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COEFFICIENTS OF STRONGLY ALPHA-CONVEX AND STRONGLY GAMMA STARLIKE FUNCTIONS

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Abstract. Let the function f be analytic in $\mathbb{D}=\{z:|z|<1\}$ and be given by $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. For $0<\beta\le 1$, denote by $\mathscr{C}(\beta)$ and $\mathscr{S}^*(\beta)$ the classes of strongly convex functions and strongly starlike functions respectively. For $0\le\alpha\le 1$, $0<\beta\le 1$ and $0\le\gamma\le 1$, let $\mathscr{M}(\alpha,\beta)$ be the class of strongly alpha-convex functions defined by

$$\left|\arg\left[(1-\alpha)\frac{zf'(z)}{f(z)}\right)+\alpha(1+\frac{zf''(z)}{f'(z)})\right]\right|<\frac{\pi\beta}{2},$$

and $\mathscr{M}^*(\gamma,\beta)$ the class of strongly gamma starlike functions defined by

$$\left|\arg\left[\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right]\right|<\frac{\pi\beta}{2}.$$

We give sharp bounds for the initial coefficients of $f\in\mathscr{M}(\alpha,\beta)$ and $f\in\mathscr{M}^*(\gamma,\beta)$, and for the initial coefficients of the inverse function f^{-1} of $f\in\mathscr{M}(\alpha,\beta)$ and $f\in\mathscr{M}^*(\gamma,\beta)$. These results generalise, improve and unify known coefficient inequalities for $\mathscr{C}(\beta)$ and $\mathscr{S}^*(\beta)$.

1. Introduction

Let $\mathscr{A}$ be the class of analytic normalized functions f, defined in the unit disk $\mathbb{D}=\{z:|z|<1\}$ given by

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

and let $\mathscr{S}$ be the subset of $\mathscr{A}$ consisting of functions which are univalent in $\mathbb{D}$.

Suppose that $f\in\mathscr{A}$. Then f is respectively strongly starlike, or strongly convex of order β in $\mathbb{D}$ if, and only if, for $0<\beta\leqslant 1$,

$$\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\pi\beta}{2},\quad\text{or}\quad\left|\arg(1+\frac{zf''(z)}{f'(z)})\right|<\frac{\pi\beta}{2}.$$

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We denote these classes by $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$ respectively, noting that $\mathcal{S}^*(1)$ is the class of starlike functions, and $\mathcal{C}(1)$ the class of convex functions, so that both $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$ are subsets of $\mathcal{S}$.

For any real number α, denote by $\mathcal{M}(\alpha)$ the class of alpha-convex, or so-called Ma-Minda functions [9] defined for $z \in \mathbb{D}$ by the relationship

$$Re\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha(1+\frac{zf''(z)}{f'(z)})\right]>0.$$

Thus $\mathcal{M}(0)$ gives the starlike functions, and $\mathcal{M}(1)$ the convex functions.

It was shown in [10] that for all real α, the class $\mathcal{M}(\alpha)$ forms a subset of the starlike functions, and is therefore a subset of $\mathcal{S}$. Finding sharp bounds for all coefficients of $f \in \mathcal{M}(\alpha)$ has received much attention, see e.g. [6, 11, 13], however a complete solution appears still to be an open problem.

Similarly, for $\gamma \geqslant 0$, denote by $\mathcal{M}^*(\gamma)$ the class of gamma starlike functions, (see e.e. [4, 5, 7]) defined for $z \in \mathbb{D}$ by the relationship

$$Re\left[\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right]>0,$$

so that again $\mathcal{M}^*(0)$ gives the starlike functions, and $\mathcal{M}^*(1)$ the convex functions. It was shown in [7] (and elsewhere), that for $\gamma \geqslant 0$, $\mathcal{M}^*(\gamma) \subset \mathcal{S}^*(1)$. However since the definition of functions in $\mathcal{M}^*(\gamma)$ requires dealing with powers, relatively little is known about the coefficients of functions in $\mathcal{M}^*(\gamma)$.

In the interests of unifying known results for $f \in \mathcal{S}^*(\beta)$ and $f \in \mathcal{C}(\beta)$, we will assume throughout this paper that $0 \leqslant \alpha \leqslant 1$, and $0 \leqslant \gamma \leqslant 1$. We also remark that for α and γ outside $[0,1]$, the methods used in this paper give incomplete results.

Preliminaries

Strongly Alpha-Convex Functions of Order β

Let f be analytic in $\mathbb{D}$ and be given by (1.1). For $0 \leqslant \alpha \leqslant 1$ and $0 < \beta \leqslant 1$, we say that f is strongly alpha-convex of order β in $\mathbb{D}$ if, and only if,

$$\left|\arg\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha(1+\frac{zf''(z)}{f'(z)})\right]\right|<\frac{\pi\beta}{2}. \tag{1.2}$$

We denote this class of functions by $\mathcal{M}(\alpha,\beta)$, so that $\mathcal{M}(0,\beta)=\mathcal{S}^*(\beta)$ and $\mathcal{M}(1,\beta)=\mathcal{C}(\beta)$. Also since $\mathcal{M}(\alpha) \subset \mathcal{S}^*(1)$, then so must $\mathcal{M}(\alpha,\beta) \subset \mathcal{S}^*(1)$ for $0 \leqslant \alpha \leqslant 1$ and $0 < \beta \leqslant 1$.

Strongly Gamma Starlike Functions of Order β

Let f be analytic in $\mathbb{D}$ and be given by (1.1). For $0 \leqslant \gamma \leqslant 1$ and $0 < \beta \leqslant 1$, we say that f is strongly gamma starlike of order β in $\mathbb{D}$ if, and only if,

$$\left|\arg\left[\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right]\right| < \frac{\pi\beta}{2}. \tag{1.3}$$

We denote this class of functions by $\mathcal{M}^*(\gamma,\beta)$, so that again $\mathcal{M}^*(0,\beta) = \mathcal{S}^*(\beta)$ and $\mathcal{M}^*(1,\beta) = \mathcal{C}(\beta)$. As was pointed out above, since $\mathcal{M}^*(\gamma) \subset \mathcal{S}^*(1)$, it follows that $\mathcal{M}^*(\gamma,\beta) \subset \mathcal{S}^*(1)$ for $0 \leqslant \gamma \leqslant 1$ and $0 < \beta \leqslant 1$.

An early paper of Brannan, Clunie and Kirwan [3] established sharp upper bounds for $|a_2|$ and $|a_3|$ when $f \in \mathcal{S}^*(\beta)$, and more recently Ali and Singh [2] obtained sharp upper bounds for $|a_4|$. Since $f \in \mathcal{C}(\beta)$ if, and only if, $zf' \in \mathcal{S}^*(\beta)$, these results provide immediate sharp upper bounds for these coefficients when $f \in \mathcal{C}(\beta)$. Since the analysis necessitates the use of powers, finding bounds for the remaining coefficients appears difficult.

In Theorems 2.1 and 2.2, we give sharp bounds for $|a_2|$, $|a_3|$ and $|a_4|$ for $f \in \mathcal{M}(\alpha,\beta)$ and for $f \in \mathcal{M}^*(\gamma,\beta)$, thus unifying and generalising the above results.

For any univalent function f, there exists an inverse function f^{-1} defined on some disc $|\omega| < r_0(f)$, with Taylor expansion

$$f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \cdots. \tag{1.4}$$

A classical theorem of Löwner [8] established sharp upper bounds for the modulus of the inverse coefficients A_n for all $n \geqslant 2$ when $f \in \mathcal{S}$, which in particular solves the problem for functions in $\mathcal{S}^*(1)$.

For $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$ with $0 < \beta < 1$, the problem of finding bounds for the inverse coefficients again seems far from simple, the only sharp results to date being those found for $f \in \mathcal{S}^*(\beta)$ by Ali [1] for $|A_n|$ when $n = 2, 3$ and 4, and in a recent paper [12], similar sharp bounds for the inverse coefficients of functions in $\mathcal{C}(\beta)$.

In Theorems 3.1 and 3.2, we will find sharp bounds for the initial coefficients of the inverse function f^{-1} of functions in $\mathcal{M}(\alpha,\beta)$ and $\mathcal{M}^*(\gamma,\beta)$, again unifying the above results.

Lemmas

Denote by $\mathcal{P}$, the class of functions p satisfying $Re\ p(z) > 0$ for $z \in \mathbb{D}$, with Taylor series

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

We shall use the following [1].

Lemma 1.1. *If $p \in \mathscr{P}$, then $|p_n| \leqslant 2$ for $n \geqslant 1$, and*

$$\left|p_2 - \frac{\mu}{2} p_1^2\right| \leqslant \max\{2,\ 2|\mu - 1|\} = \begin{cases} 2, & 0 \leqslant \mu \leqslant 2, \\ 2|\mu - 1|, & \textit{elsewhere.} \end{cases}$$

Lemma 1.2. *Let $p \in \mathscr{P}$. If $0 \leqslant B \leqslant 1$ and $B(2B-1) \leqslant D \leqslant B$, then*

$$\left|p_3 - 2Bp_1p_2 + Dp_1^3\right| \leqslant 2.$$

Lemma 1.3. *If $p \in \mathscr{P}$, and $0 \leqslant B \leqslant 1$, then*

$$\left|p_3 - 2Bp_1p_2 + Bp_1^3\right| \leqslant 2.$$

Lemma 1.4. *If $p \in \mathscr{P}$, then*

$$\left|p_3 - (1+\mu)p_1p_2 + \mu p_1^3\right| \leqslant \max\{2,\ 2|2\mu - 1|\} = \begin{cases} 2, & 0 \leqslant \mu \leqslant 1, \\ 2|2\mu - 1|, & \textit{elsewhere.} \end{cases}$$

In the following, the methods of proof develop those employed in [1, 2], and in the interests of brevity, we omit much of the elementary algebra.

Main Results

2. Coefficients of functions in $\mathscr{M}(\alpha,\beta)$ and $\mathscr{M}^*(\gamma,\beta)$

Theorem 2.1. *Let $f \in \mathscr{M}(\alpha,\beta)$ and be given by (1.1), then*

$$|a_2| \leqslant \frac{2\beta}{1+\alpha}, \qquad |a_3| \leqslant \begin{cases} \dfrac{\beta}{1+2\alpha}, & 0 < \beta \leqslant \dfrac{(1+\alpha)^2}{(3+8\alpha+\alpha^2)}, \\[2ex] \dfrac{(3+8\alpha+\alpha^2)\beta^2}{(1+\alpha)^2(1+2\alpha)}, & \dfrac{(1+\alpha)^2}{(3+8\alpha+\alpha^2)} \leqslant \beta \leqslant 1, \end{cases}$$

$$|a_4| \leqslant \begin{cases} \dfrac{2\beta}{3(1+3\alpha)}, & 0 < \beta \leqslant \Delta_1(\alpha), \\[2ex] \Gamma_1(\alpha,\beta), & \Delta_1(\alpha) \leqslant \beta \leqslant 1, \end{cases}$$

where $\Delta_1(\alpha) = \sqrt{\dfrac{2(1+\alpha)^3(1+2\alpha)}{17+109\alpha+219\alpha^2+59\alpha^3+4\alpha^4}}$ *and*

$$\Gamma_1(\alpha,\beta) = \frac{2\beta(1+17\beta^2+2\alpha^4(1+2\beta^2)+\alpha^3(7+59\beta^2)+3\alpha^2(3+73\beta^2)+\alpha(5+109\beta^2))}{9(1+\alpha)^3(1+2\alpha)(1+3\alpha)}.$$

All the inequalities are sharp.

Proof. From (1.2) we can write

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha(1+\frac{zf''(z)}{f'(z)})=p(z)^{\beta},$$

and so equating coefficients we obtain

$$\begin{aligned}
a_2 &= \frac{\beta p_1}{1+\alpha},\\
a_3 &= \frac{\beta((-1+\alpha^2(-1+\beta)+3\beta+\alpha(-2+8\beta))p_1^2+2(1+\alpha)^2p_2)}{4(1+\alpha)^2(1+2\alpha)},\\
a_4 &= \frac{\beta(4-15\beta+17\beta^2+4\alpha^4(2-3\beta+\beta^2)+\alpha^3(28-87\beta+59\beta^2))p_1^3}{36(1+\alpha)^3(1+2\alpha)(1+3\alpha)}\\
&\quad+\frac{\beta(3\alpha^2(12-51\beta+73\beta^2)+\alpha(20-93\beta+109\beta^2))p_1^3}{36(1+\alpha)^3(1+2\alpha)(1+3\alpha)}\\
&\quad+\frac{6\beta(1+\alpha)^2(-2+4\alpha^2(-1+\beta)+5\beta+3\alpha(-2+7\beta))p_1p_2}{36(1+\alpha)^3(1+2\alpha)(1+3\alpha)}\\
&\quad+\frac{12\beta(1+\alpha)^3(1+2\alpha)p_3}{36(1+\alpha)^3(1+2\alpha)(1+3\alpha)}.
\end{aligned} \tag{2.1}$$

Since $|p_1|\leqslant 2$, the first inequality follows at once, and is sharp when $p_1=2$.

Next note that the coefficient of p_1^2 in the expression for a_3 in (2.1) is positive provided $\dfrac{(1+\alpha)^2}{(3+8\alpha+\alpha^2)}\leqslant\beta\leqslant 1$, and so the second inequality for $|a_3|$ in Theorem 2.1 follows since $|p_1|\leqslant 2$ and $|p_2|\leqslant 2$.

For the first inequality we apply Lemma 1.1. Write

$$a_3=\frac{\beta}{2(1+2\alpha)}\left(p_2-\frac{\mu}{2}p_1^2\right),$$

with

$$\mu=\frac{1+\alpha^2(1-\beta)-3\beta+2\alpha(1-4\beta)}{(1+\alpha)^2},$$

so that $0\leqslant\mu\leqslant 2$ provided $0<\beta\leqslant\dfrac{(1+\alpha)^2}{(3+8\alpha+\alpha^2)}$. The inequality now follows on applying Lemma 1.1.

The first inequality for $|a_3|$ is sharp when $p_1=0$ and $p_2=2$, and the second inequality is sharp when $p_1=p_2=2$.

For a_4, first write $\Lambda_1(\alpha)=\dfrac{2(1+\alpha)(1+2\alpha)}{(1+4\alpha)(5+\alpha)}$, and note that the coefficients of p_1, p_2 and p_1^3 are positive when $\Lambda_1(\alpha)\leqslant\beta\leqslant 1$. So using $|p_n|\leqslant 2$ for $n=1,2$ and 3, gives the second inequality for $|a_4|$ when $\Lambda_1(\alpha)\leqslant\beta\leqslant 1$.

Next write

$$a_4 = \frac{\beta}{3(1+3\alpha)}\left(p_3 - 2B_1 p_1 p_2 + D_1 p_1^3\right),$$

with

$$B_1 = \frac{2+4\alpha^2(1-\beta)-5\beta+3\alpha(2-7\beta)}{4(1+\alpha)(1+2\alpha)},$$

and

$$\begin{aligned} D_1 = \frac{1}{12(1+\alpha)^3(1+2\alpha)}&\Big(4-15\beta+17\beta^2+4\alpha^4(2-3\beta+\beta^2)\\ &+\alpha^3(28-87\beta+59\beta^2)+3\alpha^2(12-51\beta+73\beta^2)\\ &+\alpha(20-93\beta+109\beta^2)\Big). \end{aligned}$$

Then $0 \leqslant B_1 \leqslant 1$ if $0 < \beta \leqslant \Lambda_1(\alpha)$, and $B_1(2B_1-1) \leqslant D_1 \leqslant B_1$ when $0 < \beta \leqslant \Delta_1(\alpha)$. Since $\Delta_1(\alpha) < \Lambda_1(\alpha)$, applying Lemma 1.2 now gives the first inequality for $|a_4|$.

Thus it remains to prove the second inequality on the interval $\Delta_1(\alpha) \leqslant \beta \leqslant \Lambda_1(\alpha)$.

We use Lemma 1.3, and the inequality $|p_1| \leqslant 2$, noting that $0 \leqslant B_1 \leqslant 1$ and $D_1 - B_1 \geqslant 0$, when $\Delta_1(\alpha) \leqslant \beta \leqslant \Lambda_1(\alpha)$ to obtain

$$\begin{aligned} |p_3 - 2B_1 p_1 p_2 + D_1 p_1^3 &= |p_3 - 2B_1 p_1 p_2 + B_1 p_1^3 + (D_1 - B_1)p_1^3| \\ &\leqslant 2 + 8(D_1 - B_1), \end{aligned}$$

from which the result follows.

The first inequality for $|a_4|$ is sharp when $p_1 = p_2 = 0$ and $p_3 = 2$, and the second inequality is sharp when $p_1 = p_2 = p_3 = 2$. □

We note at this point that when $\beta = 1$, the results in Theorem 2.1 correspond to the estimates found in [6], when $\alpha = 0$ to those in [2, 3], and when $\alpha = 1$ to those in [12].

Theorem 2.2. *Let $f \in \mathcal{M}^*(\gamma,\beta)$ and be given by* (1.1)*, then*

$$|a_2| \leqslant \frac{2\beta}{1+\gamma}, \qquad |a_3| \leqslant \begin{cases} \dfrac{\beta}{1+2\gamma}, & 0 < \beta \leqslant \dfrac{(1+\gamma)^2}{3(1+3\gamma)}, \\[2ex] \dfrac{3\beta^2(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}, & \dfrac{(1+\gamma)^2}{3(1+3\gamma)} \leqslant \beta \leqslant 1, \end{cases}$$

$$|a_4| \leqslant \begin{cases} \dfrac{2\beta}{3(1+3\gamma)}, & 0 < \beta \leqslant \Delta_1^*(\gamma), \\[2ex] \Gamma_1^*(\gamma,\beta), & \Delta_1^*(\gamma) \leqslant \beta \leqslant 1, \end{cases}$$

where $\Delta_1^*(\gamma)=\sqrt{\dfrac{2(1+5\gamma+9\gamma^2+7\gamma^3+2\gamma^4)}{17+108\gamma+283\gamma^2}}$ *and*

$$\Gamma_1^*(\gamma,\beta)=\frac{2\beta(1+7\gamma^3+2\gamma^4+17\beta^2+\gamma(5+108\beta^2)+\gamma^2(9+283\beta^2))}{9(1+\gamma)^3(1+2\gamma)(1+3\gamma)}.$$

All the inequalities are sharp.

Proof. From (1.3) we can write

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}=p(z)^{\beta}.$$

Equating coefficients gives

$$\begin{aligned}
a_2 =&\frac{\beta p_1}{1+\gamma},\\
a_3 =&\frac{\beta(-(1+\gamma^2+\gamma(2-9\beta)-3\beta)p_1^2+2(1+\gamma)^2p_2)}{4(1+\gamma)^2(1+2\gamma)},\\
a_4 =&\frac{\beta(4+8\gamma^4+\gamma^3(28-75\beta)-15\beta+17\beta^2)p_1^3}{36(1+\gamma)^3(1+2\gamma)(1+3\gamma)}\\
&+\frac{\beta(\gamma(20-105\beta+108\beta^2)+\gamma^2(36-165\beta+283\beta^2))p_1^3}{36(1+\gamma)^3(1+2\gamma)(1+3\gamma)}\\
&-\frac{6\beta(1+\gamma)^2(2+4\gamma^2+\gamma(6-25\beta)-5\beta)p_1p_2}{36(1+\gamma)^3(1+2\gamma)(1+3\gamma)}\\
&+\frac{12\beta(1+\gamma)^3(1+2\gamma)p_3}{36(1+\gamma)^3(1+2\gamma)(1+3\gamma)}.
\end{aligned}\tag{2.2}$$

Since $|p_1|\leqslant 2$, the first inequality is trivial, and is sharp when $p_1=2$.

For a_3 we use Lemma 1.1 as follows.

Write

$$a_3=\frac{\beta}{2(1+2\gamma)}\left(p_2-\frac{(1+\gamma^2+\gamma(2-9\beta)-3\beta)}{2(1+\gamma)^2}p_1^2\right).$$

Taking $\mu=\dfrac{1+\gamma^2+\gamma(2-9\beta)-3\beta}{(1+\gamma)^2}$, we note that $0\leqslant\mu\leqslant 2$ when $0<\beta\leqslant\dfrac{(1+\gamma)^2}{3(1+3\gamma)}$, and so the first inequality for $|a_3|$ follows. Applying Lemma 1.1 when μ lies outside $[0,2]$ gives the second inequality.

The first inequality for $|a_3|$ is sharp when $p_1=0$ and $p_2=2$, and the second inequality is sharp when $p_1=p_2=2$.

For the first inequality for a_4 we again use Lemma 1.2. Write

$$a_4 = \frac{\beta}{3(1+3\gamma)}\Big(p_3 - 2B_1^* p_1 p_2 + D_1^* p_1^3\Big),$$

where

$$B_1^* = \frac{2+4\gamma^2+\gamma(6-25\beta))-5\beta}{4(1+\gamma)(1+2\gamma)},$$

and

$$D_1^* = \frac{1}{12(1+\gamma)^3(1+2\gamma)}\Big(4+8\gamma^4+\gamma^3(28-75\beta)-15\beta+17\beta^2 +\gamma(20-105\beta+108\beta^2)+\gamma^2(36-165\beta+283\beta^2)\Big).$$

Write $\Lambda_1^*(\gamma) = \dfrac{2(1+\gamma)(1+2\gamma)}{5(1+5\gamma)}$. Then $0 \leqslant B_1^* \leqslant 1$ when $0 < \beta \leqslant \Lambda_1^*(\gamma)$, and $B_1^*(2B_1^*-1) \leqslant D_1^* \leqslant B_1^*$ when $0 < \beta \leqslant \Delta_1^*(\gamma)$.

Since $\Delta_1^*(\gamma) < \Lambda_1^*(\gamma)$, applying Lemma 1.2 now gives the first inequality for $|a_4|$ on the interval $0 \leqslant \beta \leqslant \Delta_1^*(\gamma)$.

Since $-2B_1^*$ and D_1^* are positive when $\Lambda_1^*(\gamma) \leqslant \beta \leqslant 1$ and $\Delta_1^*(\gamma) \leqslant \beta \leqslant 1$, the second inequality for $|a_4|$ now follows (on using the inequalities $|p_n| \leqslant 2$ for $n = 1,2$ and 3) provided $\Lambda_1^*(\gamma) \leqslant \beta \leqslant 1$, and noting that $\Delta_1^*(\gamma) \leqslant \Lambda_1^*(\gamma)$.

Thus it remains to prove the second inequality for $|a_4|$ on the interval $\Delta_1^*(\gamma) \leqslant \beta \leqslant \Lambda_1^*(\gamma)$.

Since $0 \leqslant B_1^* \leqslant 1$ when $0 < \beta \leqslant \Lambda_1^*(\gamma)$, and $D_1^* \geqslant B_1^*$ when $\Delta_1^*(\gamma) \leqslant \beta \leqslant 1$,

$$\begin{aligned}|p_3 - 2B_1^* p_1 p_2 + D_1^* p_1^3| &= |p_3 - 2B_1^* p_1 p_2 + B_1^* p_1^3 + (D_1^* - B_1^*) p_1^3| \\ &\leqslant 2 + 8(D_1^* - B_1^*),\end{aligned}$$

when $\Delta_1^*(\gamma) \leqslant \beta \leqslant \Lambda_1^*(\gamma)$, which on substituting for D_1^* and B_1^*, and using Lemma 1.3, proves the inequality for $|a_4|$ on the interval $\Delta_1^*(\gamma) \leqslant \beta \leqslant \Lambda_1^*(\gamma)$.

The first inequality for $|a_4|$ is sharp when $p_1 = p_2 = 0$ and $p_3 = 2$, and the second inequality is sharp when $p_1 = p_2 = p_3 = 2$. □

We note at this point that when $\beta = 1$, the results in Theorem 2.2 complete the partial solution given in [4]. When $\alpha = 0$ the results correspond to those in [2, 3], and when $\alpha = 1$ to those in [12].

3. Inverse coefficients of functions in $\mathcal{M}(\alpha,\beta)$ and $\mathcal{M}^*(\gamma,\beta)$

First note that since $f\left(f^{-1}(\omega)\right)=\omega$, comparing coefficients in (1.1) and (1.4) gives

$$\begin{aligned} A_2 &= -a_2,\\ A_3 &= 2a_2^2-a_3,\\ A_4 &= -5a_2^3+5a_2a_3-a_4. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let $f\in\mathcal{M}(\alpha,\beta)$ and the coefficients of the inverse function f^{-1} be given by (1.4), then*

$$|A_2|\leqslant\frac{2\beta}{1+\alpha},\qquad |A_3|\leqslant\begin{cases}\dfrac{\beta}{1+2\alpha}, & 0<\beta\leqslant\dfrac{(1+\alpha)^2}{(5+8\alpha-\alpha^2)},\\[2ex] \dfrac{(5+8\alpha-\alpha^2)\beta^2}{(1+\alpha)^2(1+2\alpha)}, & \dfrac{(1+\alpha)^2}{(5+8\alpha-\alpha^2)}\leqslant\beta\leqslant 1,\end{cases}$$

$$|A_4|\leqslant\begin{cases}\dfrac{2\beta}{3(1+3\alpha)}, & 0<\beta\leqslant\Delta_2(\alpha),\\[2ex] \Gamma_2(\alpha,\beta), & \Delta_2(\alpha)\leqslant\beta\leqslant 1,\end{cases}$$

where

$$\Delta_2(\alpha)=\sqrt{\frac{(1+\alpha)^3(1+2\alpha)}{31+122\alpha+87\alpha^2-38\alpha^3+2\alpha^4}}$$

and

$$\Gamma_2(\alpha,\beta)=\frac{2\beta(1+62\beta^2+\alpha^3(7-76\beta^2)+\alpha^4(2+4\beta^2)+3\alpha^2(3+58\beta^2)+\alpha(5+244\beta^2))}{9(1+\alpha)^3(1+2\alpha)(1+3\alpha)}.$$

All the inequalities are sharp.

Proof. From (2.1) and (3.1) we obtain

$$\begin{aligned} A_2 &= -\frac{\beta p_1}{1+\alpha},\\ A_3 &= \frac{\beta((1-\alpha^2(-1+\beta)+5\beta+2\alpha(1+4\beta))p_1^2-2(1+\alpha)^2p_2)}{4(1+\alpha)^2(1+2\alpha)},\\ A_4 &= -\beta\Big(\frac{(2+15\beta+31\beta^2+\alpha^3(14+24\beta-38\beta^2)+2\alpha^4(2-3\beta+\beta^2))p_1^3}{18(1+\alpha)^3(1+2\alpha)(1+3\alpha)}\\ &\quad+\frac{(3\alpha^2(6+27\beta+29\beta^2)+2\alpha(5+33\beta+61\beta^2))p_1^3}{18(1+\alpha)^3(1+2\alpha)(1+3\alpha)}\\ &\quad+\frac{6(1+\alpha)^2(-1+2\alpha^2(-1+\beta)-5\beta-3\alpha(1+4\beta))p_1p_2}{18(1+\alpha)^3(1+2\alpha)(1+3\alpha)}+\frac{p_3}{3(1+3\alpha)}\Big). \end{aligned}$$

Again, the first inequality is trivial, and is sharp when $p_1 = 2$.

For A_3 we write

$$A_3 = -\frac{\beta}{2(1+2\alpha)}\Big(p_2 - \frac{(1+\alpha^2(1-\beta)+5\beta+2\alpha(1+4\beta))}{2(1+\alpha)^2}p_1^2\Big),$$

and apply Lemma 1.1 with $\mu = \dfrac{1+\alpha^2(1-\beta)+5\beta+2\alpha(1+4\beta)}{(1+\alpha)^2}$. Then since $0 \leqslant \mu \leqslant 2$ when $0<\beta \leqslant \dfrac{(1+\alpha)^2}{(5+8\alpha-\alpha^2)}$, the inequalities for $|A_3|$ follows at once.

The first inequality for $|A_3|$ is sharp when $p_1 = 0$ and $p_2 = 2$, and the second inequality is sharp when $p_1 = p_2 = 2$.

To find the bound for the first inequality for A_4, we follow the same method employed in Theorem 2.1 and use Lemma 1.2 so that

$$A_4 = \frac{\beta}{3(1+3\alpha)}\Big(p_3 - 2B_2p_1p_2 + D_2p_1^3\Big),$$

with

$$B_2 = \frac{1+2\alpha^2(1-\beta)+5\beta+3\alpha(1+4\beta)}{2(1+\alpha)(1+2\alpha)},$$

and

$$D_2 = \frac{1}{6(1+\alpha)^3(1+2\alpha)}\Big(2+15\beta+31\beta^2+\alpha^3(14+24\beta-38\beta^2) +2\alpha^4(2-3\beta+\beta^2)+3\alpha^2(6+27\beta+29\beta^2)+2\alpha(5+33\beta+61\beta^2)\Big).$$

Write $\Lambda_2(\alpha) = \dfrac{(1+\alpha)(1+2\alpha)}{(5+12\alpha-2\alpha^2)}$. Then $0 \leqslant B_2 \leqslant 1$ provided $0<\beta \leqslant \Lambda_2(\alpha)$, and $B_2(2B_2-1) \leqslant D_2 \leqslant B_2$ when $0<\beta\leqslant \Delta_2(\alpha)$. Then since $\Delta_2(\alpha) < \Lambda_2(\alpha)$, applying Lemma 1.2 now gives the first inequality for $|A_4|$.

For the second inequality we use Lemma 1.3 and write

$$p_3 - 2B_2p_1p_2 + D_2p_1^3 = p_3 - 2B_2p_1p_2 + B_2p_1^3 + (D_2 - B_2)p_1^3.$$

Since $D_2 \geqslant B_2$ when $\Delta_2(\alpha) \leqslant \beta \leqslant 1$, and $0 \leqslant B_2 \leqslant 1$ provided $0<\beta\leqslant\Lambda_2(\alpha)$, the second inequality for $|A_4|$ follows on the interval $\Delta_2(\alpha)\leqslant\beta\leqslant\Lambda_2(\alpha)$ by applying Lemma 1.3, and noting that $|p_1| \leqslant 2$.

Thus it remains to establish the second inequality for $|A_4|$ on the interval $\Lambda_2(\alpha)\leqslant\beta\leqslant 1$.

We apply Lemma 1.4 as follows.

Write

$$p_3 - 2B_2p_1p_2 + D_2p_1^3 = p_3 - (1+\mu)p_1p_2 + \mu p_1^3 + (D_2-\mu)p_1^3.$$

so that $\mu = \dfrac{(5+12\alpha-2\alpha^2)\beta}{(1+\alpha)(1+2\alpha)}$.

Then $0 \leqslant \mu \leqslant 1$ is false on $\Lambda_2(\alpha) \leqslant \beta \leqslant 1$, and so since $D_2 \geqslant \mu$ on $0 < \beta \leqslant 1$, applying Lemma 1.4 and using the fact that $|p_1| \leqslant 2$, we obtain the second inequality for $|A_4|$ on the interval $\Lambda_2(\alpha) \leqslant \beta \leqslant 1$, after substituting for D_2 and μ.

The first inequality for $|A_4|$ is sharp when $p_1 = p_2 = 0$ and $p_3 = 2$, and the second inequality is sharp when $p_1 = p_2 = p_3 = 2$. □

We note again that when $\beta = 1$, the results in Theorem 3.1 correspond to the estimates found in [6], when $\alpha = 0$ to those in [1], and when $\alpha = 1$ to those in [12].

Theorem 3.2. *Let $f \in \mathcal{M}^*(\gamma,\beta)$ and the coefficients of the inverse function f^{-1} be given by* (1.4)*, then*

$$|A_2| \leqslant \frac{2\beta}{1+\gamma}, \qquad |A_3| \leqslant \begin{cases} \dfrac{\beta}{1+2\gamma}, & 0 < \beta \leqslant \dfrac{(1+\gamma)^2}{5+7\gamma}, \\[2ex] \dfrac{(5+7\gamma)\beta^2}{(1+\gamma)^2(1+2\gamma)}, & \dfrac{(1+\gamma)^2}{5+7\gamma} \leqslant \beta \leqslant 1, \end{cases}$$

$$|A_4| \leqslant \begin{cases} \dfrac{2\beta}{3(1+3\gamma)}, & 0 < \beta \leqslant \Delta_2^*(\gamma), \\[2ex] \Gamma_2^*(\gamma,\beta), & \Delta_2^*(\gamma) \leqslant \beta \leqslant 1, \end{cases}$$

where $\Delta_2^*(\gamma) = \sqrt{\dfrac{(1+\gamma)^3}{31+37\gamma}}$ *and*

$$\Gamma_2^*(\gamma,\beta) = \frac{2\beta(1+3\gamma^2+\gamma^3+62\beta^2+\gamma(3+74\beta^2))}{9(1+\gamma)^3(1+3\gamma)}.$$

All the inequalities are sharp.

Proof. From (2.2) and (3.1) we obtain

$$\begin{aligned} A_2 &= -\frac{\beta p_1}{1+\gamma}, \\ A_3 &= -\frac{\beta((1+\gamma^2+5\beta+\gamma(2+7\beta))p_1^2 - 2(1+\gamma)^2 p_2)}{4(1+\gamma)^2(1+2\gamma)}, \\ A_4 &= \beta\Big(\frac{(2+2\gamma^3+15\beta+31\beta^2+3\gamma^2(2+5\beta)+\gamma(6+30\beta+37\beta^2)p_1^3}{18(1+\gamma)^3(1+3\gamma)} \\ &\quad - \frac{(1+\gamma+5\beta)p_1p_2}{3(1+\gamma)(1+3\gamma)} + \frac{p_3}{3(1+3\gamma)}\Big). \end{aligned}$$

Again, the first inequality is trivial, and is sharp when $p_1 = 2$.

For A_3 we write

$$A_3 = \frac{\beta}{2(1+2\gamma)}\Big(p_2 - \frac{(1+\gamma^2+5\beta+\gamma(2+7\beta))}{2(1+\gamma)^2}p_1^2\Big).$$

Now apply Lemma 1.1 with $\mu = \dfrac{1+\gamma^2+5\beta+\gamma(2+7\beta)}{(1+\gamma)^2}$, so that $0 \leqslant \mu \leqslant 2$ provided $0 < \beta \leqslant \dfrac{(1+\gamma)^2}{5+7\gamma}$ and the inequalities for $|A_3|$ follow as before.

The first inequality for $|A_3|$ is sharp when $p_1 = 0$ and $p_2 = 2$, and the second inequality is sharp when $p_1 = p_2 = 2$.

For A_4 we follow the same methods as previously used, and write

$$A_4 = \frac{\beta}{3(1+3\gamma)}\Big(p_3 - 2B_2^* p_1 p_2 + D_2^* p_1^3\Big),$$

with

$$B_2^* = \frac{1+\gamma+5\beta}{2(1+\gamma)},$$

and

$$D_2^* = \frac{2+2\gamma^3+15\beta+31\beta^2+3\gamma^2(2+5\beta)+\gamma(6+30\beta+37\beta^2)}{6(1+\gamma)^3}.$$

Write $\Lambda_2^*(\gamma) = \dfrac{1+\gamma}{5}$. Then $0 \leqslant B_2^* \leqslant 1$ provided $0 < \beta \leqslant \Lambda_2^*(\gamma)$, and $B_2^*(2B_2^*-1) \leqslant D_2^* \leqslant B_2^*$ when $0 < \beta \leqslant \Delta_2^*(\gamma)$. Since $\Delta_2^*(\gamma) < \Lambda_2^*(\gamma)$, applying Lemma 1.2 now gives the first inequality for $|A_4|$.

For the second inequality we again use Lemma 1.3, and write

$$\begin{aligned} p_3 - 2B_2^* p_1 p_2 + D_2^* p_1^3 &= p_3 - 2B_2^* p_1 p_2 + B_2^* p_1^3 \\ &\quad + (D_2^* - B_2^*) p_1^3. \end{aligned}$$

Since $D_2^* \geqslant B_2^*$ when $\Delta_2^*(\gamma) \leqslant \beta \leqslant 1$, and $0 \leqslant B_2^* \leqslant 1$ when $0 < \beta \leqslant \Lambda_2^*(\gamma)$, the second inequality for $|A_4|$ follows on the interval $\Delta_2^*(\gamma)\beta \leqslant \Lambda_2^*(\gamma)$ on applying Lemma 1.3, and noting that $|p_1| \leqslant 2$.

Thus it remains to prove the second inequality on the interval $\Lambda_2^*(\gamma) \leqslant \beta \leqslant 1$.

We proceed as in the proof of Theorem 3.1, and again write

$$p_3 - 2B_2^* p_1 p_2 + D_2^* p_1^3 = p_3 - (1+\mu) p_1 p_2 + \mu p_1^3 + (D_2^* - \mu) p_1^3.$$

with $\mu = \dfrac{5\beta}{1+\gamma}$. Since $0 \leqslant \mu \leqslant 1$ is false on the interval $\Lambda_2^*(\gamma) \leqslant \beta \leqslant 1$, applying Lemma 1.4, and substituting for D_2^* and μ, gives the inequality for $|A_4|$ on this interval.

The first inequality for $|A_4|$ is sharp when $p_1 = p_2 = 0$ and $p_3 = 2$, and the second inequality is sharp when $p_1 = p_2 = p_3 = 2$. □

We note that when $\beta = 1$, the results in Theorem 3.2 complete the partial solution given in [5]. When $\alpha = 0$, the results correspond to those [1], and when $\alpha = 1$ to those in [12].

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