



EXTREME MONOPHONIC GRAPHS AND EXTREME GEODESIC GRAPHS

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Abstract. For a connected graph $G = (V, E)$ of order at least two, a chord of a path P is an edge joining two non-adjacent vertices of P . A path P is called a monophonic path if it is a chordless path. A monophonic set of G is a set S of vertices such that every vertex of G lies on a monophonic path joining some pair of vertices in S . The monophonic number of G is the minimum cardinality of its monophonic sets and is denoted by $m(G)$. A geodetic set of G is a set S of vertices such that every vertex of G lies on a geodesic joining some pair of vertices in S . The geodetic number of G is the minimum cardinality of its geodetic sets and is denoted by $g(G)$. The number of extreme vertices in G is its extreme order $ex(G)$. A graph G is an extreme monophonic graph if $m(G) = ex(G)$ and an extreme geodesic graph if $g(G) = ex(G)$. Extreme monophonic graphs of order p with monophonic number p and $p-1$ are characterized. It is shown that every pair a, b of integers with $0 \leq a \leq b$ is realized as the extreme order and monophonic number, respectively, of some graph. For positive integers r, d and $k \geq 3$ with $r < d$, it is shown that there exists an extreme monophonic graph G of monophonic radius r , monophonic diameter d , and monophonic number k . Also, we give a characterization result for a graph G which is both extreme geodesic and extreme monophonic.

1. Introduction

By a *graph* $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q , respectively. For basic graph theoretic terminology we refer to Harary [4]. For vertices x and y in a connected graph G , the *distance* $d(x, y)$ is the length of a shortest x - y path in G . An x - y path of length $d(x, y)$ is called an x - y *geodesic*. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v . A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete. An extreme vertex of G is also called a *simplicial vertex* of G . The number of extreme vertices in G is its *extreme order* $ex(G)$.

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A *geodetic set* of G is a set S of vertices such that every vertex of G lies on a geodesic path joining some pair of vertices in S . The *geodetic number* of G is the minimum cardinality of its geodetic sets and is denoted by $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g -set*. The geodetic number of a graph was introduced in [1, 5] and further studied in [2]. A graph G is an *extreme geodesic graph* if $g(G) = ex(G)$. Extreme geodesic graphs were introduced and studied in [4].

A *chord* of a path P is an edge joining two non-adjacent vertices of P . A path P is called a *monophonic path* if it is a chordless path. A set S of vertices of G is a *monophonic set* if each vertex v of G lies on an $x - y$ monophonic path for some $x, y \in S$. The minimum cardinality of a monophonic set of G is the *monophonic number* of G and is denoted by $m(G)$. The monophonic number of a graph was studied in [9].

For any two vertices u and v in a connected graph G , the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G . The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max \{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m(G)$ of G is $rad_m(G) = \min \{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m(G)$ of G is $diam_m(G) = \max \{e_m(v) : v \in V(G)\}$. A vertex u in G is a *monophonic eccentric vertex* of a vertex v in G if $e_m(u) = d_m(u, v)$. The monophonic distance was introduced in [7] and further studied in [8].

The following theorems will be used in the sequel.

Theorem 1.1 ([2]). *Each extreme vertex of a graph G belongs to every geodetic set of G .*

Theorem 1.2 ([4]). *Let G be a connected graph with at least three vertices. The following statements are equivalent:*

- (i) G is a block.
- (ii) Every two vertices of G lie on a common cycle.

Theorem 1.3 ([9]). *Each extreme vertex of a graph G belongs to every monophonic set of G .*

Theorem 1.4 ([9]). *No cutvertex of a graph G belongs to any minimum monophonic set of G .*

Theorem 1.5 ([9]). *Let G be a connected graph with cutvertices and let S be a monophonic set of G . If x is a cutvertex of G , then every component of $G - x$ contains an element of S .*

Theorem 1.6 ([5]). *Let G be a connected graph with cutvertices and let S be a geodetic set of G . If x is a cutvertex of G , then every component of $G - x$ contains an element of S .*

2. Extreme monophonic graphs

Definition 2.1. A graph G is said to be an *extreme monophonic graph* if $m(G) = ex(G)$.

For the graph G given in Figure 2.1, v_1 and v_7 are the only two extreme vertices so that $ex(G) = 2$. The set $S = \{v_1, v_7\}$ is a minimum monophonic set of G so that $m(G) = ex(G) = 2$. Therefore, G is an extreme monophonic graph.

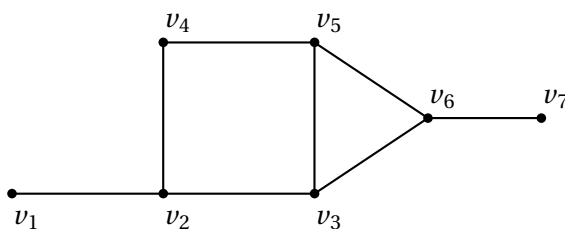


Figure 2.1: G .

For any nontrivial tree T with k end vertices, $ex(T) = k$ and by Theorems 1.3 and 1.4, $m(T) = k$. Thus any nontrivial tree is an extreme monophonic graph. It is obvious that the cycle C_p ($p \geq 4$) and the complete bipartite graph $K_{r,s}$ ($2 \leq r \leq s$) are not extreme monophonic graphs. By Theorem 1.3, we see that for any connected graph G of order p , $0 \leq ex(G) \leq m(G) \leq p$. It is an easy consequence of Theorem 1.3 that a connected graph G of order $p \geq 2$ is an extreme monophonic graph with monophonic number p if and only if $G = K_p$.

The following theorem gives a characterization of extreme monophonic graphs of order p with monophonic number $p - 1$.

Theorem 2.2. *Let G be a connected graph of order $p \geq 3$. Then G is an extreme monophonic graph with $m(G) = p - 1$ if and only if $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$.*

Proof. Let $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$. Since G has exactly one cutvertex and all other vertices are extreme, it follows from Theorems 1.3, 1.4 that $m(G) = p - 1$. Conversely, let G be an extreme monophonic graph with $m(G) = p - 1$. Then there exists exactly one non-extreme vertex, say x , in G . If $p = 3$, then G is a path of order 3 and hence $G = K_1 + \cup 2K_1$. Let $p \geq 4$. We claim that x is the cutvertex of G . Otherwise, x lies on a smallest cycle, say C , of length at least 4. Then the neighbors of x on C do not belong to a minimum monophonic set of G and hence $m(G) \leq p - 2$, which is a contradiction. Hence $G - x$ has at least two components. We prove that each component is complete and x is adjacent to every vertex of each component. Suppose there exists a component B , which is not complete. Let u and v be two vertices in B such that $d(u, v) \geq 2$. Then by Theorem 1.2, both u and v lie on a common cycle and hence u and v lie on a smallest cycle of length at least 4. Hence $m(G) \leq p - 2$, which is a contradiction.

Thus each component of $G - x$ is complete. Now, if x is not adjacent to some vertex of a component, then x lies on a monophonic path of length at least 3 so that $m(G) \leq p - 2$, which is a contradiction. Hence $G = K_1 \cup m_j K_j$, where K_1 is the vertex x and $\sum m_j \geq 2$. \square

For any connected graph G , we have $0 \leq ex(G) \leq m(G)$ and $2 \leq m(G) \leq p$. In view of this, we have the following realization result.

Theorem 2.3. For every pair a, b of integers with $0 \leq a \leq b$ and $b \geq 2$, there exists a connected graph G with $ex(G) = a$ and $m(G) = b$.

Proof. We consider two cases, according to whether $a = 0$ or $a \geq 1$.

Case (i) $a = 0$. Let $F_1, F_2, \dots, F_i, \dots, F_b$ be such that each F_i is a copy of C_4 with vertices $v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}$ for $i = 1, 2, \dots, b$. Let G be the graph obtained by identifying the vertices $v_{i,1}$ of F_i ($1 \leq i \leq b$). The graph G is shown in Figure 2.2. Since $b \geq 2$, G has the unique cutvertex, say x .

Clearly, no vertex of G is an extreme vertex and so $ex(G) = 0$. By Theorem 1.5, every monophonic set of G contains at least one vertex from each component of $G - x$ and so $m(G) \geq b$. Let $S = \{v_{1,3}, v_{2,3}, v_{3,3}, \dots, v_{b,3}\}$. Since S is a monophonic set of G , it follows that $m(G) = |S| = b$.

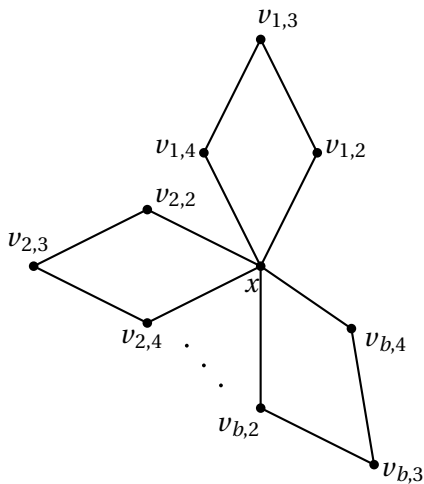


Figure 2.2: G

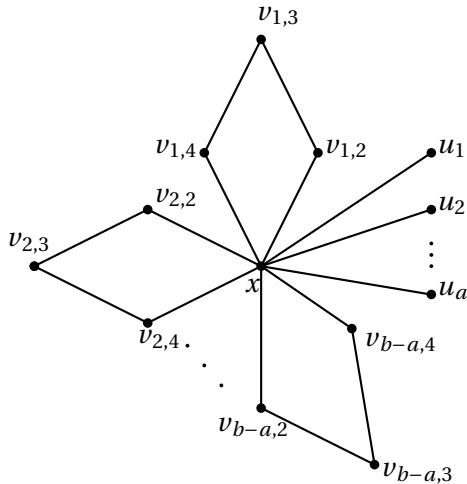


Figure 2.3: G

Case (ii) $a \geq 1$. If $a = b$, then the complete graph $G = K_a$ has the desired properties. If $a < b$, then we construct a graph G as follows. Let $F_1, F_2, \dots, F_i, \dots, F_{b-a}$ be such that each F_i is a copy of C_4 with vertices $v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}$ for $i = 1, 2, \dots, b - a$. Let $K_{1,a}$ be the star at x and let $U = \{u_1, u_2, \dots, u_a\}$ be the set of endvertices of $K_{1,a}$. Let G be the graph obtained by identifying the vertices $v_{i,1}$ of F_i ($1 \leq i \leq b - a$) and the vertex x of $K_{1,a}$. The graph G is shown in Figure 2.3 and it is clear that $ex(G) = a$.

Subcase (ii). k is odd. Let G be the graph obtained from C by joining every pair of vertices of $\{x_2, x_3, \dots, x_a\}$, every pair of vertices of $\{x_b, x_{b+1}, \dots, x_n\}$ and also adding the edge x_2x_n . The graph G is shown in Figure 2.5 for $k = 5$ and $r = 3$. Then $S = \{x_1, x_3, x_4, \dots, x_{a-1}, x_{b+1}, x_{b+2}, \dots, x_{n-1}\}$ is the set of all extreme vertices of G with $|S| = k$. It is clear that $1 \leq d(x_1, x_i) \leq r$ for $i = 2, 3, \dots, n$ and $d(x_1, x_{\frac{n+1}{2}}) = d(x_1, x_{\frac{n+3}{2}}) = r$. Hence eccentricity of x_1 is r . Similarly, it can be verified that eccentricity of each of the remaining vertices is also r . Hence $rad(G) = diam(G) = r$. It is clear that S is the unique minimum monophonic set of G so that $m(G) = k$ and G is an extreme monophonic graph.

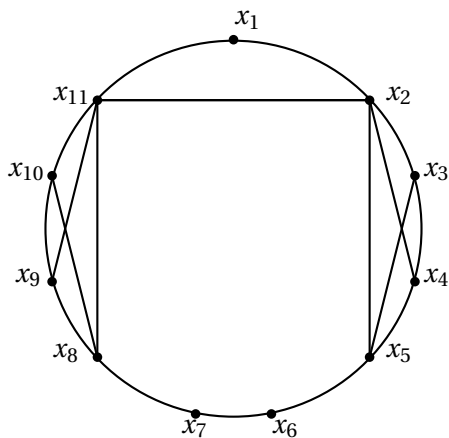


Figure 2.5: G

Case 2. $r < d \leq 2r$. Let $P_{d+1} : u_1, u_2, \dots, u_r, u_{r+1}, \dots, u_{d+1}$ be a path of order $d + 1$. Add $k - 2$ new vertices v_1, v_2, \dots, v_{k-2} to P_{d+1} and join each $v_i (1 \leq i \leq k - 2)$ to u_d , there by producing the tree G of Figure 2.6. Then $S = \{u_1, u_{d+1}, v_1, v_2, \dots, v_{k-2}\}$ is the set of all extreme vertices of G with $|S| = k$. It is clear that $e(u_{r+1}) = r, e(u_1) = d$ and $r \leq e(x) \leq d$ for all other vertices x in G . Then $rad(G) = r$ and $diam(G) = d$. It is clear that S is the unique minimum monophonic set of G and so $m(G) = k$. □

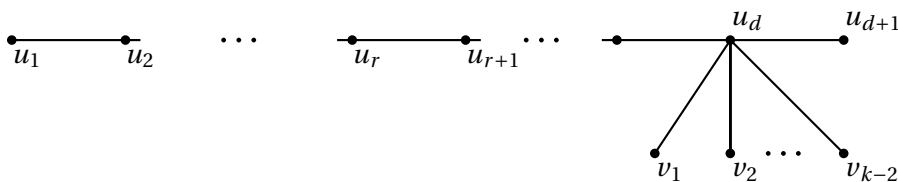


Figure 2.6: G

For any connected graph G , $rad_m(G) \leq diam_m(G)$. Santhakumaran and Titus [7] showed that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This theorem can be extended to extreme monophonic graphs so that the monophonic number can also be prescribed when $rad_m(G) < diam_m(G)$.

Theorem 2.5. *For positive integers r, d and $k \geq 3$ with $r < d$, there exists an extreme monophonic graph G with $rad_m(G) = r$, $diam_m(G) = d$ and $m(G) = k$.*

Proof. We prove this theorem by considering two cases.

Case 1. $r = 1$. Then $d \geq 2$. Let $C_{d+2} : v_1, v_2, \dots, v_{d+2}, v_1$ be a cycle of order $d + 2$. Let G be the graph obtained by adding $k - 2$ new vertices u_1, u_2, \dots, u_{k-2} to C_{d+2} and joining each of the vertices $u_1, u_2, \dots, u_{k-2}, v_3, v_4, \dots, v_{d+1}$ to the vertex v_1 . The graph G is shown in Figure 2.7. It is clear that $e_m(v_1) = 1$, $e_m(v_2) = e_m(v_{d+2}) = d$ and $2 \leq e_m(x) \leq d - 1$ for all other vertices x in G . Hence $rad_m(G) = 1$ and $diam_m(G) = d$. Let $S = \{u_1, u_2, \dots, u_{k-2}, v_2, v_{d+2}\}$ be the set of all extreme vertices of G . By Theorem 1.3, every monophonic set of G contains S . It is clear that S is a monophonic set of G and so $ex(G) = m(G) = k$.

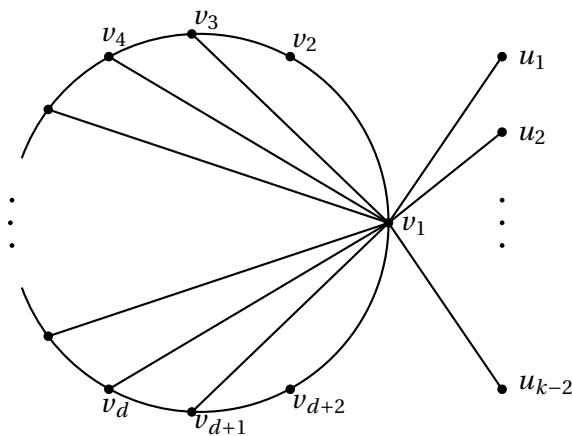


Figure 2.7: G

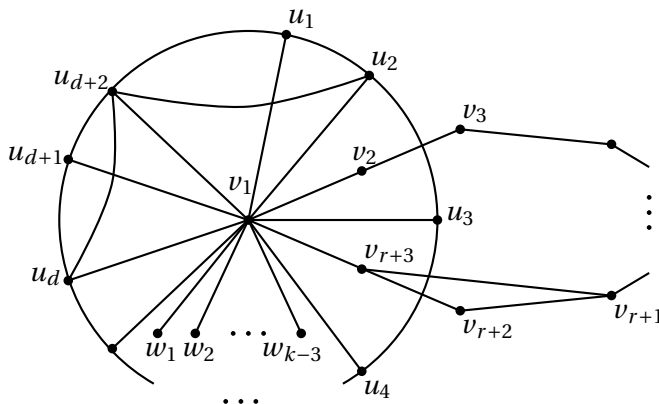


Figure 2.8: G

Case 2. $r \geq 2$. Let $C : v_1, v_2, \dots, v_{r+3}, v_1$ be a cycle of order $r + 3$ and let $W = K_1 + C_{d+2}$ be a wheel with $V(C_{d+2}) = \{u_1, u_2, \dots, u_{d+2}\}$ and let the vertex of K_1 be v_1 . Let H be the graph obtained from this by joining the vertices v_{r+1} and v_{r+3} of C , and also joining the vertex u_{d+2} with the vertices u_2 and u_d of W . Now, add $k - 3$ new vertices w_1, w_2, \dots, w_{k-3} to the graph H and join each vertex $w_i (1 \leq i \leq k - 3)$ to the vertex v_1 and obtain the graph G of Figure 2.8. It is easily verified that $r \leq e_m(x) \leq d$ for any vertex x in G and $e_m(v_2) = r, e_m(u_1) = d$. Then $rad_m(G) = r$ and $diam_m(G) = d$. Let $S = \{w_1, w_2, \dots, w_{k-3}, v_{r+2}, u_1, u_{d+1}\}$ be the set of all extreme vertices of G . By Theorem 1.3, every monophonic set of G contains S . It is clear that S is a monophonic set of G and so $ex(G) = m(G) = k$. □

Problem 2.6. For any three positive integers r, d and $k \geq 3$ with $r = d$, does there exist an extreme monophonic graph G with $rad_m(G) = r, diam_m(G) = d$ and $m(G) = k$?

Theorem 2.7. For any three positive integers d, k and p with $2 \leq d < p, 2 \leq k < p$ and $p - d - k + 2 \geq 0$, there exists an extreme monophonic graph G of order p , monophonic diameter d and $m(G) = k$.

Proof. Let K_{p-d+1} be the complete graph with vertex set $\{w_1, w_2, \dots, w_{p-d-k+2}, v_1, v_2, \dots, v_{k-1}\}$. Now, add a new vertex x to K_{p-d+1} and let H be the graph obtained from K_{p-d+1} by joining x with each $w_i (1 \leq i \leq p - d - k + 2)$. Let $P : u_0, u_1, \dots, u_{d-2}$ be a path of length $d - 2$. Let G be the graph obtained from H and P by identifying u_0 with v_1 . Then G has order p , monophonic diameter d and the graph G is shown in Figure 2.9. Let $S = \{x, u_{d-2}, v_2, v_3, \dots, v_{k-1}\}$ be the set of extreme vertices of G . It is clear that S is a monophonic set of G and so by Theorem 1.3, $m(G) = ex(G) = k$. Thus G is an extreme monophonic graph with monophonic number k . □

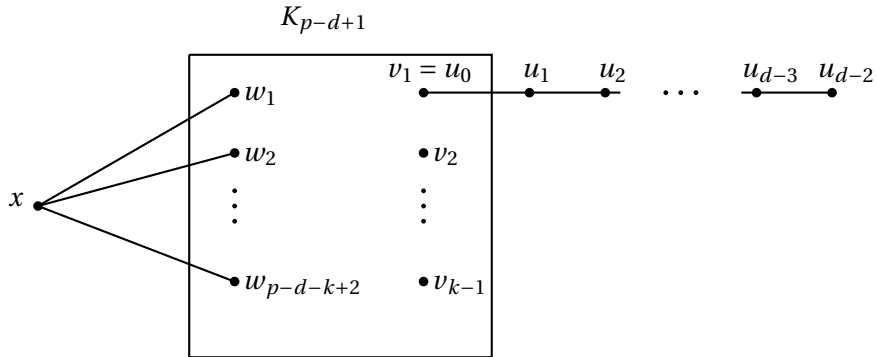


Figure 2.9: G

In the following theorem we construct a non-extreme monophonic graph G of order p , monophonic diameter $diam_m(G) = d$ and monophonic number $m(G) = k$ with data same as in Theorem 2.7.

Theorem 2.8. *For any three positive integers d, k and p with $2 \leq d < p, 2 \leq k < p$ and $p - d - k \geq 1$, there exists a non-extreme monophonic graph G of order p , $diam_m(G) = d$ and $m(G) = k$.*

Proof. Let $W = K_{p-d-k} + C_{d+2}$ be a general wheel with $V(K_{p-d-k}) = \{v_1, v_2, \dots, v_{p-d-k}\}$ and $V(C_{d+2}) = \{u_1, u_2, \dots, u_{d+2}\}$. Now, add $k - 2$ new vertices w_1, w_2, \dots, w_{k-2} to W and join each vertex $w_i (1 \leq i \leq k - 2)$ to every vertex of K_{p-d-k} . The graph G is shown in Figure 2.10. It is easily verified that $1 \leq e_m(x) \leq d$ for any vertex x in G and $e_m(u_1) = d$. Then $diam_m(G) = d$. Let $S = \{w_1, w_2, \dots, w_{k-2}\}$ be the set of all extreme vertices of G and so $ex(G) = k - 2$. By Theorem 1.3, every monophonic set of G contains S . It is clear that S is not a monophonic set of G . Let $T = S \cup \{u_2, u_{d+2}\}$. It is clear that T is a minimum monophonic set of G and so $m(G) = k$. Since $ex(G) = k - 2 \neq m(G)$, G is a non-extreme monophonic graph with $diam_m(G) = d$ and $m(G) = k$. \square

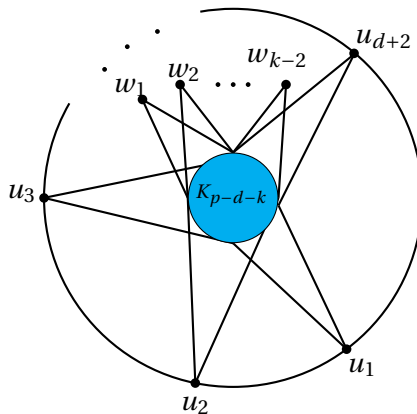


Figure 2.10: G

3. Extreme monophonic graphs and extreme geodesic graphs

Since every geodetic set is a monophonic set and every extreme vertex belongs to every geodetic set, it follows that every extreme geodesic graph is an extreme monophonic graph. However, the converse need not be true. For the graph G given in Figure 2.1, $S = \{v_1, v_7\}$ is a minimum monophonic set so that $m(G) = ex(G) = 2$. Therefore, G is an extreme monophonic graph. Since S is not a geodetic set, G is not an extreme geodesic graph. If $m(G) = g(G) = ex(G)$, then the graph G is both extreme monophonic and extreme geodesic. We observe that if G is extreme geodesic, then it is also extreme monophonic. In view of this observation, we have the following characterization theorem.

Theorem 3.1. *A graph G is both extreme geodesic and extreme monophonic if and only if every non-extreme vertex of G lies on an $x - y$ geodesic for some extreme vertices x and y in G .*

Proof. Let S be the set of all extreme vertices of G . If every non-extreme vertex of G lies on an $x - y$ geodesic for some extreme vertices x and y in G , then by Theorems 1.1 and 1.3, S is both minimum geodetic set and minimum monophonic set of G . Hence G is both extreme geodesic and extreme monophonic graph.

Conversely, let G be both extreme geodesic and extreme monophonic graph. Then $ex(G) = g(G) = m(G)$. Since $ex(G) = g(G)$, S is the unique minimum geodetic set. Hence every non-extreme vertex of G lies on an $x - y$ geodesic for some vertices $x, y \in S$. \square

Theorem 3.2. *For every pair a, b of integers with $2 \leq a \leq b \leq 2a$, there exists a connected graph G which is neither extreme geodesic nor extreme monophonic such that $m(G) = a$ and $g(G) = b$.*

Proof. For each integer i with $1 \leq i \leq b - a$, let $C_i : v_{i,0}, v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}, v_{i,0}$ be a cycle of order 5 and for each integer j with $1 \leq j \leq 2a - b$, let $C'_j : u_{j,0}, u_{j,1}, u_{j,2}, u_{j,3}, u_{j,0}$ be a cycle of order 4. Let G be the graph obtained by identifying $v_{i,0}$ of C_i ($1 \leq i \leq b - a$) and $u_{j,0}$ of C'_j ($1 \leq j \leq 2a - b$). The graph G is shown in Figure 3.1. It is clear that no vertex of G is an extreme vertex of G . The set $M_1 = \{u_{1,2}, u_{2,2}, \dots, u_{2a-b,2}, v_{1,2}, v_{2,2}, \dots, v_{b-a,2}\}$ is a monophonic set of G and so by Theorem 1.5, $m(G) = a$. Also, the set $S_1 = \{u_{1,2}, u_{2,2}, \dots, u_{2a-b,2}, v_{1,2}, v_{1,3}, v_{2,2}, v_{2,3}, \dots, v_{b-a,2}, v_{b-a,3}\}$ is a geodetic set of G and so by Theorem 1.6, $g(G) = b$. Hence G is neither an extreme geodesic nor an extreme monophonic graph. \square

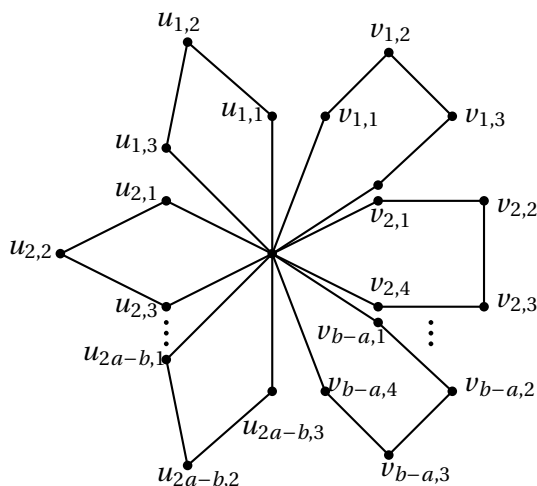


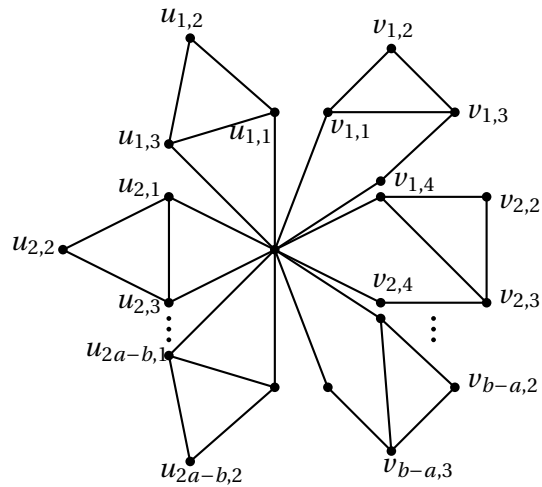
Figure 3.1: G

Problem 3.3. For every pair a, b of integers with $2 \leq a \leq b$ and $b > 2a$, does there exist a connected graph G which is neither extreme geodesic nor extreme monophonic such that $m(G) = a$ and $g(G) = b$?

Theorem 3.4. For every pair a, b of integers with $2 \leq a < b \leq 2a$, there exists a connected graph G which is extreme monophonic but not extreme geodesic such that $m(G) = a$ and $g(G) = b$.

Proof. For each integer i with $1 \leq i \leq b - a$, let $C_i : v_{i,0}, v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}, v_{i,0}$ be a cycle of order 5 and for each integer j with $1 \leq j \leq 2a - b$, let $C'_j : u_{j,0}, u_{j,1}, u_{j,2}, u_{j,3}, u_{j,0}$ be a cycle of order 4. Let H be the graph obtained from C_i and C'_j by identifying $v_{i,0}$ of each C_i ($1 \leq i \leq b - a$) and $u_{j,0}$ of each C'_j ($1 \leq j \leq 2a - b$). Now, let G be the graph obtained from H by (i) joining each $u_{j,1}$ with $u_{j,3}$ ($1 \leq j \leq 2a - b$), and (ii) joining each $v_{i,1}$ with $v_{i,3}$ ($1 \leq i \leq b - a$). The graph G is shown in Figure 3.2. Let $S = \{u_{1,2}, u_{2,2}, \dots, u_{2a-b,2}, v_{1,2}, v_{2,2}, \dots, v_{b-a,2}\}$ be the set of all extreme vertices of G . Since S is a monophonic set of G , it follows from Theorem 1.5 that $m(G) = a$. It is clear that S is not a geodetic set of G and $S_1 = S \cup \{v_{1,3}, v_{2,3}, \dots, v_{b-a,3}\}$ is a geodetic set of G and so by Theorem 1.6, $g(G) = b$. Hence G is an extreme monophonic graph but it is not an extreme geodesic graph. \square

If G is an extreme monophonic graph and it is not an extreme geodesic graph, then $m(G) < g(G)$. This leads to the following problem.

Figure 3.2: G

Problem 3.5. For every pair a, b of integers with $2 \leq a < b$ and $b > 2a$, does there exist a connected graph G which is extreme monophonic but not extreme geodesic such that $m(G) = a$ and $g(G) = b$?

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