{1}{2}\left[g(x)+\frac{g(a)+g(b)}{2}\right]-\left(x-\frac{a+b}{2}\right)g'(x)\right\}\left(\frac{1}{b-a}\int_a^b f(t)dt\right)\Bigg|$$
$$\le M(x)\left[\|f''\|_\infty\left(\frac{1}{b-a}\int_a^b |g(t)|dt\right)+\|g''\|_\infty\left(\frac{1}{b-a}\int_a^b |f(t)|dt\right)\right], \tag{5.1}$$

$$\Bigg|\left(\frac{1}{b-a}\int_a^b f(t)dt\right)g(x)+\left(\frac{1}{b-a}\int_a^b g(t)dt\right)f(x)-f(x)g(x)$$
$$-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}g(x)+\frac{g(a)+g(b)}{2}f(x)\right]-\left(x-\frac{a+b}{2}\right)(fg)'(x)\Bigg|$$
$$\le M(x)[\|f''\|_\infty|g(x)|+\|g''\|_\infty|f(x)|], \tag{5.2}$$

for $x\in[a,b]$, *where*

$$M(x)=\frac{1}{2(b-a)}\int_a^b |p(x,t)|\left|t-\frac{a+b}{2}\right|dt. \tag{5.3}$$

From the hypotheses, we have the following identities (see [3, pp.231-232]):

$$\frac{1}{b-a}\int_a^b f(t)dt=\frac{1}{2}\left[f(x)+\frac{f(a)+f(b)}{2}\right]-\left(x-\frac{a+b}{2}\right)f'(x)$$
$$+\frac{1}{2(b-a)}\int_a^b p(x,t)\left(t-\frac{a+b}{2}\right)f''(t)dt, \tag{5.4}$$

$$\frac{1}{b-a}\int_a^b g(t)dt=\frac{1}{2}\left[g(x)+\frac{g(a)+g(b)}{2}\right]-\left(x-\frac{a+b}{2}\right)g'(x)$$
$$+\frac{1}{2(b-a)}\int_a^b p(x,t)\left(t-\frac{a+b}{2}\right)g''(t)dt, \tag{5.5}$$

for $x\in[a,b]$. Multiplying (5.4) and (5.5) respectively by $\frac{1}{b-a}\int_a^b g(t)dt$ and $\frac{1}{b-a}\int_a^b f(t)dt$, adding and following the proof of inequality (2.1) (or (2.3)) we get the inequality (5.1).

Multiplying both sides of (5.4) and (5.5) respectively by $g(x)$ and $f(x)$, adding the resulting identities and following the proof of inequality (2.2) (or (2.4)) we get the desired inequality in (5.2).

Remark 2. We note that in [3, pp.232-234] the authors have evaluated the integral in (5.3) and obtain

$$M(x)=\frac{1}{3(b-a)}\left|x-\frac{a+b}{2}\right|^3+\frac{(b-a)^2}{48},$$

for $x\in[a,b]$. If we take $g(x)=1$ and hence $g'(x)=0$, $g''(x)=0$, in Theorem 3, then we get the inequality established by Dragomir and Sofo in [3, p.230].

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