TAMKANG JOURNAL OF MATHEMATICS
Volume 35, Number 3, Autumn 2004

APPROXIMATE FIXED POINTS ON ALMOST CONVEX SETS

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Abstract. In this paper, we deduce a maximal element theorem on multimaps and an approximate fixed point theorem on almost convex sets. This generalizes the well-known Himmelberg fixed point theorem and also unifies recent results of Park and Tan [14] and Sy and Park [16].

The celebrated Knaster-Kuratowski-Mazurkiewicz (simply KKM) principle is a versatile tool to obtain fixed point theorems on convex subsets of topological vector spaces. For examples, Park and Tan ([13], [14]) gave simple proofs of the generalizations of fixed point theorems due to Schauder, Tychonoff, and Himmelberg by applying the KKM principle directly.

In an earlier work of Sy and Park [16], the KKM principle is applied to obtain a new non-compact version of the Fan-Browder fixed point theorem, from which an approximate fixed point theorem is deduced. In this paper, we follow the method of [16] and obtain a maximal element theorem and an approximate fixed point theorem which unify those in Park and Tan ([13], [14]) and Sy and Park [16].

A *multimap* (or simply, a *map*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of a set Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^-(y) := \{x \in X : y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) : x \in A\}$.

For a set D, let $\langle D\rangle$ denote the set of nonempty finite subsets of D.

Let X be a subset of a vector space and D a nonempty subset of X. We call (X, D) a *convex space* if co$D \subset X$ and X has a topology that induces the Euclidean topology on the convex hulls of any $N \in \langle D\rangle$; see [5], [6]. If $X = D$ is convex, then $X = (X, X)$ becomes a convex space in the sense of Lassonde [4]. If X is compact, then the convex space (X, D) is said to be compact. Every nonempty convex subset X of a topological vector space is a convex space with respect to any nonempty subset D of X, and the converse is known to be not true.

The following version of the Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem for convex spaces is known.

Theorem 1. *Let (X, D) be a convex space and $F : D \multimap X$ a multimap such that*

(1) *$F(z)$ is open [resp. closed] for each $z \in D$; and*

Received March 14, 2003.
2000 *Mathematics Subject Classification.* 47H04, 47H10, 52H07, 54C60, 54H25.
Key words and phrases. KKM maps, almost convex spaces, the Himmelberg fixed point theorem.

(2) *F is a KKM map (that is, co$N \subset F(N)$ for each $N \in \langle D \rangle$).*

Then $\{F(z)\}_{z \in D}$ has the finite intersection property. (More precisely, for any $N \in \langle D \rangle$, we have co$N \cap [\bigcap_{z \in N} F(z)] \neq \emptyset$.*)*

The closed version is essentially due to Fan [1] and the open version is motivated from the works of Kim [3] and Shih-Tan [15], who showed that the original KKM theorem holds for open valued KKM maps on a simplex. Later, Lassonde [5] showed that the closed and open versions of Theorem 1 can be derived from each other. More general versions of Theorem 1 were recently known; for example, see Park ([10] - [12]).

From Theorem 1, Sy and Park [16] obtained the following.

Theorem 2. *Let (X, D) be a convex space and $P : X \multimap D$ a multimap. If there exist $z_1, z_2, \ldots, z_n \in D$ and nonempty open [resp. closed] subsets $G_i \subset P^-(z_i)$ for each $i = 1, 2, \ldots, n$ such that co$\{z_1, z_2, \ldots, z_n\} \subset \bigcup_{i=1}^n G_i$, then the map co$P : X \multimap X$ has a fixed point $x_0 \in X$ (that is, $x_0 \in$ co$P(x_0)$).*

From Theorem 2, we have the following.

Theorem 3. *Let (X, D) be a compact convex space and $P : X \multimap D$ a map such that*
(1) $x \notin$ co $P(x)$ for all $x \in X$; and
(2) $P^-(y)$ is open for all $y \in D$.
Then there exists an $\overline{x} \in X$ such that $P(\overline{x}) = \emptyset$.

Proof. Suppose $P(x) \neq \emptyset$ for all $x \in X$. Then $X = \bigcup_{y \in D} P^-(y)$. Since X is compact, $X = \bigcup_{y \in N} P^-(y)$ for some $N \in \langle D \rangle$. Then by Theorem 2, co P has a fixed point, which contradicts (1).

For $X = D$, Theorem 3 reduces to theorems of Toussaint [17] and Yannelis and Prabhakar [18] on the existence of maximal elements.

A nonempty subset Y of a topological vector space E is said to be *almost convex* if for any neighborhood V of the origin O in E and for any finite set $\{y_1, y_2, \ldots, y_n\} \subset Y$, there exists a finite set $\{z_1, z_2, \ldots, z_n\} \subset Y$ such that, for each $i \in \{1, 2, \ldots, n\}$, we have $z_i - y_i \in V$ and co$\{z_1, z_2, \ldots, z_n\} \subset Y$; see [2].

From Theorem 2, we deduce the following approximate fixed point theorem.

Theorem 4. *Let X be a subset of a topological vector space E and Y an almost convex dense subset of X. Let $F : X \multimap X$ be a lower [resp. upper] semi-continuous map such that (1) F has nonempty values, (2) $F(y)$ is convex for all $y \in Y$, and (3) $F(X)$ is totally bounded. Then for every open [resp. closed] convex neighborhood V of the origin O of E, there exists a point $x_V \in Y$ such that*

$$F(x_V) \cap (x_V + V) \neq \emptyset.$$

Proof. Let V be the given symmetric neighborhood of O in E. Then there exists a neighborhood U of O such that $U + U \subset V$. Since $F(X)$ is totally bounded in X, there exists a finite subset $\{x_1, x_2, \ldots, x_n\} \subset F(X)$ such that $F(X) \subset \bigcup_{i=1}^n (x_i + U)$. Moreover, since Y is almost convex and dense in X, there exists a finite subset $D := \{y_1, y_2, \ldots, y_n\}$ of Y such that $x_i - y_i \in U$ for each $i \in \{1, 2, \ldots, n\}$ and $Z := \mathrm{co}\{y_1, y_2, \ldots, y_n\} \subset Y$.

Since $x_i + U = y_i + (x_i - y_i) + U \subset y_i + U + U \subset y_i + V$, we have $F(Z) \subset F(X) \subset \bigcup_{i=1}^n (y_i + V)$. Define a map $P : Z \multimap D$ by $P(z) := (F(z) - V) \cap D$ for $z \in Z$. Then each $P(z)$ is nonempty. Note that for each $y \in D$, we have

$$\begin{aligned} P^-(y) &= \{z \in Z : y \in P(z)\} \\ &= \{z \in Z : y \in (F(z) - V) \cap D\} \\ &= \{z \in Z : F(z) \cap (y + V) \neq \emptyset\}. \end{aligned}$$

If F is lower semi-continuous and V is open, then each $P^-(y)$ is open in Z. If F is upper semi-continuous and V is closed, then each $P^-(y)$ is closed in Z.

Note that for each $z \in Z$, we have a $y \in D$ such that $z \in P^-(y)$. Therefore, $Z \subset Y = \bigcup_{y \in D} P^-(y)$. Hence, by Theorem 2, $\mathrm{co}P : Z \multimap Z$ has a fixed point $x_V \in Z \subset Y$, that is, $x_V \in \mathrm{co}P(x_V)$. Since $x_V \in Z \subset Y$, $F(x_V)$ is convex and hence, $x_V \in \mathrm{co}P(x_V) \subset \mathrm{co}[(F(x_V) - V) \cap D] \subset (F(x_V) - V) \cap Z$, which readily implies $F(x_V) \cap (x_V + V) \neq \emptyset$.

If $X = Y$ is almost convex, then Theorem 4 improves Theorem 5 obtained in Sy and Park [16].

We now deduce the following result due to Park and Tan [14].

Theorem 5. (Park and Tan [14], Theorem 1) *Let X be a subset of a locally convex Hausdorff topological vector space E and Y an almost convex dense subset of X. Let $T : X \multimap X$ be a compact upper semi-continuous multimap with nonempty closed values such that $T(y)$ is convex for all $y \in Y$. Then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.*

Proof. For each neighborhood V of O, there exist $x_V, y_V \in X$ such that $y_V \in T(x_V)$ and $y_V \in x_V + V$. Since $T(X)$ is relatively compact, we may assume that the net $\{y_V\}$ converges to some $x_0 \in X$. Since E is Hausdorff, the net $\{x_V\}$ also converges to x_0. Because T is upper semi-continuous with closed values, the graph of T is closed in $X \times T(X)$ and hence we have $x_0 \in T(x_0)$. This proves the theorem.

In particular, for $X = Y$, we obtain

Theorem 6. (Park and Tan [14], Theorem 2) *Let X be an almost convex subset of a locally convex Hausdorff topological vector space. Then any compact upper semi-continuous multimap $T : X \multimap X$ with nonempty closed convex values has a fixed point in X.*

If X itself is convex, Theorem 6 reduces to the Himmelberg fixed point theorem.

From Theorem 4 or from Theorem 5, we obtain

Theorem 7. (Park and Tan [13], Theorem 1) *Let X be an almost convex subset of a locally convex Hausdorff topological vector space E and $f : X \multimap X$ a compact continuous map. Then f has a fixed point.*

Further, from the lower semi-continuous case of Theorem 4, we deduce the following.

Theorem 8. *Let X be a subset of a topological vector space and Y an almost convex dense subset of X. Let $F : X \multimap X$ be a multimap such that*
(1) *$F(x)$ is nonempty for each $x \in X$;*
(2) *$F(y)$ is convex for each $y \in Y$;*
(3) *$F^-(z)$ is open for each $z \in X$; and*
(4) *$F(X)$ is totally bounded.*
Then for any convex neighborhood V of O in E, there exists a point $x_V \in X$ such that $F(x_V) \cap (x_V + V) \neq \emptyset$.

Proof. Simply F is lower semi-continuous.

If $X = Y$ is convex, then Theorem 8 reduces to Sy and Park ([16], Theorem 7).

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