TAMKANG JOURNAL OF MATHEMATICS
Volume 47, Number 3, 357-371, September 2016
doi:10.5556/j.tkjm.47.2016.2035

This paper is available online at http://journals.math.tku.edu.tw/index.php/TKJM/pages/view/onlinefirst

TWIN SIGNED ROMAN DOMINATION NUMBERS IN DIRECTED GRAPHS

ASGHAR BODAGHLI, SEYED MAHMOUD SHEIKHOLESLAMI AND LUTZ VOLKMANN

Abstract. Let D be a finite simple digraph with vertex set $V(D)$ and arc set $A(D)$. A twin signed Roman dominating function (TSRDF) on the digraph D is a function $f : V(D) \to \{-1,1,2\}$ satisfying the conditions that (i) $\sum_{x\in N^-[v]} f(x) \ge 1$ and $\sum_{x\in N^+[v]} f(x) \ge 1$ for each $v \in V(D)$, where $N^-[v]$ (resp. $N^+[v]$) consists of v and all in-neighbors (resp. out-neighbors) of v, and (ii) every vertex u for which $f(u) = -1$ has an in-neighbor v and an out-neighbor w for which $f(v) = f(w) = 2$. The weight of an TSRDF f is $\omega(f) = \sum_{v\in V(D)} f(v)$. The twin signed Roman domination number $\gamma^*_{sR}(D)$ of D is the minimum weight of an TSRDF on D. In this paper, we initiate the study of twin signed Roman domination in digraphs and we present some sharp bounds on $\gamma^*_{sR}(D)$. In addition, we determine the twin signed Roman domination number of some classes of digraphs.

1. Introduction

Let D be a finite simple directed graph with vertex set $V(D)$ and arc set $A(D)$ (briefly V and A). The integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ are the *order* and the *size* of the digraph D respectively. A digraph without directed cycles of length 2 is an *oriented graph*. For an arc $(u, v) \in A(D)$, the vertex v is an *out-neighbor* of u and u is an *in-neighbor* of v, and we also say that v is out-dominated by u or u is in-dominated by v. For every vertex v, we denote the set of in-neighbors and out-neighbors of v by $N^-(v) = N^-_D(v)$ and $N^+(v) = N^+_D(v)$, respectively. Let $N^-_D[v] = N^-[v] = N^-(v) \cup \{v\}$ and $N^+_D[v] = N^+[v] = N^+(v) \cup \{v\}$. We write $d^+_D(v)$ for the outdegree of a vertex v and $d^-_D(v)$ for its indegree. The *minimum* and *maximum indegree* and *minimum* and *maximum outdegree* of D are denoted by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively. A digraph D is called *regular* or r-*regular* if $\delta^-(D) = \delta^+(D) = \Delta^-(D) = \Delta^+(D) = r$. For a subset $S \subseteq V$, let $d^-_S(v)$ (resp. $d^+_S(v)$) denote the number of in-neighbors (resp. out-neighbors) of v in S. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. If $X \subseteq V(D)$ and $v \in V(D)$, then $A(X, v)$ is the set of arcs from

Received September 25, 2015, accepted November 27, 2015.
2010 *Mathematics Subject Classification.* 05C69.
Key words and phrases. Twin signed Roman dominating function, twin signed Roman domination number, directed graph.
Corresponding author: S. M. Sheikholeslami.

X to v. We denote by $A(X,Y)$ the set of arcs from a subset X to a subset Y. We denote by D^{-1} the digraph obtained from D by reversing the arcs of D. For a real-valued function $f : V \longrightarrow \mathbb{R}$ the weight of f is $\omega(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, so $\omega(f) = f(V)$. Consult [10] for the notation and terminology which are not defined here.

A *signed Roman dominating function* (abbreviated SRDF) on D is defined as a function $f : V \longrightarrow \{-1,1,2\}$ such that (i) $f(N^-[v]) = \sum_{x \in N^-[v]} f(x) \geq 1$ for each vertex $v \in V$ and (ii) every vertex u for which $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$. The weight of an SRDF f on a digraph D is $\omega(f) = \sum_{v \in V(D)} f(v)$. The *signed Roman domination number* $\gamma_{sR}(D)$ of D is the minimum weight of an SRDF on D. A $\gamma_{sR}(D)$-function is a signed Roman dominating function on D of weight $\gamma_{sR}(D)$. For an SRDF f on D, let $V_i = V_i(f) = \{v \in V \mid f(v) = i\}$. A signed Roman dominating function $f : V \longrightarrow \{-1,1,2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of V. For notational convenience, we let $V_{1,2} = V_1 \cup V_2$, $|V_{1,2}| = n_{1,2}$ and $|V_i| = n_i$ for $i = -1,1,2$. Then, $n_{1,2} = n_1 + n_2$ and $n = n_1 + n_2 + n_{-1}$. Furthermore, let $D_{1,2} = D[V_1 \cup V_2]$ be the subdigraph induced by the set $V_{1,2} = V_1 \cup V_2$ and let $D_{1,2}$ have size $m_{1,2}$. For $i = 1,2$, if $V_i \neq \emptyset$, let $D_i = D[V_i]$ be the subdigraph induced by the set V_i and let D_i have size m_i. Hence, $m_{1,2} = m_1 + m_2 + |A(V_1,V_2)| + |A(V_2,V_1)|$. The signed Roman domination number of a digraph was introduced by Sheikholeslami and Volkmann in [8] and has been studied in [9].

A signed Roman dominating function of D is called a *twin signed Roman dominating function* (briefly TSRDF) if it also is a signed Roman dominating function of D^{-1}, i.e., $f(N^+[v]) \geq 1$ for every $v \in V$ and every vertex u for which $f(u) = -1$ has an out-neighbor v for which $f(v) = 2$. The twin signed Roman domination number for a digraph D is $\gamma^*_{sR}(D) = \min\{\omega(f) \mid f$ is a TSRDF of $D\}$. Since every TSRDF of D is an SRDF on both D and D^{-1} and since the constant function 1 is a TSRDF of D, we have

$$\max\{\gamma_{sR}(D), \gamma_{sR}(D^{-1})\} \leq \gamma^*_{sR}(D) \leq n. \tag{1.1}$$

In this paper, we initiate the study of the twin signed Roman domination number and establish some sharp lower bounds on twin signed Roman domination number of digraphs.

We make use of the following results in this paper.

Observation 1. *If $f = (V_{-1}, V_1, V_2)$ is a TSRDF on a digraph D of order n, then*

(a) $n = |V_1| + |V_2| + |V_{-1}|$.

(b) *Every vertex in V_{-1} is in-dominated and out-dominated by some vertex of V_2.*

(c) $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.

(d) *$V_1 \cup V_2$ is a dominating set of D.*

(e) *$\gamma^*_{sR}(D) = n - k$ if and only if $|V_2| = 2|V_{-1}| - k$.*

Proof. Since (a), (b), (c), (d) are trivial, we only prove (e). If $\gamma^*_{sR}(D) = n-k$, then we deduce from (a) and (c) that $|V_1|+|V_2|+|V_{-1}|-k = n-k = |V_1|+2|V_2|-|V_{-1}|$ which implies that $|V_2| = 2|V_{-1}|-k$. The proof of the inverse is similar. □

As we observed in (1.1), $\gamma^*_{sR}(D) \geq \max\{\gamma_{sR}(D), \gamma_{sR}(D^{-1})\}$. Now we show that the difference $\gamma^*_{sR}(D) - \max\{\gamma_{sR}(D), \gamma_{sR}(D^{-1})\}$ can be arbitrarily large.

Theorem 2. *For every positive integer k, there exists a digraph D such that*

$$\gamma^*_{sR}(D) - \max\{\gamma_{sR}(D), \gamma_{sR}(D^{-1})\} \geq 4k-2.$$

Proof. Let $k \geq 1$ be an integer, and let D be a digraph with vertex set

$$V(D) = \{x, y, u_1, u_2, \ldots, u_{2k}, v_1, v_2, \ldots, v_{2k}\}$$

and arc set

$$A(D) = \{(x, u_i), (v_i, u_i), (y, v_{k+i}), (u_{k+i}, x), (u_{k+i}, v_{k+i}), (v_i, y) \mid 1 \leq i \leq k\}.$$

Obviously, $D \cong D^{-1}$ and so, $\gamma_{sR}(D) = \gamma_{sR}(D^{-1})$. It is easy to verify that the function $f : V(D) \to \{-1, 1, 2\}$ defined by $f(x) = f(y) = 2, f(u_i) = f(v_{k+i}) = -1$ for $1 \leq i \leq k$ and $f(u) = 1$ otherwise, is an SRDF of D and so $\gamma_{sR}(D) \leq 4$. Now let g be a $\gamma^*_{sR}(D)$-function. Since $N^+[u] = \{u\}$ for each $u \in \{u_i, v_{k+i} \mid 1 \leq i \leq k\}$ and $N^-[u] = \{u\}$ for each $u \in \{u_{k+i}, v_i \mid 1 \leq i \leq k\}$, we must have $g(u) \geq 1$ for each $u \in V(D) - \{x, y\}$. It follows that $\gamma^*_{sR}(D) \geq 4k+2$. Thus $\gamma^*_{sR}(D) - \max\{\gamma_{sR}(D), \gamma_{sR}(D^{-1})\} \geq 4k-2$, and the proof is complete. □

Observation 3. *Let D be a digraph of order $n \geq 2$. Then $\gamma^*_{sR}(D) \geq 3-n$, with equality if and only if D is a directed cycle of order 2.*

Proof. Let f be a $\gamma^*_{sR}(D)$-function. If $f(x) \geq 1$ for each $x \in V(D)$, then $\gamma^*_{sR}(D) \geq n \geq 3-n$. Suppose next that $f(v) = -1$ for at least one vertex $v \in V(D)$. Then there exists a vertex $w \in V(D)$ with $f(w) = 2$ and therefore $\gamma^*_{sR}(D) \geq 2-(n-1) = 3-n$, and the desired bound is proved.

Assume now that $\gamma^*_{sR}(D) = 3-n$, and let f be a $\gamma^*_{sR}(D)$-function. This implies that D has exactly one vertex w with $f(w) = 2$ and $n-1$ vertices $y_1, y_2, \ldots, y_{n-1}$ such that $f(y_i) = -1$ for $1 \leq i \leq n-1$. By the definition, w is an in-neighbor as well as an out-neighbor of y_i for $1 \leq i \leq n-1$. This implies $n = 2$, and thus D is a directed 2-cycle of order 2.

Conversely, if D is a directed 2-cycle of order 2 with the vertex set $\{v, w\}$, then define the function $g : V(D) \to \{-1, 1, 2\}$ by $g(v) = -1$ and $g(w) = 2$. Then g is a TSRDF on D of weight $1 = 3-n$ and so $\gamma^*_{sR}(D) = 3-n$. □

Observation 4. *If D is a digraph of order $n \geq 3$, then $\gamma^*_{sR}(D) \geq 5-n$.*

Proof. Let f be a $\gamma^*_{sR}(D)$-function. If $f(x) \geq 1$ for each $x \in V(D)$, then $\gamma^*_{sR}(D) \geq n \geq 5-n$. Suppose next that $f(v) = -1$ for at least one vertex $v \in V(D)$. Then there exist a vertex $w \in V(D)$ with $f(w) = 2$. If there is a second vertex $z \neq w$ with $f(z) = 2$, then we deduce that $\gamma^*_{sR}(D) \geq 4-(n-2) \geq 6-n$. Thus we suppose that $f(x) \leq 1$ for $x \in V(D)-\{w\}$. If $f(x) = 1$ for all vertices $x \in V(D)-\{v, w\}$, then we obtain $\gamma^*_{sR}(D) \geq 1+(n-2) \geq 5-n$. So assume that there is a second vertex $u \neq v$ with $f(u) = -1$. Then w is an in-neighbor as well as an out-neighbor of u and v. Consequently, there exists an in-neighbor z_1 and an out-neighbor z_2 ($z_1 = z_2$ is possible) such that $f(z_1) = f(z_2) = 1$. This leads to $\gamma^*_{sR}(D) \geq 3-(n-2) = 5-n$, and the proof is complete. □

2. Special families of digraphs

In this section we determine the twin signed Roman domination number of some classes of digraphs, including directed paths, directed cycles, acyclic tournaments and circulant tournaments.

Proposition 5. *If P_n is a directed path of order $n \geq 3$, then $\gamma^*_{sR}(P_n) = \lfloor \frac{n}{2} \rfloor + 2$.*

Proof. Let $P_n = v_1 v_2 \dots v_n$ be a directed path where $(v_i, v_{i+1}) \in A(P_n)$ for $1 \leq i \leq n-1$. First we show that $\gamma^*_{sR}(P_n) \geq \lfloor \frac{n}{2} \rfloor + 2$. Let f be a $\gamma^*_{sR}(P_n)$-function. Clearly, $f(v_1) \geq 1$, $f(v_n) \geq 1$ and if $f(v_i) = -1$, then $f(v_{i+1}) = f(v_{i-1}) = 2$ for each $2 \leq i \leq n-1$. We proceed by induction on n. Assume that $n = 3$. If $f(v_2) \geq 1$, then $\gamma^*_{sR}(P_3) = \omega(f) \geq 3$ as desired. If $f(v_2) = -1$, then we must have $f(v_1) = f(v_3) = 2$ that implies $\gamma^*_{sR}(P_3) = \omega(f) \geq 3$. If $n = 4$, then as above we can see that $\gamma^*_{sR}(P_4) \geq 4$. Let $n \geq 5$, and let the statement be true for any directed path of order less than n. If $f(v_{n-1}) \geq 1$, then clearly the function f, restricted to $P_n - v_n$ is a TSRDF and it follows from the induction hypothesis that $\gamma^*_{sR}(P_n) = \omega(f) \geq f(v_n) + \gamma^*_{sR}(P_{n-1}) \geq 1 + \lfloor \frac{n-1}{2} \rfloor + 2 \geq \lfloor \frac{n}{2} \rfloor + 2$. Assume that $f(v_{n-1}) = -1$. Then $f(v_n) = f(v_{n-2}) = 2$ and clearly the function f, restricted to $P_n - \{v_n, v_{n-1}\}$ is a TSRDF and it follows from the induction hypothesis that $\gamma^*_{sR}(P_n) = \omega(f) \geq f(v_n) + f(v_{n-1}) + \gamma^*_{sR}(P_{n-2}) \geq 1 + \lfloor \frac{n-2}{2} \rfloor + 2 = \lfloor \frac{n}{2} \rfloor + 2$. Thus $\gamma^*_{sR}(P_n) \geq \lfloor \frac{n}{2} \rfloor + 2$.

Now we show that $\gamma^*_{sR}(P_n) \leq \lfloor \frac{n}{2} \rfloor + 2$. If n is odd, then define $g : V(P_n) \longrightarrow \{-1, 1, 2\}$ by $g(v_{2i-1}) = 2$ for $1 \leq i \leq (n+1)/2$ and $g(v_{2i}) = -1$ for $1 \leq i \leq (n-1)/2$, and if n is even then define $g : V(P_n) \longrightarrow \{-1, 1, 2\}$ by $g(v_n) = 1, g(v_{2i-1}) = 2$ for $1 \leq i \leq n/2$ and $g(v_{2i}) = -1$ for $1 \leq i \leq (n-2)/2$. It is easy to see that g is a TSRDF of P_n of weight $\lfloor \frac{n}{2} \rfloor + 2$ and so $\gamma^*_{sR}(P_n) = \lfloor \frac{n}{2} \rfloor + 2$. This completes the proof. □

The proof of next result can be found in [8].

Proposition A. *Let C_n be a directed cycle of order $n \geq 2$. Then $\gamma_{sR}(C_n) = n/2$ when n is even and $\gamma_{sR}(C_n) = (n+3)/2$ when n is odd.*

Proposition 6. *Let C_n be a directed cycle of order $n \geq 2$. Then $\gamma^*_{sR}(C_n) = \gamma_{sR}(C_n)$.*

Proof. By (1.1) and Proposition A, we have

$$\gamma^*_{sR}(C_n) \geq \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n+3)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Let $C_n = v_1 v_2 \ldots v_n v_1$. If n is even, then the function $g : V(C_n) \longrightarrow \{-1, 1, 2\}$ defined by $g(v_{2i-1}) = -1$ and $g(v_{2i}) = 2$ for $1 \leq i \leq n/2$, is clearly a TSRDF on C_n of weight $n/2$. Thus $\gamma^*_{sR}(C_n) = \gamma_{sR}(C_n)$ in this case. Let n be odd and define $g : V(C_n) \longrightarrow \{-1, 1, 2\}$ by $g(v_{2i-1}) = -1$ and $g(v_{2i}) = 2$ for $1 \leq i \leq (n-1)/2$ and $g(v_n) = 2$. Obviously, g is a TSRDF on C_n of weight $(n+3)/2$ and so $\gamma^*_{sR}(C_n) = \gamma_{sR}(C_n)$. □

A *tournament* is a digraph in which for every pair u, v of different vertices, either $(u, v) \in A(D)$ or $(v, u) \in A(D)$, but not both. Here we determine the exact value of the twin signed Roman domination number for particular types of tournaments.

The *acyclic tournament* $AT(n)$ of order n has the vertex set $V(AT(n)) = \{u_1, u_2, \ldots, u_n\}$. An arc goes from u_i into u_j if and only if $i < j$.

Let n be an odd positive integer such that $n = 2r + 1$ with a positive integer r. We define the *circulant tournament* $\mathrm{CT}(n)$ with n vertices as follows. The vertex set of $\mathrm{CT}(n)$ is $V(\mathrm{CT}(n)) = \{u_0, u_1, \ldots, u_{n-1}\}$. For each i, the arcs are going from u_i to the vertices $u_{i+1}, u_{i+2}, \ldots, u_{i+r}$, where the indices are taken modulo n.

Proposition 7. *For $n \geq 3$,*

$$\gamma^*_{sR}(AT(n)) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 4 & \text{if } n \text{ is even.} \end{cases}$$

Proof. First let n be odd. Define $f : V(\mathrm{AT}(n)) \to \{-1, 1, 2\}$ by $f(u_1) = f(u_n) = 2, f(v_{2i}) = -1$ for $1 \leq i \leq (n-1)/2$ and $f(x) = 1$ otherwise. It is easy to see that f is a TSRDF on $AT(n)$ of weight 3 which implies $\gamma^*_{sR}(\mathrm{AT}(n)) \leq 3$. Now we show that $\gamma^*_{sR}(\mathrm{AT}(n)) \geq 3$. Assume that g is a $\gamma^*_{sR}(\mathrm{AT}(n))$-function. Since $N^-[u_1] = \{u_1\}$ and $N^+[u_n] = \{u_n\}$, we have $g(u_1) \geq 1$ and $g(u_n) \geq 1$. If $V_{-1} = \emptyset$, then the result is immediate. Suppose that $u_i \in V_{-1}$ for some $2 \leq i \leq n-1$. Then $\gamma^*_{sR}(\mathrm{AT}(n)) = \omega(g) = g(N^-[u_i]) + g(N^+[u_i]) - g(u_i) \geq 3$ and hence $\gamma^*_{sR}(\mathrm{AT}(n)) = 3$.
Now let $n \geq 4$ be even. Define $f : V(\mathrm{AT}(n)) \to \{-1, 1, 2\}$ by $f(u_1) = f(u_n) = 2, f(u_{2i}) = -1$ for $1 \leq i \leq (n-2)/2$ and $f(x) = 1$ otherwise. It is easy to see that f is a TSRDF on $AT(n)$ of weight 4 which implies that $\gamma^*_{sR}(\mathrm{AT}(n)) \leq 4$. To prove $\gamma^*_{sR}(\mathrm{AT}(n)) \geq 4$, let g be a $\gamma^*_{sR}(\mathrm{AT}(n))$-function. If $V_{-1} = \emptyset$, then the result is immediate. Suppose that $V_{-1} \neq \emptyset$. If $g(u_i) = 2$ for some $2 \leq i \leq n-1$, then we have $\gamma^*_{sR}(\mathrm{AT}(n)) = \omega(g) = g(N^-[u_{i-1}]) + g(N^+[u_{i+1}]) + g(u_i) \geq 4$ as desired. Henceforth, $g(u_i) \leq 1$ for each $2 \leq i \leq n-1$ and $g(u_1) = g(u_n) = 2$. We claim that $|V_{-1}| \leq |V_1|$. Assume, to the contrary, $|V_{-1}| > |V_1|$. Since $|V_{-1}| + |V_1| = n-2$ is even, we deduce

that $|V_{-1}| \geq |V_1|+2$. But then $g(N^-[u_{n-1}]) \leq 0$ which is a contradiction. Thus $|V_{-1}| \leq |V_1|$ and so $\gamma^*_{sR}(\mathrm{AT}(n)) = \omega(g) = 2|V_2|+|V_1|-|V_{-1}| \geq 4$ as desired. Therefore $\gamma^*_{sR}(\mathrm{AT}(n)) = 4$ in this case. This completes the proof. □

The proof of the next result can be found in [8].

Proposition B. *Let $n = 2r+1$ where r is a positive integer. If $r \neq 2$, then $\gamma_{sR}(\mathrm{CT}(n)) = 3$ and $\gamma_{sR}(\mathrm{CT}(5)) = 4$.*

Proposition 8. *Let $n \geq 3$ and $n = 2r+1$, where r is a positive integer. Then*

$$\gamma^*_{sR}(CT(n)) = \begin{cases} 5 & \text{if } n = 5, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. First let $r \neq 2$. If $r = 1$, then the result is immediate by Proposition 6. Suppose that $r \geq 3$. By (1.1) and Proposition B, we have $\gamma^*_{sR}(\mathrm{CT}(n)) \geq 3$. If r is an odd integer, then define the function $f : V(CT(n)) \to \{-1,1,2\}$ by $f(u_{\frac{r+1}{2}}) = f(u_{r+\frac{r+1}{2}}) = 2$, $f(u_i) = -1$ for $0 \leq i \leq \frac{r-1}{2}$ or $r+1 \leq i \leq r+\frac{r-1}{2}$ and $f(u_i) = 1$ otherwise. Assume now that r is even. If $r = 4$, then define $f(u_3) = f(u_4) = f(u_7) = f(u_8) = 2$ and $f(x) = -1$ otherwise. If $r \geq 6$, then define the function $f : V(CT(n)) \to \{-1,1,2\}$ by $f(u_{\frac{r+2}{2}}) = f(u_{\frac{r+4}{2}}) = f(u_{\frac{3r+2}{2}}) = f(u_{\frac{3r+4}{2}}) = 2, f(u_i) = -1$ for $0 \leq i \leq \frac{r}{2}$ or $r+1 \leq i \leq r+\frac{r}{2}$ and $f(u_i) = 1$ otherwise. Obviously, f is a TSRDF on $CT(n)$ with $\omega(f) = 3$ in each case. It follows that $\gamma^*_{sR}(CT(n)) = 3$.
Now, let $r = 2$ and f be a $\gamma^*_{sR}(\mathrm{CT}(5))$-function. If $V_{-1} = \emptyset$, then $\gamma^*_{sR}(\mathrm{CT}(5)) \geq 5$. Let $V_{-1} \neq \emptyset$. Assume, without loss of generality, that $f(u_0) = -1$. Since f is a TSRDF, we must have $f(u_1)+f(u_2) \geq 3$ and $f(u_3)+f(u_4) \geq 3$. This yields $\gamma^*_{sR}(\mathrm{CT}(5)) \geq 5$. Now the result follows by (1.1) and the proof is complete. □

3. Bounds

In this section we establish some lower sharp bounds on the twin signed Roman domination number in terms of order and size of a digraph.

Theorem 9. *If D is a digraph of order $n \geq 2$, then*

$$\gamma^*_{sR}(D) \geq \frac{3}{\sqrt{2}}\sqrt{n} - n.$$

Furthermore, this bound is sharp.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma^*_{sR}(D)$-function. If $V_{-1} = \emptyset$, then $\gamma^*_{sR}(D) \geq n \geq \frac{3}{\sqrt{2}}\sqrt{n} - n$ for $n \geq 2$. Hence we may assume that $V_{-1} \neq \emptyset$. Since each vertex of V_{-1} has at least one in-neighbor and one out-neighbor in V_2, it follows from the Pigeonhole Principle that we have

$$|A(v, V_{-1})| + |A(V_{-1}, v)| \geq \frac{2|V_{-1}|}{|V_2|} = \frac{2n_{-1}}{n_2}$$

for at least one vertex $v \in V_2$. Therefore $2 \le f(N^+[v]) + f(N^-[v]) \le 4n_2 + 2n_1 - \frac{2n_{-1}}{n_2}$, and so $2n_2^2 + n_1 n_2 - n_{-1} - n_2 \ge 0$. It follows from Observation 1 (part (a)) that $2n_2^2 + n_1 n_2 + n_1 - n \ge 0$. Since $n_2 \ge 1$ and n_1 is an non-negative integer, $\frac{5}{3}n_1 n_2 - \frac{1}{9}n_1 \ge 0$ and $n_1^2 \ge n_1$. Thus

$$2n_2^2 + \frac{8}{3}n_1 n_2 + \frac{8}{9}n_1^2 - n \ge (2n_2^2 + n_1 n_2 + n_1 - n) + (\frac{5}{3}n_1 n_2 - \frac{1}{9}n_1) \ge 0$$

and thus $3n_2 + 2n_1 \ge 3\sqrt{n/2}$. Therefore $\gamma^*_{sR}(D) = 2n_2 + n_1 - n_{-1} = 3n_2 + 2n_1 - n \ge 3\sqrt{n/2} - n$, which establishes the desired bound.

To prove the sharpness, let K_k^* be the complete digraph with $k \ge 1$ vertices $v_1, v_2, \ldots, v_k$. To each v_i add $2k-1$ vertices $w_1^i, w_2^i, \ldots, w_{2k-1}^i$ such that w_j^i is an in-neighbor as well as an out-neighbor of v_i for $1 \le i \le k$ and $1 \le j \le 2k-1$. Let D be the resulting digraph of order $2k^2$. Then the function $f : V(D) \to \{-1, 1, 2\}$ defined by $f(v) = 2$ if $v \in V(K_k^*)$ and $f(x) = -1$ otherwise, is a TSRDF of weight $2k - k(2k-1) = 3k - 2k^2 = 3\sqrt{n(D)/2)} - n(D)$. Therefore $\gamma^*_{sR}(D) \le 3\sqrt{n(D)/2} - n(D)$ and so $\gamma^*_{sR}(D) = 3\sqrt{n(D)/2} - n(D)$. □

Next we establish some sharp bounds on the twin signed Roman domination numbers of oriented graphs.

Theorem 10. *If D is an oriented graph of order $n \ge 2$, then*

$$\gamma^*_{sR}(D) \ge \frac{3}{2}(-1 + \sqrt{1+4n}) - n.$$

Furthermore, this bound is sharp.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma^*_{sR}(D)$-function. If $V_{-1} = \emptyset$, then $\gamma^*_{sR}(D) \ge n \ge \frac{3}{2}(-1 + \sqrt{1+4n}) - n$ for $n \ge 2$. Hence, we may assume that $V_{-1} \ne \emptyset$. Since each vertex in V_{-1} has at least one in-neighbor and one out-neighbor in V_2, it follows from the Pigeonhole Principle that for at least one vertex $v \in V_2$ the sum of its in-neighbors and its out-neighbors is at least $\frac{2|V_{-1}|}{|V_2|} = \frac{2n_{-1}}{n_2}$. Therefore, $2 \le f(N^+[v]) + f(N^-[v]) \le 2n_2 + n_1 - \frac{2n_{-1}}{n_2} + 2$, and so $2n_2^2 + n_1 n_2 - 2n_{-1} \ge 0$. It follows from Observation 1 (part (a)) that $4n_2^2 + 2n_1 n_2 + 4n_2 + 4n_1 - 4n \ge 0$. Since $n_2 \ge 1$ and n_1 is an non-negative integer, $\frac{10}{3}n_1 n_2 - \frac{4}{9}n_1 \ge 0$. Therefore,

$$\begin{aligned} 4\left(n_2 + \frac{2}{3}n_1 + \frac{1}{2}\right)^2 - 1 - 4n &= 4n_2^2 + \frac{16}{9}n_1^2 + 1 + \frac{16}{3}n_1 n_2 + \frac{8}{3}n_1 + 4n_2 - 4n - 1 \\ &\ge (4n_2^2 + 2n_1 n_2 + 4n_2 + 4n_1 - 4n) + (\frac{10}{3}n_1 n_2 - \frac{4}{9}n_1) \\ &\ge 0, \end{aligned}$$

or equivalently, $3n_2 + 2n_1 \ge \frac{3}{2}(-1 + \sqrt{1+4n})$. Thus

$$\gamma^*_{sR}(D) = 2n_2 + n_1 - n_{-1} = 3n_2 + 2n_1 - n \ge \frac{3}{2}(-1 + \sqrt{1+4n}) - n,$$

as desired.

To prove the sharpness, let k be an odd integer and K_k a complete graph. Assume that $V(G) = \{v_1, v_2, \dots, v_k\}$ is the vertex set of K_k and let $\{C_1, C_2, \dots, C_{(k-1)/2}\}$ be a Hamiltonian factorization of K_k. Suppose, without loss of generality, that $C_1 = (v_1 v_2 \dots v_k)$. Let G_k be the graph obtained from K_k by adding k paths of length 2 between v_i and v_{i+1} for each $1 \le i \le n$, where $v_{n+1} = v_1$. Clearly, G_k has $k + k^2$ vertices. Suppose that H_k is an orientation of G_k so that:

1. C_i is a directed cycle in H_k for each i,
2. $(v_i, v_{i+1}) \in A(H_k)$ for $i = 1, 2, \dots, n$,
3. k paths of length 2 between v_i and v_{i+1} are directed paths from v_i to v_{i+1}.

Then the function $f : V(G) \to \{-1, 1, 2\}$ defined by $f(v) = 2$ if $v \in V(K_k)$ and $f(x) = -1$ otherwise, is a TSRDF of weight $2k - k^2 = \frac{3}{2}(-1 + \sqrt{1 + 4n(G)}) - n(G)$. Therefore $\gamma^*_{sR}(D) \le \frac{3}{2}(-1 + \sqrt{1 + 4n(G)}) - n(G)$ and so $\gamma^*_{sR}(D) = \frac{3}{2}(-1 + \sqrt{1 + 4n(G)}) - n(G)$. □

Let $f = (V_{-1}, V_1, V_2)$ be a signed Roman dominating function on D. We recall that $D_{1,2} = D[V_1 \cup V_2]$ is the subdigraph induced by the set $V_{1,2} = V_1 \cup V_2$ and $D_{1,2}$ has size $m_{1,2}$. Also, for $i = 1, 2$, if $V_i \neq \emptyset$, then $D_i = D[V_i]$ is the subdigraph induced by the set V_i and D_i has size m_i. Hence, $m_{1,2} = m_1 + m_2 + |A(V_1, V_2)| + |A(V_2, V_1)|$.

Theorem 11. *Let D be an oriented graph of order n and size m without isolated vertices. Then*

$$\gamma^*_{sR}(D) \ge \frac{3n}{2} - m.$$

Proof. Let D be an orientation of G and $f = (V_{-1}, V_1, V_2)$ be a $\gamma^*_{sR}(D)$-function. By the definition of a TSRDF, each vertex in V_{-1} has at least an in-neighbor and an out-neighbor in V_2, and so $|A(V_{-1}, V_{12})| + |A(V_{12}, V_{-1})| \ge |A(V_{-1}, V_2)| + |A(V_2, V_{-1})| \ge 2n_{-1}$. For each vertex $v \in V_2$, we have $f(v) + 2d^+_{V_2}(v) + d^+_{V_1}(v) - d^+_{V_{-1}}(v) = f(N^+[v]) \ge 1$, and $f(v) + 2d^-_{V_2}(v) + d^-_{V_1}(v) - d^-_{V_{-1}}(v) = f(N^-[v]) \ge 1$. So $f(v) + 2d^+_{V_2}(v) + d^+_{V_1}(v) - 1 \ge d^+_{V_{-1}}(v)$ and $f(v) + 2d^-_{V_2}(v) + d^-_{V_1}(v) - 1 \ge d^-_{V_{-1}}(v)$. Hence,

$$\begin{aligned}
2n_{-1} &\le |A(V_{-1}, V_2)| + |A(V_2, V_{-1})| \\
&= \sum_{v \in V_2} (d^+_{V_{-1}}(v) + d^-_{V_{-1}}(v)) \\
&\le \sum_{v \in V_2} (f(v) + 2d^+_{V_2}(v) + d^+_{V_1}(v) - 1 + f(v) + 2d^-_{V_2}(v) + d^-_{V_1}(v) - 1) \\
&= \sum_{v \in V_2} (2d^+_{V_2}(v) + 2d^-_{V_2}(v) + d^+_{V_1}(v) + d^-_{V_1}(v) + 2f(v) - 2)
\end{aligned}$$

$$
\begin{aligned}
&= 4m_2 + |A(V_1,V_2)| + |A(V_2,V_1)| + 2n_2 \\
&= 4m_{12} - 3(|A(V_1,V_2)| + |A(V_2,V_1)|) - 4m_1 + 2n_2,
\end{aligned}
$$

and so

$$m_{12} \geq \frac{3(|A(V_1,V_2)| + |A(V_2,V_1)|) + 4m_1 + 2n_{-1} - 2n_2}{4}.$$

Thus

$$
\begin{aligned}
m &\geq m_{12} + |A(V_{-1},V_{12})| + |A(V_{12},V_{-1})| + m_{-1} \\
&\geq \frac{3(|A(V_1,V_2)| + |A(V_2,V_1)|) + 4m_1 + 2n_{-1} - 2n_2}{4} + 2n_{-1} + m_{-1} \\
&= \frac{3(|A(V_1,V_2)| + |A(V_2,V_1)|) + 4m_1 + 10n_{-1} - 2n_{12} + 2n_1}{4} + m_{-1} \\
&= \frac{3(|A(V_1,V_2)| + |A(V_2,V_1)|) + 4m_1 + 10n - 12n_{12} + 2n_1}{4} + m_{-1}
\end{aligned}
$$

or equivalently, $n_{12} \geq (3(|A(V_1,V_2)| + |A(V_2,V_1)|) + 4m_1 + 10n - 4m + 2n_1 + 4m_{-1})/12$. So

$$
\begin{aligned}
\gamma^*_{sR}(D) &= 2n_2 + n_1 - n_{-1} \\
&= 3n_{12} - (n_2 + 2n_1) - n_{-1} \\
&= 3n_{12} - n_1 - (n_1 + n_2 + n_{-1}) \\
&= 3n_{12} - n_1 - n \\
&\geq \frac{(3(|A(V_1,V_2)| + |A(V_2,V_1)|) + 4m_1 + 10n - 4m + 2n_1 + 4m_{-1})}{4} - n_1 - n \\
&= \frac{3n - 2m}{2} + \frac{3(|A(V_1,V_2)| + |A(V_2,V_1)|) + 4m_1 - 2n_1 + 4m_{-1}}{4}.
\end{aligned}
$$

Let $I = (3(|A(V_1,V_2)| + |A(V_2,V_1)|) + 4m_1 - 2n_1 + 4m_{-1})/4$. It suffices to show that $I \geq 0$, since then $\gamma^*_{sR}(D) \geq \frac{3n-2m}{2}$ as desired. If $n_1 = 0$, then $I = m_{-1} \geq 0$. Henceforth, we may assume that $n_1 \geq 1$. If $v \in V_1$, $d^+_{V_{12}}(v) = 0$ and $d^-_{V_{12}}(v) = 0$, then since there is no isolated vertex in G, we deduce that every in-neighbor and out-neighbor of v belongs to V_{-1}. But then $f(N^-[v]) \leq 0$ and $f(N^+[v]) \leq 0$, a contradiction. Hence $|A(v, V_1 \cup V_2)| + |A(V_1 \cup V_2, v)| \geq 1$ for each $v \in V_1$. Therefore,

$$
\begin{aligned}
4I &= 3(|A(V_1,V_2)| + |A(V_2,V_1)|) - 2n_1 + 4m_1 + 4m_{-1} \\
&= |A(V_1,V_2)| + |A(V_2,V_1)| + 4m_{-1} + \\
&+ 2\sum_{v \in V_1} (|A(v,V_2)| + |A(V_2,v)|) + 2\sum_{v \in V_1} (|A(v,V_1)| + |A(V_1,v)|) - 2n_1 \\
&\geq |A(V_1,V_2)| + |A(V_2,V_1)| + 4m_{-1} > 0
\end{aligned}
$$

Therefore $\gamma^*_{sR}(D) > \frac{3n-2m}{2}$ and the proof is complete. □

Corollary 12. Let T be a tree of order $n \geq 3$. If $\overrightarrow{T}$ is an orientation of T, then $\gamma^*_{sR}(\overrightarrow{T}) \geq \lceil \frac{n}{2} \rceil + 1$.

Proposition 5 shows that Corollary 12 and therefore Theorem 11 is sharp for odd n.

4. Twin Signed Roman domination in oriented graphs

Let G be the complete bipartite graph $K_{3,5}$ with bipartite sets $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{u_1, u_2, u_3, u_4, u_5\}$. Let D_1 be an orientation of G such that all arcs go from V_1 to V_2 and D_2 be an orientation of G such that $A(D_2) = \{(v_i, u_j), (u_j, v_r) \mid i = 1, 2, r = 3 \text{ and } 1 \leq j \leq 5\}$. It is easy to see that $\gamma^*_{sR}(D_1) = 8$ and $\gamma^*_{sR}(D_2) = 5$. Thus two distinct orientations of a graph can have distinct twin signed Roman domination numbers. Motivated by this observation, we define lower orientable twin signed Roman domination number $\mathrm{dom}^*_{sR}(G)$ and upper orientable twin signed Roman domination number $\mathrm{Dom}^*_{sR}(G)$ of a graph G as follows:

$$\mathrm{dom}^*_{sR}(G) = \min\{\gamma^*_{sR}(D) \mid \text{D is an orientation of } G\},$$

and

$$\mathrm{Dom}^*_{sR}(G) = \max\{\gamma^*_{sR}(D) \mid \text{D is an orientation of } G\}.$$

Corresponding concepts have been defined and studied for orientable domination (out-domination) [7], twin signed total domination [4] and twin domination number [6].

In this section, we determine the orientable twin signed Roman domination number of complete bipartite graphs and complete graphs. Let $m \leq n$ and $K_{m,n}$ be the bipartite graph with bipartite sets $X = \{u_1, u_2, \ldots, u_m\}$ and $Y = \{v_1, v_2, \ldots, v_n\}$.

Proposition 13. For $n \geq 2$,

$$\mathrm{dom}^*_{sR}(K_{2,n}) = \begin{cases} 3 & \text{if } n = 3 \\ 2 & \text{otherwise.} \end{cases}$$

Proof. The result is immediate for $n = 2$ by Proposition 6. Suppose that $n \geq 3$. First let $n \geq 4$. Let D be an orientation of $K_{2,n}$ such that $\gamma^*_{sR}(D) = \mathrm{dom}^*_{sR}(K_{2,n})$ and let f be a $\gamma^*_{sR}(D)$-function. Assume, without loss of generality, that $f(u_1) \leq f(u_2)$. It follows from $f(N^+[u_1]) \geq 1$ and $f(N^-[u_1]) \geq 1$ that

$$2 \leq f(N^+[u_1]) + f(N^-[u_1]) = \sum_{i=1}^{n} f(v_i) + 2f(u_1) = \gamma^*_{sR}(D) + f(u_1) - f(u_2).$$

Since $f(u_1) \leq f(u_2)$, we deduce that $\gamma^*_{sR}(D) \geq 2$.

Now we show that $\mathrm{dom}^*_{sR}(K_{2,n}) \leq 2$. Let D be an orientation of $K_{2,n}$ such that

$$A(D) = \{(v_1, u_1), (u_2, v_1)\} \cup \{(u_1, v_j), (v_j, u_2) \mid 2 \leq j \leq n\}.$$

First let n be even. Define $f : V(K_{2,n}) \to \{-1,1,2\}$ by $f(v_1) = -1, f(u_1) = f(u_2) = 2$ and $f(v_j) = (-1)^{j+1}$ for $2 \le j \le n$. It is easy to see that f is a TSRDF of D of weight 2 and so $\mathrm{dom}^*_{sR}(K_{2,n}) \le 2$ in this case. Now let n be odd. Define $f : V(K_{2,n}) \to \{-1,1,2\}$ by $f(v_1) = f(v_2) = f(v_3) = -1, f(u_1) = f(u_2) = f(v_4) = 2$ and $f(v_j) = (-1)^j$ for $5 \le j \le n$. Clearly, f is a TSRDF of D of weight 2. Hence, $\mathrm{dom}^*_{sR}(K_{2,n}) = 2$.

Suppose that $n = 3$. If $f(u_1) = -1$ (the case $f(u_2) = -1$ is similar), then at least for two i, $f(v_i) = 2$, say $i = 1,2$ and $f(v_3) \ge 1$. Thus we have $\gamma^*_{sR}(K_{2,3}) \ge 5-2 = 3$. Henceforth, we assume $f(u_i) \ge 1$ for $i = 1,2$. Let $f(v_i) = -1$ for some $1 \le i \le 3$. Then $f(u_1) = f(u_2) = 2$. Since either $d^+(u_1) > d^-(u_1)$ or $d^-(u_1) > d^+(u_1)$, we have $\sum_{i=1}^{3} f(v_i) \ge -1$. Thus $\gamma^*_{sR}(K_{2,3}) \ge 3$. To prove $\gamma^*_{sR}(K_{2,3}) \le 3$, let D be an orientation of $K_{2,3}$ such that

$$A(D) = \{(v_i, u_1), (u_2, v_i) \mid i = 1,2\} \cup \{(u_1, v_3), (v_3, u_2)\}.$$

Now define $g : V(D) \to \{-1,1,2\}$ by $g(u_1) = g(u_2) = 2$, $f(v_2) = +1$ and $f(v_1) = f(v_3) = -1$. Obviously, g is a TSRDF on D of weight 3 and the proof is complete. □

Proposition 14. $\mathrm{dom}^*_{sR}(K_{3,3}) = 5$.

Proof. First we show that $\mathrm{dom}^*_{sR}(K_{3,3}) \ge 5$. Let D be an orientation of $K_{3,3}$ with $\gamma^*_{sR}(D) = \mathrm{dom}^*_{sR}(K_{3,3})$, and let f be a $\gamma^*_{sR}(D)$-function. If $f(x) \ge 1$ for each $x \in V(K_{3,3})$, then we are done. Assume, without loss of generality, that $f(u_1) = -1$. Since f is a TSRDF of D, u_1 has an in-neighbor and an out-neighbor with label 2. Suppose $f(v_1) = f(v_2) = 2$. We deduce from $f(N^+[u_1]) \ge 1$ and $f(N^-[u_1]) \ge 1$ that $f(v_3) \ge 1$ and so

$$\sum_{i=1}^{3} f(v_i) \ge 5. \tag{3.1}$$

If $f(v_3) = 1$, then it follows from $f(N^+[v_3]) \ge 1$ and $f(N^-[v_3]) \ge 1$ that $\sum_{i=1}^{3} f(u_i) \ge 0$ and so $\gamma^*_{sR}(D) = \sum_{i=1}^{3} f(u_i) + \sum_{i=1}^{3} f(v_i) \ge 5$ as desired. Let $f(v_3) = 2$. We deduce from $f(N^+[v_3]) \ge 1$ and $f(N^-[v_3]) \ge 1$ that $\sum_{i=1}^{3} f(u_i) \ge -1$ and so $\gamma^*_{sR}(D) = \sum_{i=1}^{3} f(u_i) + \sum_{i=1}^{3} f(v_i) \ge 5$. Thus $\mathrm{dom}^*_{sR}(K_{3,3}) \ge 5$.

To prove $\mathrm{dom}^*_{sR}(K_{3,3}) \le 5$, let D be an orientation of $K_{3,3}$ such that

$$A(D) = \{(u_1, v_i), (u_2, v_i), (v_i, u_3) \mid 1 \le i \le 3\}$$

and define $f : V(K_{3,3}) \to \{-1,1,2\}$ by $f(u_1) = f(u_3) = f(v_1) = 2, f(u_2) = 1, f(v_2) = f(v_3) = -1$. It is easy to see that f is a TSRDF of D of weight 5 and so $\mathrm{dom}^*_{sR}(K_{3,3}) \le 5$. Thus $\mathrm{dom}^*_{sR}(K_{3,3}) = 5$ and the proof is complete. □

Proposition 15. For $n \ge 4$, $\mathrm{dom}^*_{sR}(K_{3,n}) = 4$.

Proof. First we show that $\mathrm{dom}^*_{sR}(K_{3,n}) \geq 4$. Let D be an orientation of $K_{3,n}$ such that $\gamma^*_{sR}(D) = \mathrm{dom}^*_{sR}(K_{3,n})$, and let f be a $\gamma^*_{sR}(D)$-function such that $|V_{-1} \cap Y|$ is as large as possible. Suppose $f(X) = \sum_{i=1}^{3} f(u_i)$ and $f(Y) = \sum_{i=1}^{n} f(v_i)$. If $V_{-1} = \emptyset$, then $\gamma^*_{sR}(D) = n+3 > 4$ and we are done. Assume that $V_{-1} \neq \emptyset$. First let $|V_{-1} \cap Y| \geq 1$. Assume, without loss of generality, that $v_1 \in V_{-1} \cap Y$. Since f is a TSRDF of D, v_1 has an in-neighbor and an out-neighbor with label 2. Suppose $f(u_1) = f(u_2) = 2$. It follows from $f(N^+[u_3]) \geq 1$ and $f(N^-[u_3]) \geq 1$ that $f(Y) \geq 2 - 2f(u_3)$. Hence

$$\gamma^*_{sR}(D) = f(X) + f(Y) \geq (4 + f(u_3)) + (2 - 2f(u_3)) = 6 - f(u_3) \geq 4.$$

Now let $|V_{-1} \cap Y| = 0$. Then $V_{-1} \subseteq X$. Assume, without loss of generality, that $u_1 \in V_{-1}$. As above, we may assume that $f(v_1) = f(v_2) = 2$. If $f(v_i) = 1$ for some $3 \leq i \leq n$, then we must have $f(X) \geq 2 - 2f(v_i) \geq 0$ that implies $\gamma^*_{sR}(D) = f(X) + f(Y) \geq n + 2 \geq 6$. Otherwise, we have $\gamma^*_{sR}(D) = f(X) + f(Y) \geq 2n - 3 \geq 5$. Thus $\gamma^*_{sR}(D) \geq 4$ in all cases.

Now we prove that $\mathrm{dom}^*_{sR}(K_{3,n}) \leq 4$. First let n be even. Let D be an orientation of $K_{3,n}$ such that

$$A(D) = \{(u_1, v_1), (u_3, v_1), (v_1, u_2), (v_2, u_3)\} \cup \{(v_i, u_1), (u_2, v_i), (u_3, v_j) \mid 2 \leq i \leq n, 3 \leq j \leq n\}.$$

Define $f : V(K_{3,n}) \to \{-1, 1, 2\}$ by $f(v_1) = f(v_2) = -1, f(u_1) = f(u_2) = f(u_3) = 2$ and $f(v_j) = (-1)^j$ for $3 \leq j \leq n$. It is easy to see that f is a TSRDF of D of weight 4 and so $\mathrm{dom}^*_{sR}(K_{3,n}) \leq 4$. Hence $\mathrm{dom}^*_{sR}(K_{3,n}) = 4$ in this case. Now let n be odd. Let D be an orientation of $K_{3,n}$ such that

$$A(D) = \{(v_1, u_1), (u_2, v_1), (u_3, v_1)\} \cup \{(u_1, v_i), (v_i, u_2), (v_i, u_3) \mid 2 \leq i \leq n\}.$$

Define $f : V(K_{3,n}) \to \{-1, 1, 2\}$ by $f(v_1) = f(v_2) = f(v_3) = f(v_4) = -1, f(v_5) = 2, f(u_1) = f(u_2) = f(u_3) = 2$ and $f(v_j) = (-1)^j$ for $6 \leq j \leq n$. It is easy to see that f is a TSRDF of D of weight 4 and so $\mathrm{dom}^*_{sR}(K_{3,n}) \leq 4$. Thus $\mathrm{dom}^*_{sR}(K_{3,n}) = 4$ and the proof is complete. □

Proposition 16. If $m = 4, 5$, then $\mathrm{dom}^*_{sR}(K_{m,n}) = m + 2$.

Proof. First we show that $\mathrm{dom}^*_{sR}(K_{m,n}) \geq m + 2$. Let D be an orientation of $K_{m,n}$ such that $\gamma^*_{sR}(D) = \mathrm{dom}^*_{sR}(K_{m,n})$ and let f be a $\gamma^*_{sR}(D)$-function such that $|V_{-1} \cap Y|$ is as large as possible. The result is immediate if $|V_{-1}| = 0$. Suppose that $|V_{-1}| \geq 1$. If $|V_{-1} \cap X| \neq 0$ and $|V_{-1} \cap Y| \neq 0$, then $f(X) \geq 4$ and $f(Y) \geq 4$. Thus $\gamma^*_{sR}(K_{m,n}) = f(X) + f(Y) \geq 8 > m + 2$. Suppose that $|V_{-1} \cap X| = 0$ (the case $|V_{-1} \cap Y| = 0$ is similar). If $|V_1 \cap X| \geq 1$, then clearly $f(Y) \geq 0$. On the other hand, it follows from $V_{-1} \cap Y \neq \emptyset$ that $|V_2 \cap X| \geq 2$ and so $f(X) \geq m + 2$. Therefore $\gamma^*_{sR}(K_{m,n}) = f(X) + f(Y) \geq m + 2$. Let $f(u_i) = 2$ for each $1 \leq i \leq m$. Then we have $f(X) \geq 2m$ and $f(Y) \geq -2$. Consequently, $\gamma^*_{sR}(K_{m,n}) = f(X) + f(Y) \geq 2m - 2 \geq m + 2$.

Now we show that $\mathrm{dom}^*_{sR}(K_{m,n}) \le m+2$. Let D be an orientation of $K_{m,n}$ such that

$$A(D) = \{(u_1, v_i), (v_i, u_2) \mid 1 \le i \le n\} \cup \{(u_i, v_j) \mid 3 \le i \le m, 1 \le j \le n\}.$$

Define $f : V(K_{m,n}) \to \{-1, 1, 2\}$ by $f(u_1) = f(u_2) = 2$, $f(u_i) = 1$ for $3 \le i \le m$ and $f(v_i) = (-1)^{i+1}$ for $1 \le i \le n$ when n is even, and $f(v_1) = 2$, $f(v_2) = f(v_3) = -1$, $f(v_i) = (-1)^{i+1}$ for $4 \le i \le n$ if n is odd. It is easy to see that f is a TSRDF of D of weight $m+2$ and so $\mathrm{dom}^*_{sR}(K_{m,n}) \le m+2$. Thus $\mathrm{dom}^*_{sR}(K_{m,n}) = m+2$ and the proof is complete. □

Proposition 17. For $n \ge m \ge 6$, $\mathrm{dom}^*_{sR}(K_{m,n}) = 8$.

Proof. First we show that $\mathrm{dom}^*_{sR}(K_{m,n}) \ge 8$. Let D be an orientation of $K_{m,n}$ such that $\gamma^*_{sR}(D) = \mathrm{dom}^*_{sR}(K_{m,n})$, and let $f = (V_{-1}, V_1, V_2)$ be a $\gamma^*_{sR}(D)$-function. If $V_{-1} = \emptyset$, we are done. Let $V_{-1} \ne \emptyset$. If $|V_{-1} \cap X| \ne 0$ and $|V_{-1} \cap Y| \ne 0$, then clearly $f(X) \ge 4$ and $f(Y) \ge 4$ that implies $\gamma^*_{sR}(K_{m,n}) = f(X) + f(Y) \ge 8$. Suppose that $|V_{-1} \cap Y| = 0$ (the case $|V_{-1} \cap X| = 0$ is similar). Since each vertex with label -1 has an in-neighbor and an out-neighbor with label 2, we have $f(Y) \ge n+2$. If $V_1 \cap Y \ne \emptyset$, then $f(X) \ge 0$ that implies $\gamma^*_{sR}(K_{m,n}) = f(X) + f(Y) \ge n+2 \ge 8$. Otherwise, $f(v_i) = 2$ for each $1 \le i \le n$. It follows that $f(Y) \ge 2n$ and $f(X) \ge -2$. Consequently, $\gamma^*_{sR}(K_{m,n}) = f(X) + f(Y) \ge 2n-2 > 8$.

To prove $\mathrm{dom}^*_{sR}(K_{m,n}) \le 8$, let D be an orientation of $K_{m,n}$ such that

$$A(D) = \{(u_i, v_1), (v_2, u_i) \mid 1 \le i \le m\} \cup \{(u_1, v_j), (v_j, u_2), (u_i, v_j) \mid 3 \le i \le m, 3 \le j \le n\}.$$

If m and n are even, then define $f : V(K_{m,n}) \to \{-1, 1, 2\}$ by $f(u_1) = f(u_2) = f(v_1) = f(v_2) = 2$, $f(u_i) = (-1)^i$ for each $3 \le i \le m$ and $f(v_j) = (-1)^j$ for each $3 \le j \le n$. If m and n are odd, then define $f : V(K_{m,n}) \to \{-1, 1, 2\}$ by $f(u_1) = f(u_2) = f(u_3) = f(v_1) = f(v_2) = f(v_3) = 2$, $f(u_4) = f(u_5) = f(v_4) = f(v_5) = -1$, $f(u_i) = (-1)^i$ for each $6 \le i \le m$ and $f(v_j) = (-1)^j$ for each $6 \le j \le n$. If m is even and n is odd (the case m is odd and n is even is similar), then define $f : V(K_{m,n}) \to \{-1, 1, 2\}$ by $f(u_1) = f(u_2) = f(v_1) = f(v_2) = f(v_3) = 2$, $f(v_4) = f(v_5) = -1$, $f(u_i) = (-1)^i$ for each $3 \le i \le m$ and $f(v_j) = (-1)^j$ for each $6 \le j \le n$. It is easy to see that f is a TSRDF of D of weight 8 and so $\mathrm{dom}^*_{sR}(K_{m,n}) \le 8$. Thus $\mathrm{dom}^*_{sR}(K_{m,n}) = 8$ and the proof is complete. □

Proposition 18. If G is a bipartite graph of order n, then $\mathrm{Dom}^*_{sR}(G) = n$.

Proof. Let X and Y be the partite sets of G. Let D be an orientation of G such that $A(D) = \{(u, v) \mid u \in X \text{ and } v \in Y\}$ and let f be a $\gamma^*_{sR}(D)$-function. Since $d^-(u) = 0$ for each $u \in X$ and $d^+(v) = 0$ for each $v \in Y$, we must have $f(x) \ge 1$ for each $x \in V(G)$. Hence, $\mathrm{Dom}^*_{sR}(G) \ge \omega(f) = n$ and the result follows by (1.1). □

Proposition 19. For $n \geq 3$,

$$\mathrm{dom}^*_{sR}(K_n) = \begin{cases} 4 & \text{if } n = 4,6 \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_n. First let $n \neq 4,6$. Assume that D is an orientation of K_n such that $\gamma^*_{sR}(D) = \mathrm{dom}^*_{sR}(K_n)$ and f is a $\gamma^*_{sR}(D)$-function. If $V_{-1} = \emptyset$, then clearly $\mathrm{dom}^*_{sR}(K_n) \geq 3$. Hence, we may suppose that $f(v_i) = -1$ for some i, say $i = 1$. Since $f(N^-[v_1]) \geq 1$ and $f(N^+[v_1]) \geq 1$, we have that

$$\mathrm{dom}^*_{sR}(K_n) = f(N^+[v_1]) + f(N^-[v_1]) - f(v_1) \geq 1 + 1 - (-1) = 3.$$

Now we show that $\mathrm{dom}^*_{sR}(K_n) \leq 3$. If n is odd, then $\mathrm{dom}^*_{sR}(K_n) \leq 3$ by Proposition 7. Let n be even and D be an orientation of K_n such that

$$\begin{aligned} A(D) =&\{(v_1,v_3),(v_1,v_5),(v_1,v_7),(v_2,v_1),(v_2,v_6),(v_2,v_8),(v_3,v_2,),(v_3,v_4),(v_3,v_6),(v_3,v_8),\\ &(v_4,v_1),(v_4,v_2),(v_4,v_6),(v_4,v_7),(v_5,v_2),(v_5,v_3),(v_5,v_4),(v_5,v_6),(v_5,v_7),(v_5,v_8),\\ &(v_6,v_1),(v_7,v_2),(v_7,v_3),(v_7,v_6),(v_7,v_8),(v_8,v_1),(v_8,v_4),(v_8,v_6)\}\\ &\cup\{(v_j,v_i),(v_i,v_2) \mid j = 1,3,\ldots,8 \text{ and } 9 \leq i \leq n\} \cup \{(v_i,v_{i+1}),(v_j,v_{j+1}),(v_i,v_j),\\ &(v_i,v_{j+1}),(v_{i+1},v_j),(v_{i+1},v_{j+1}) \mid 9 \leq i < j \leq n \text{ and } i,j \text{ are odd}\}. \end{aligned}$$

Define $f : V(K_n) \to \{-1,1,2\}$ by $f(v_1) = f(v_2) = f(v_3) = 2$, $f(v_4) = 1$, $f(v_5) = f(v_6) = f(v_7) = f(v_8) = -1$ and $f(v_i) = (-1)^{i+1}$ for $9 \leq i \leq n$. It is easy to verify that f is a TSRDF of D of weight 3 and so $\mathrm{dom}^*_{sR}(K_n) \leq 3$. Thus $\mathrm{dom}^*_{sR}(K_n) = 3$ in this case.

It is not hard to see that $\mathrm{dom}^*_{sR}(K_4) = 4$. Let $n = 6$. It follows from Proposition 7 that $\mathrm{dom}^*_{sR}(K_6) \leq 4$. Assume that D is an orientation of K_6 such that $\gamma^*_{sR}(D) = \mathrm{dom}^*_{sR}(K_6)$ and f is a $\gamma^*_{sR}(D)$-function. Since $\omega(f) \leq 4$, we have $V_{-1} \neq \emptyset$. Since each vertex with label -1 must have an in-neighbor and an out-neighbor with label 2, we may assume, without loss of generality, that $f(v_1) = -1$ and $f(v_2) = f(v_3) = 2$. As above, we obtain $\mathrm{dom}^*_{sR}(K_6) \geq 3$ that implies $f(v_4) + f(v_5) + f(v_6) \geq 0$. If $f(v_4) + f(v_5) + f(v_6) = 0$, then, without loss of generality, we may suppose that $f(v_4) = f(v_5) = -1$, $f(v_6) = 2$. The digraph induced by v_1, v_4, v_5 has at least one vertex with in-degree one and out-degree one, say x. Since $f(N^+[x]) \geq 1$ and $f(N^-[x]) \geq 1$, the vertex x should have at least two in-neighbors and two out-neighbors in $\{v_2, v_3, v_6\}$ that is impossible. Thus $f(v_4) + f(v_5) + f(v_6) \geq 1$ that implies $\mathrm{dom}^*_{sR}(K_6) \geq 4$. This completes the proof. □

References

[1] H. Abdollahzadeh Ahangar, J. Amjadi, S. M. Sheikholeslami, L. Volkmann and Y. Zhao, *Signed Roman edge domination numbers in graphs*, J. Comb. Optim., **31** (2016), 333–346.

[2] H. Abdollahzadeh Ahangar, M. A. Henning, Y. Zhao, C. Löwenstein and V. Samodivkin, *Signed Roman domination in graphs*, J. Comb. Optim., **27** (2014), 241–255.

[3] S. Arumugam, K. Ebadi and L. Sathikala, *Twin domination and twin irredudance in digraphs*, Appl. Anal. Discrete Math., **7** (2013), 275–284.

[4] M. Atapour, A. Bodaghli and S. M. Sheikholeslami, *Twin signed total domination numbers in directed graphs*, Ars Combin. (to appear)

[5] E. W. Chambers, B. Kinnersley, N. Prince and D. B. West, *Extremal problems for Roman domination*, SIAM J. Discrete Math., **23** (2009), 1575–1586.

[6] G. Chartrand, P. Dankelmann, M. Schultz, H.C. Swart, *Twin domination in digraphs*, Ars Combin., **67** (2003), 105–114.

[7] G. Chartrand, D. W. VanderJagt, B. Q. Yue, *Orientable domination in graphs*, Congr. Numer., **119** (1996), 51–63.

[8] S. M. Sheikholeslami and L. Volkmann, *Signed Roman domination in digraphs*, J. Comb. Optim., **30** (2015), 456–467.

[9] S. M. Sheikholeslami and L. Volkmann, *The signed Roman domatic number of a digraph*, Electronic Journal of Graph Theory and Applications, **3** (2015), 85–93.

[10] D. B. West, Introduction to Graph Theory, Prentice-Hall, Inc, 2000.

Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I.R. Iran.

E-mail: bodaghli@azaruniv.edu

Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, I.R. Iran.

E-mail: s.m.sheikholeslami@azaruniv.edu

Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany.

E-mail: volkm@math2.rwth-aachen.de