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THE EQUIVALENCE OF MANN ITERATION AND ISHIKAWA ITERATION FOR A LIPSCHITZIAN $\psi-$ UNIFORMLY PSEUDOCONTRACTIVE AND $\psi-$ UNIFORMLY ACCRETIVE MAPS

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Abstract. We show that certain Ishikawa iteration and the corresponding Mann iteration schemes are equivalent when applied to Lipschitzian and $\psi-$uniformly pseudocontractive or Lipschitzian $\psi-$ uniformly accretive maps.

1. Introduction

In previous paper [10] the authors established the equivalence of certain Mann and Ishikawa iteration procedures for Lipschitzian strongly pseudocontractive and strongly accretive maps. This paper is an extension of some of that work to more general classes of maps.

Let X be a real Banach space, B a nonempty, convex subset of X, T a selfmap of B. The Mann iteration scheme, (see [4]), we shall use is one defined by

$$\begin{aligned} &u_0 \in B, \\ &u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n, \quad n = 0, 1, 2, \dots . \end{aligned} \tag{1}$$

The Ishikawa iteration scheme is defined, (see [2]), by

$$\begin{aligned} x_0 &\in B, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n = 0, 1, 2, \dots . \end{aligned} \tag{2}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and satisfy

$$\lim_{n\to\infty} \alpha_n = 0,\ \lim_{n\to\infty} \beta_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty,\ \sum_{n=1}^{\infty} \alpha_n^2 < \infty,\ \sum_{n=1}^{\infty} \alpha_n \beta_n < \infty. \tag{3}$$

Moreover, the sequence $\{\alpha_n\}$ from (1) is the same as that in (2). For $\beta_n = 0, \forall n \in N$ we get from (2), Mann iteration.

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The map $J : X \to 2^{X^*}$ given by $Jx := \{f \in X^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\}, \forall x \in X$,is called *the normalized duality mapping*. The Hahn-Banach theorem assures that $Jx \neq \emptyset, \forall x \in X$.

Let $\Psi := \{\psi \mid \psi : [0,\infty) \to [0,\infty)$ is a nondecreasing map such that $\psi(0) = 0\}$.

Definition 1.([5]) Let X be a real Banach space. Let B be a nonempty subset of X. A map $T : B \to B$ is called $\psi-$uniformly pseudocontractive if there exists a map $\psi \in \Psi$ and a $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \psi(\|x-y\|), \forall x, y \in B. \tag{4}$$

The map $S : B \to B$ is called $\psi-$uniformly accretive if there exists a map $\psi \in \Psi$ and a $j(x-y) \in J(x-y)$ such that

$$\langle Sx - Sy, j(x-y)\rangle \geq \psi(\|x-y\|), \forall x, y \in B. \tag{5}$$

Taking $\psi(a) := \psi(a) \cdot a, \forall a \in [0,\infty)$, $(\psi \in \Psi)$, we get the usual definitions of $\psi-$strongly pseudocontractivity and $\psi-$strongly accretivity. Taking $\psi(a) := \gamma \in (0,1)$, $\forall a \in [0,\infty)$, $(\psi \in \Psi)$, we get the usual definitions of strongly pseudocontractivity and strongly accretivity.

Let $T : [2,\infty) \to \mathbb{R}$, $T(x) = (x-2)^3/\left(1 + (x-2)^2\right), \psi(a) = a^2/\left(1 + a^2\right), \psi \in \Psi$; in [5] was shown that T is $\psi-$uniformly pseudocontractive map without being $\psi-$strongly pseudocontractive. Let $T : [0,\infty) \to [0,\infty), T(x) = x/(1+x), \psi(a) = a^2/(1+a), \psi \in \Psi$; in [7] was shown that T is $\psi-$strongly psudocontractive map without being strongly pseudocontractive.

Let us denote by I the identity map.

Remark 2. (i) T is $\psi-$uniformly pseudocontractive if and only if $S = (I - T)$ is $\psi-$uniformly accretive.

(ii) T is $\psi-$strongly pseudocontractive map if and only if $(I - T)$ is $\psi-$strongly accretive.

Let $F(T)$ denote the fixed point set of T. In [10] the equivalence between the convergence of Mann and Ishikawa iterations for a Lipschitzian, strongly pseudocontractive map was proved. In this paper we show that the convergence of Mann iteration is equivalent to the convergence of Ishikawa iteration, for Lipschitzian, $\psi-$uniformly pseudocontractions. We also prove a similar result for $\psi-$uniformly accretive maps. This equivalence allows us that in practical problem to consider only iteration (1) which is more convenient to use. Iteration (2) will have the same behavior.

The following Lemma is from [11].

Lemma 3. *Let* $\{a_n\}$ *be a nonnegative sequence satisfying*

$$a_{n+1} \leq a_n + b_n,\ \forall n \in N. \tag{6}$$

If $\sum_{n=1}^{\infty} b_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists.*

The following lemma will be useful.

Lemma 4. *Let* $\psi \in \Psi$, *and let* $\{\mu_n\}$ *and* $\{\lambda_n\}$ *be two nonnegative sequences in* $(0,1)$ *satisfying the conditions*

$$\sum_{n=1}^{\infty} \mu_n = \infty; \ \ \sum_{n=1}^{\infty} \mu_n^2 < \infty; \ \ \sum_{n=1}^{\infty} \lambda_n < \infty. \tag{7}$$

Let $\{a_n\}$ *be a nonnegative sequence which satisfies the inequality*

$$a_{n+1} \ \le \ \left(1 \ - \ \frac{\psi(a_{n+1})}{(1 \ + \ \psi(a_{n+1}) \ + \ a_{n+1})}\, \mu_n\right) \ a_n \ + \ \lambda_n, \ \forall n \in \mathbb{N} \tag{8}$$

Then $\lim_{n\to\infty} a_n = 0$.

Proof. From (8)

$$a_{n+1} \le a_n + \lambda_n, \ \forall n \in \mathbb{N}. \tag{9}$$

Lemma 3 assures the existence of $\lim_{n\to\infty} a_n$. Because $\{a_n\}$ is bounded and $\psi \in \Psi$

$$\exists D > 0 : \ \ a_n \le D, \ \ \ \forall n \in \mathbb{N} \ \ \Rightarrow \ \ \psi(a_n) \le \psi(D), \ \ \ \forall n \in \mathbb{N}. \tag{10}$$

Let $A = \lim_{n\to\infty} a_n$. Then $A \ge 0$. Suppose $A > 0$

$$\exists n_0 : \ \ a_n \ge \frac{A}{2}, \ \ \forall n \ge n_0 \ \ \ \Rightarrow \ \ \ \psi(a_n) \ge \psi(\frac{A}{2}), \ \ \ \forall n \ge n_0. \tag{11}$$

Using (10), (11) and (8),

$$\begin{aligned} \mu_n \, \frac{\psi(\frac{A}{2}) \ \frac{A}{2}}{(\ 1 \ + \ \psi(D) \ + \ D\ )} \ \ \ &\le \ \mu_n \, \frac{\psi(a_{n+1}) \ \ a_n}{(\ 1 + \psi(a_{n+1}) + a_{n+1}\ )} \\ &\le \ a_n \ - \ a_{n+1} \ + \ \lambda_n, \ \forall n \ge n_0. \end{aligned} \tag{12}$$

It then follows that

$$\frac{\psi(\frac{A}{2}) \ \frac{A}{2}}{(\ 1 \ + \ \psi(D) \ + \ D\ )} \ \sum_{j=n_0}^{n} \mu_j \ \le \ a_{n_0} \ + \ \sum_{j=n_0}^{n} \lambda_n \ < \infty. \tag{13}$$

Contradicting the fact that $\sum_{n=1}^{\infty} \mu_n = \infty$. Thus $A = 0$.

2. Main Results

We are able now to prove the following result:

Theorem 5. *Let* X *be a real Banach space,* B *be a nonempty, bounded, convex and closed subset of* X, *and* $T : B \to B$ *be a Lipschitzian,* $\psi-$ *uniformly pseudocontractive map. If* $\{\alpha_n\}, \{\beta_n\}$ *satisfy* (3), *with* $u_0 = x_0 \in B$, *then the following are equivalent:*

(i) *Mann iteration* (1) *converges (to* $x^* \in F(T)$*),*
(ii) *Ishikawa iteration* (2) *converges (to the same* $x^* \in F(T)$*).*

Proof. For all $x, y \in X$ we have

$$\begin{aligned} \langle Tx - Ty, j(x-y)\rangle &\le \|x-y\|^2 - \psi(\|x-y\|) \\ \|x-y\|^2 - \langle Tx - Ty, j(x-y)\rangle &\ge \psi(\|x-y\|) \\ \langle x-y, j(x-y)\rangle - \langle Tx - Ty, j(x-y)\rangle &\ge \psi(\|x-y\|) \\ \langle (I-T)x - (I-T)y, j(x-y)\rangle &\ge \psi(\|x-y\|). \end{aligned} \tag{14}$$

Using the fact that $1 + \psi(a) + \alpha^2 \ge a^2$, for all $a \ge 0$, the following inequality is satisfied

$$\psi(a) \ \ge \ \frac{\psi(a)}{1+\psi(a)+\alpha^2}\, a^2.$$

Taking $a := \|x-y\|$we obtain

$$\begin{aligned} &\langle\, (I-T)\, x \ - \ (I-T)\, y\ ,\ j(x-y)\,\rangle \\ &\quad \ge\ \psi(\, \|x-y\|\,)\ \ge\ \frac{\psi(\|x-y\|)}{1\ +\ \psi(\|x-y\|)\ +\ \|x-y\|^2}\ \|x-y\|^2 \\ &\quad = \sigma(x,y)\ \ \|x-y\|^2, \end{aligned} \tag{15}$$

where

$$\sigma(x,y) := \frac{\psi(\|x-y\|)}{1\ +\ \psi(\|x-y\|)\ +\ \|x-y\|^2} \in [0,1),\ \forall x,y \in X.$$

Hence for each $x, y \in X$:

$$\langle\, (I-T)\, x \ - \ (I-T)\, y\ ,\ j(x-y)\,\rangle\ \ge\ \sigma(x,y)\ \|x-y\|^2, \tag{16}$$

or,

$$\langle(\ (I-T)\ x - \sigma(x,y)\ x\ ) - \ (\ (I-T)\ y - \sigma(x,y)\ y\ )\ ,\ j(x-y)\ \rangle \ge 0.$$

From Lemma 1.1 of [3], for each $x, y \in X$:

$$\exists j(x) \in J(x) : \langle y, j(x)\rangle \ge 0 \iff \|x\| \le \|x + \alpha y\|, \forall \alpha > 0.$$

In the above equivalence set

$$\begin{aligned} &x := \|x-y\|, \\ &\alpha := r, \\ &y := ((I-T)x - \sigma(x,y)x) - ((I-T)y - \sigma(x,y)y). \end{aligned}$$

Then it follows that for each $x, y \in X$ and $r > 0$:

$$\|x-y\| \le ||x - y + r[((I-T)x - \sigma(x,y)x) - ((I-T)y - \sigma(x,y)y)]||. \tag{17}$$

From (1) we have, for each $n \in \mathbb{N}$:

$$\begin{aligned}
u_n &= u_{n+1} + \alpha_n u_n - \alpha_n Tu_n \\
&= u_{n+1} + 2\alpha_n u_n - \alpha_n u_n - \alpha_n Tu_n \\
&= u_{n+1} + 2\alpha_n u_n - \alpha_n u_n - \alpha_n Tu_n + 2\alpha_n u_{n+1} - 2\alpha_n u_{n+1} \\
&= (1+\alpha_n) u_{n+1} + \sigma(x_{n+1}, u_{n+1}) \alpha_n u_{n+1} - \sigma(x_{n+1}, u_{n+1}) \alpha_n u_{n+1} \\
&\quad + (-2\alpha_n u_{n+1} + 2\alpha_n u_n) - \alpha_n u_n - \alpha_n Tu_n + \alpha_n u_{n+1} \\
&= (1+\alpha_n) u_{n+1} + \sigma(x_{n+1}, u_{n+1}) \alpha_n [(1-\alpha_n) u_n + \alpha_n Tu_n] \\
&\quad - \sigma(x_{n+1}, u_{n+1}) \alpha_n u_{n+1} + ((-2\alpha_n)(1-\alpha_n) u_n - 2\alpha_n^2 Tu_n + 2\alpha_n u_n) \\
&\quad - \alpha_n u_n - \alpha_n Tu_n + \alpha_n u_{n+1} \\
&= (1+\alpha_n) u_{n+1} - \sigma(x_{n+1}, u_{n+1}) \alpha_n u_{n+1} + \alpha_n u_{n+1} - \alpha_n Tu_{n+1} + \alpha_n Tu_{n+1} \\
&\quad + \sigma(x_{n+1}, u_{n+1}) \alpha_n [u_n - \alpha_n u_n + \alpha_n Tu_n] \\
&\quad + \left(2\alpha_n^2 u_n - 2\alpha_n u_n - 2\alpha_n^2 Tu_n + 2\alpha_n u_n \right) - \alpha_n u_n - \alpha_n Tu_n \\
&= (1+\alpha_n) u_{n+1} + \alpha_n [(I - T) u_{n+1} - \sigma(x_{n+1}, u_{n+1}) u_{n+1}] + \alpha_n Tu_{n+1} \\
&\quad - \sigma(x_{n+1}, u_{n+1}) \alpha_n^2 u_n + \sigma(x_{n+1}, u_{n+1}) \alpha_n^2 Tu_n + 2\alpha_n^2 u_n - 2\alpha_n^2 Tu_n \\
&\quad + \sigma(x_{n+1}, u_{n+1}) \alpha_n u_n - 2\alpha_n u_n + 2\alpha_n u_n - \alpha_n u_n - \alpha_n Tu_n \\
&= (1+\alpha_n) u_{n+1} + \alpha_n [(I-T) u_{n+1} - \sigma(x_{n+1}, u_{n+1}) u_{n+1}] + \alpha_n Tu_{n+1} \\
&\quad + \left[2\alpha_n^2 u_n - 2\alpha_n^2 Tu_n - \sigma(x_{n+1}, u_{n+1}) \alpha_n^2 u_n + \sigma(x_{n+1}, u_{n+1}) \alpha_n^2 Tu_n \right] \\
&\quad + \sigma(x_{n+1}, u_{n+1}) \alpha_n u_n - \alpha_n u_n - \alpha_n Tu_n \\
&= (1+\alpha_n) u_{n+1} + \alpha_n [(I-T)u_{n+1} - \sigma(x_{n+1}, u_{n+1}) u_{n+1}] \\
&\quad + (2 - \sigma(x_{n+1}, u_{n+1})) \alpha_n^2 (u_n - Tu_n) \\
&\quad - (1 - \sigma(x_{n+1}, u_{n+1})) \alpha_n u_n + \alpha_n (Tu_{n+1} - Tu_n) \\
&= (1+\alpha_n) u_{n+1} + \alpha_n [(I-T)u_{n+1} - \sigma(x_{n+1}, u_{n+1}) u_{n+1}] \\
&\quad - (1 - \sigma(x_{n+1}, u_{n+1})) \alpha_n u_n \\
&\quad + (2 - \sigma(x_{n+1}, u_{n+1})) \alpha_n^2 (u_n - Tu_n) + \alpha_n (Tu_{n+1} - Tu_n). \qquad (18)
\end{aligned}$$

Analogously, from (2) we have

$$\begin{aligned}
x_n &= x_{n+1} + \alpha_n x_n - \alpha_n Ty_n \\
&= (1+\alpha_n) x_{n+1} + \alpha_n [(I-T)x_{n+1} - \sigma(x_{n+1}, u_{n+1}) x_{n+1}] \\
&\quad - (1 - \sigma(x_{n+1}, u_{n+1})) \alpha_n x_n \\
&\quad + (2 - \sigma(x_{n+1}, u_{n+1})) \alpha_n^2 (x_n - Ty_n) + \alpha_n (Tx_{n+1} - Ty_n). \qquad (19)
\end{aligned}$$

Hence

$$\begin{aligned}
x_n - u_n &= (1+\alpha_n)(x_{n+1} - u_{n+1}) + \alpha_n[((I-T)x_{n+1} - \sigma(x_{n+1}, u_{n+1})x_{n+1}) \\
&\quad - ((I-T)u_{n+1} - \sigma(x_{n+1}, u_{n+1})u_{n+1})] - (1 - \sigma(x_{n+1}, u_{n+1}))\alpha_n(x_n - u_n) \\
&\quad + (2 - \sigma(x_{n+1}, u_{n+1}))\alpha_n^2(x_n - Ty_n - u_n + Tu_n) \\
&\quad + \alpha_n(Tx_{n+1} - Ty_n - Tu_{n+1} + Tu_n). \qquad (20)
\end{aligned}$$

Using the norm and relation (20), we obtain

$$\begin{aligned}\|x_n - u_n\| \geq (1+\alpha_n)\|(x_{n+1} - u_{n+1}) + \frac{\alpha_n}{1+\alpha_n}[((I-T)x_{n+1} - \sigma(x_{n+1}, u_{n+1})x_{n+1}) \\ - ((I-T)u_{n+1} - \sigma(x_{n+1}, u_{n+1})u_{n+1})]\| - (1-\sigma(x_{n+1}, u_{n+1}))\alpha_n \|x_n - u_n\| \\ -(2-\sigma(x_{n+1}, u_{n+1}))\alpha_n^2 \|x_n - Ty_n - u_n + Tu_n\| \\ -\alpha_n \|Tx_{n+1} - Ty_n - Tu_{n+1} + Tu_n\|.\end{aligned}$$

Using (17) with

$$\begin{aligned}r &:= \frac{\alpha_n}{1+\alpha_n}, \\ x &:= x_n, \\ y &:= u_n,\end{aligned}$$

we get

$$\begin{aligned}\|x_n - u_n\| \geq (1+\alpha_n)\|x_{n+1} - u_{n+1}\| - (1-\sigma(x_{n+1}, u_{n+1}))\alpha_n \|x_n - u_n\| \\ -(2-\sigma(x_{n+1}, u_{n+1}))\alpha_n^2 \|x_n - Ty_n - u_n + Tu_n\| \\ -\alpha_n \|Tx_{n+1} - Ty_n - Tu_{n+1} + Tu_n\|. \quad (21)\end{aligned}$$

Thus we have

$$\begin{aligned}(1+\alpha_n)\|x_{n+1} - u_{n+1}\| \leq [1 + (1-\sigma(x_{n+1}, u_{n+1}))\alpha_n]\|x_n - u_n\| \\ +(2-\sigma(x_{n+1}, u_{n+1}))\alpha_n^2 \|x_n - Ty_n - u_n + Tu_n\| \\ +\alpha_n \|Tx_{n+1} - Ty_n - Tu_{n+1} + Tu_n\|. \quad (22)\end{aligned}$$

That is

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| \leq \left[\frac{1 + (1-\sigma(x_{n+1}, u_{n+1}))\alpha_n}{1+\alpha_n}\right]\|x_n - u_n\| \\ +\frac{1}{1+\alpha_n}(2-\sigma(x_{n+1}, u_{n+1}))\alpha_n^2 \|x_n - Ty_n - u_n + Tu_n\| \\ +\frac{\alpha_n}{1+\alpha_n}\|Tx_{n+1} - Ty_n - Tu_{n+1} + Tu_n\|. \quad (23)\end{aligned}$$

Sine B is bounded and T is Lipschitzian, there exists a positive constant M such that, for all $\{x_n\}, \{u_n\} \subset B$,

$$\max\{\|x_n\|, \|Ty_n\|, \|u_n\|, \|Tu_n\|\} \leq M, \text{ for all } n \in \mathbb{N}.$$

Hence, for each n,

$$\|x_{n+1} - u_{n+1}\| \leq \left[\frac{1 + (1-\sigma(x_{n+1}, u_{n+1}))\alpha_n}{1+\alpha_n}\right]\|x_n - u_n\|$$

$$
\begin{aligned}
&+\frac{1}{1+\alpha_n}(2-\sigma(x_{n+1},u_{n+1}))\alpha_n^2 4M \\
&+\frac{\alpha_n}{1+\alpha_n}\left\|Tx_{n+1}-Ty_n-Tu_{n+1}+Tu_n\right\| \\
\le &\left[\frac{1+(1-\sigma(x_{n+1},u_{n+1}))\alpha_n}{1+\alpha_n}\right]\|x_n-u_n\|+\frac{8M}{1+\alpha_n}\alpha_n^2 \\
&+\frac{\alpha_n}{1+\alpha_n}\left\|Tx_{n+1}-Ty_n-Tu_{n+1}+Tu_n\right\|. \qquad (24)
\end{aligned}
$$

We have

$$
\begin{aligned}
&\|Tx_{n+1}-Ty_n-Tu_{n+1}+Tu_n\| \\
&\quad\le \|Tx_{n+1}-Ty_n\|+\|Tu_{n+1}-Tu_n\| \\
&\quad\le L\,\|x_{n+1}-y_n\|+L\,\|u_{n+1}-u_n\| \\
&\quad\le L\left((\|-\alpha_n x_n+\beta_n x_n+\alpha_n Ty_n-\beta_n Tx_n\|)+\alpha_n\|Tu_n-u_n\|\right) \\
&\quad\le L\left((\alpha_n\|-x_n+Ty_n\|+\beta_n\|x_n-Tx_n\|)+\alpha_n\|Tu_n-u_n\|\right) \\
&\quad\le L((\alpha_n(\|x_n\|+\|Ty_n\|)+\beta_n(\|x_n\|+\|Tx_n\|))+\alpha_n(\|Tu_n\|+\|u_n\|)) \\
&\quad\le L\left(\alpha_n 2M+\beta_n 2M+\alpha_n 2M\right) \\
&\quad= 2LM(2\alpha_n+\beta_n)\to 0 \text{ as } n\to\infty. \qquad (25)
\end{aligned}
$$

Using the fact that $(1+\alpha_n)^{-1}\le 1-\alpha_n+\alpha_n^2$, and $(1+\alpha_n)^{-1}\le 1, \forall n\in N$, substituting (25) into (24), yields

$$
\begin{aligned}
\|x_{n+1}-u_{n+1}\| &\le \left[(1+(1-\sigma(x_{n+1},u_{n+1}))\alpha_n)\left(1-\alpha_n+\alpha_n^2\right)\right]\|x_n-u_n\| \\
&\quad+\alpha_n 2LM(2\alpha_n+\beta_n)+8M\alpha_n^2 \\
&= [1-\alpha_n+\alpha_n^2+\alpha_n-\alpha_n^2+\alpha_n^3-\sigma(x_{n+1},u_{n+1})\alpha_n+\sigma(x_{n+1},u_{n+1})\alpha_n^2 \\
&\quad-\sigma(x_{n+1},u_{n+1})\alpha_n^3]\,\|x_n-u_n\|+8M\alpha_n^2+\alpha_n 2LM(2\alpha_n+\beta_n) \\
&\le [1-\sigma(x_{n+1},u_{n+1})\alpha_n]\,\|x_n-u_n\| \\
&\quad+\alpha_n^2\left(\alpha_n+(1-\alpha_n)\,\sigma(x_{n+1},u_{n+1})\right)\|x_n-u_n\| \\
&\quad+8M\alpha_n^2+\alpha_n 2LM(2\alpha_n+\beta_n) \\
&\le [1-\sigma(x_{n+1},u_{n+1})\alpha_n]\,\|x_n-u_n\|+\alpha_n^2\,(1+1)\,\|x_n-u_n\| \\
&\quad+8M\alpha_n^2+\alpha_n 2LM(2\alpha_n+\beta_n) \\
&\le [1-\sigma(x_{n+1},u_{n+1})\alpha_n]\,\|x_n-u_n\|+\alpha_n^2 4M+8M\alpha_n^2+\alpha_n 2LM(2\alpha_n+\beta_n) \\
&= [1-\sigma(x_{n+1},u_{n+1})\alpha_n]\,\|x_n-u_n\|+\alpha_n^2 12M+\alpha_n 2LM(2\alpha_n+\beta_n) \\
&= (1-\alpha_n\sigma(x_{n+1},u_{n+1}))\,\|x_n-u_n\|+\lambda_n. \qquad (26)
\end{aligned}
$$

Define

$$
\begin{aligned}
&\mu_n := \alpha_n, \\
&\lambda_n := \alpha_n^2\ 12\ M\ +\ \alpha_n 2LM\ (2\alpha_n+\beta_n) \qquad (27) \\
&a_n := \|x_n-u_n\|.
\end{aligned}
$$

Since $\{\alpha_n\} \subset (0,1)$, $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$, if follows from Lemma 4 that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{28}$$

Suppose now that $\lim_{n\to\infty} u_n = x^*$ then

$$\|x_n - x^*\| \le \|x_n - u_n\| + \|u_n - x^*\|, \tag{29}$$

and $\lim_{n\to\infty} x_n = x^*$. For the converse we suppose that $\lim_{n\to\infty} x_n = x^*$. Relation (28) and the following inequality

$$\|u_n - x^*\| \le \|x_n - u_n\| + \|x_n - x^*\| \tag{30}$$

implies that $\lim_{n\to\infty} u_n = x^*$.

The above result does not completely generalize the main result from [10] because in [10] B is not assumed to be bounded.

Theorem 6. *Let K be a closed convex subset of an arbitrary Banach space X and let T be a Lipschitzian pseudocontractive selfmap of K. Let us consider Mann iteration and Ishikawa iteration with the same initial point and satisfying the conditions* $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Let $x^* \in F(T)$. Then the following are equivalent:*
(i) *Mann iteration* (1) *converges to* x^*,
(ii) *Ishikawa iteration* (2) *converges to* x^*.

3. The $\psi-$ Uniformly Accretive Case

Let S be a $\psi-$ uniformly accretive and Lipschitzian map. Suppose that the equation $Sx = f$ has a solution for a given $f \in X$. It easy to see that

$$Tx = x + f - Sx, \forall x \in X, \tag{31}$$

is a $\psi-$uniformly pseudocontractive and Lipschitzian map. Moreover, a fixed point for T is a solution of $Sx = f$, and conversely. Theorem 3 assures that the convergence of Mann and Ishikawa iterations to the fixed point of T are equivalent. The map T is assumed to be $\psi-$uniformly pseudocontractive and Lipschitzian. A similar result holds for the convergence of Mann and Ishikawa iterations to the solution of $Sx = f$. The map S is assumed $\psi-$ uniformly accretive. For this case we need to know that $(I - S)$ must have a bounded range. It is well known that if T is bounded $(I-T)$ could be unbounded. For example take $T : \mathbb{R} \to \mathbb{R}$ with $T(x) = (1/2)\cos x$. From [1], $(I-T)(x) = x-(1/2)\cos x$, is Lipschitzian and strongly accretive. For the same $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ as in (3), iterations (1) and (2) become

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n \left(f + (I - S)y_n\right), \\ y_n &= (1-\beta_n)x_n + \beta_n \left(f + (I - S)x_n\right), \quad n = 0, 1, 2, \dots, \end{aligned} \tag{32}$$

and

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n \left(f + (I-S)u_n\right), n = 0,1,2,\dots . \tag{33}$$

We are able now to give the following result

Theorem 7. *Let X be a real Banach space, B be a nonempty, bounded, convex and closed subset of X, and $S: B \to B$ be a Lipschitzian, $\psi-$ uniformly accretive map, with $(I-S)(B)$ bounded, and suppose that there exists solutions for $Sx = f$. If the sequences $\{\alpha_n\}$, $\{\beta_n\}$ satisfy condition* (3)*, and $u_0 = x_0 \in B$, then the following two assertions are equivalent:*

(i) *Mann iteration* (33) *converges to a solution x^* of $Sx = f$,*

(ii) *Ishikawa iteration* (32) *converges to the same solution x^* of $Sx = f$.*

For the multivalued case we have

Definition 8. Let X be a real Banach space. Let B be a nonempty subset. A map $T : B \to 2^B$ is called uniformly pseudocontractive if there exists $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle \xi - \theta, j(x-y)\rangle \leq \|x-y\|^2 - \psi(\|x-y\|), \tag{34}$$

for all $x, y \in B, \xi \in Tx, \theta \in Ty$.

Let $S : X \to 2^X$. The map S is called uniformly accretive if there exists $\gamma \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle \xi - \theta, j(x-y)\rangle \geq \psi(\|x-y\|), \tag{35}$$

for all $x, y \in X$, $\xi \in Sx$, $\theta \in Sy$.

We remark that all the results from this paper hold in the multivalued case, provided that these multivalued maps admit single valued selections.

5. Remarks on the Convergence of Mann and Ishikawa Iterations for $\psi-$ Uniformly Pseudocontractive and $\psi-$ Uniformly Accretive Maps

Taking $\psi(a) := \psi(a)\cdot a$, $\forall a \in [0,\infty)$ in (4) and (5) from Definition 1 we obtain the definition of $\psi-$strongly pseudocontractive and $\psi-$strongly accretive map, (see [8, 9]).

For all $x, y \in X$ we have

$$\begin{aligned} \|x-y\|^2 - \psi(\|x-y\|)\|x-y\| &\geq \langle Tx - Ty, j(x-y)\rangle \\ \|x-y\|^2 - \langle Tx-Ty, j(x-y)\rangle &\geq \psi(\|x-y\|)\|x-y\| \\ \langle x-y, j(x-y)\rangle - \langle Tx-Ty, j(x-y)\rangle &\geq \psi(\|x-y\|)\|x-y\| \\ \langle (I-T)x - (I-T)y, j(x-y)\rangle &\geq \psi(\|x-y\|)\|x-y\| . \end{aligned} \tag{36}$$

Furthermore

$$\begin{aligned}\langle (I-T)x-(I-T)y, j(x-y)\rangle &\geq \psi(\|x-y\|)\,\|x-y\| \\ &\geq \frac{\psi(\|x-y\|)}{1+\psi(\|x-y\|)+\|x-y\|}\,\|x-y\|^2 \\ &= \sigma(x,y)\,\|x-y\|^2, \end{aligned} \tag{37}$$

where

$$\sigma(x,y)=\frac{\psi(\|x-y\|)}{1+\psi(\|x-y\|)+\|x-y\|}\in[0,1), \forall x,y\in X.$$

Hence we get $\forall x, y \in X$:

$$\langle (I-T)x-(I-T)y, j(x-y)\rangle \geq \sigma(x,y)\,\|x-y\|^2,$$

$$\langle ((I-T)x-\sigma(x,y)x)-((I-T)y-\sigma(x,y)y)\,, j(x-y)\rangle \geq 0. \tag{38}$$

Observe that (37) is (15) and formula (38) is (16) but with a different $\sigma(x, y)$. Using the same argument as for (15) and (16) it follows that for all $x, y \in X$ and $r > 0$:

$$\|x-y\| \leq \|x-y+r\,[((I-T)x-\sigma(x,y)x)-((I-T)y-\sigma(x,y)y)]\|\,. \tag{39}$$

Let us consider the case when T is $\psi-$uniformly accretive or $\psi-$uniformly pseudo-contractive map. For the convergence of Mann or Ishikawa iteration, using inequality (17) (which is similar to (39) used in [8] and [9]), one can easily see that the proofs are exactly the same as those in [7], [8] or [9].

Conclusion. *All the results from* [7], [8], [9], *concerning the convergences of Mann-Ishikawa iteration hold if we replace the "$\psi-$strongly" with "$\psi-$uniformly".*

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