NEW CLASSES OF \( k \)-UNIFORMLY CONVEX
AND STARLIKE FUNCTIONS

ESSAM AQLAN, JAY M. JAHANGIRI AND S. R. KULKARNI

Abstract. Certain classes of analytic functions are defined which will generalize new, as well as well-known, classes of \( k \)-uniformly convex and starlike functions. We provide necessary and sufficient coefficient conditions, distortion bounds, extreme points and radius of starlikeness for these classes.

1. Introduction

Let \( A \) denote the family of functions \( f \) that are analytic in the open unit disc \( \Delta = \{ z : |z| < 1 \} \) and consider the subclass \( T \) consisting of functions \( f \) in \( A \) which are univalent in \( \Delta \) and are of the form \( f(z) = z - \sum_{m=2}^{\infty} a_m z^m \) where \( a_m \geq 0 \). The class \( T \) was introduced and studied by Silverman [9]. For \( 0 \leq \lambda \leq 1 \), \( 0 \leq \beta < 1 \) and \( k \geq 0 \) we let \( U(k, \beta, \lambda) \) consist of functions \( f \) in \( T \) satisfying the condition

\[
\Re \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f'(z) + \lambda z f'(z)} \right) \geq k \left| \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f'(z) + \lambda z f'(z)} - 1 \right| + \beta. \tag{1.1}
\]

The family \( U(k, \beta, \lambda) \) is of special interest for it contains many well-known, as well as new, classes of analytic univalent functions. In particular, \( U(0, \beta, 0) \) is the family of functions starlike of order \( \beta \) and \( U(0, \beta, 1) \) is the family of functions convex of order \( \beta \). For \( U(k, 0, 0) \) and \( U(k, 0, 1) \) we, respectively, obtain the classes of \( k \)-uniformly starlike and \( k \)-uniformly convex functions. The case for \( \beta \) to be other than zero, i.e. \( \beta \in (0, 1) \), is of special interest. For instance, if \( \beta \in (0, 1) \) then these classes are generalized to \( U(k, \beta, 0) \) and \( U(k, \beta, 1) \) of \( k \)-uniformly starlike functions of order \( \beta \) and \( k \)-uniformly convex functions of order \( \beta \). More generally speaking, as \( \beta \) and \( \lambda \) vary, the family \( U(k, \beta, \lambda) \) provides a transition from the class \( k \)-uniformly starlike functions of order \( \beta \) and type \( \lambda \) to the class of \( k \)-uniformly convex functions of order \( \beta \) and type \( \lambda \) in \( \Delta \). The main feature of the elements of these classes is the fact that they map circular arcs with center at any point \( \zeta \) in the open unit disk \( \Delta \) onto convex arcs or arcs starlike with respect to \( f(\zeta) \), respectively. We remark that the classes of uniformly convex and uniformly starlike functions were introduced by Goodman [3,4] and later generalized by Kanas et

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al. in [5, 6, 7, 8]. In this paper we provide necessary and sufficient coefficient conditions, distortion bounds, extreme points, radius of starlikeness and convexity, closure theorem for functions in \( U(k, \beta, \lambda) \).

2. Main Results

Our first theorem is on the necessary and sufficient coefficient requirements for functions to be in the class \( U(k, \beta, \lambda) \).

**Theorem 1.** \( f \in U(k, \beta, \lambda) \) if and only if

\[
\sum_{m=2}^{\infty} (1 + m\lambda - \lambda)(m(1 + k) - (k + \beta))a_m \leq 1 - \beta
\]  

(2.1)

where \( 0 \leq \beta < 1 \), \( k \geq 2 \), \( 0 \leq \lambda \leq 1 \), and \( -\pi < \theta \leq \pi \).

**Proof.** We have \( f \in U(k, \beta, \lambda) \) if and only if the condition (1.1) is satisfied. Upon using the fact that

\[
\Re w > k|w - 1| + \beta \Leftrightarrow \Re \{w(1 + ke^{i\theta}) - ke^{i\theta}\} > \beta
\]

the condition (1.1) may be written as

\[
\Re \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda zf'(z)}(1 + ke^{i\theta}) - ke^{i\theta}\right) \geq \beta
\]

or equivalently

\[
\Re \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda zf'(z)}(1 + ke^{i\theta}) - ke^{i\theta}((1 - \lambda)f(z) + \lambda zf'(z))\right) \geq \beta.
\]  

(2.2)

Now we let \( A(z) = [zf'(z) + \lambda z^2 f''(z)](1 + ke^{i\theta}) - ke^{i\theta}((1 - \lambda)f(z) + \lambda zf'(z)) \) and let \( B(z) = (1 - \lambda)f(z) + \lambda zf'(z) \). Then (2.2) is equivalent to

\[
|A(z) + (1 - \beta)B(z)| \geq |A(z) - (1 + \beta)B(z)| \text{ for } 0 \leq \beta < 1.
\]

For \( A(z) \) and \( B(z) \) as above, we have

\[
|A(z) + (1 - \beta)B(z)| = [(2 - \beta)|z| - \sum_{m=2}^{\infty} (m + m\lambda(m - 1) + (1 - \beta)(1 - \lambda + m\lambda))a_m z^m
\]

\[
-ke^{i\theta} \left( \sum_{m=2}^{\infty} [m + m\lambda(m - 1) - (1 - \lambda + m\lambda)]a_m z^m \right)
\]

\[
\geq (2 - \beta)|z| - \sum_{m=2}^{\infty} [m + m\lambda(m - 1) + (1 - \beta)(1 - \lambda + m\lambda)]a_m |z|^m
\]

\[
-ke^{i\theta} \left( \sum_{m=2}^{\infty} [m + m\lambda(m - 2) - 1 + \lambda]a_m |z|^m \right)
\]
and, similarly,

\[
|A(z) - (1 + \beta)B(z)| \leq \beta|z| + \sum_{m=2}^{\infty} [m + m\lambda(m - 1) - (1 + \beta)(1 - \lambda + m\lambda)]a_m|z|^m
\]

\[
+ k \sum_{m=2}^{\infty} [m + m\lambda(m - 1) - (1 - \lambda + m\lambda)]a_m|z|^m.
\]

Therefore,

\[
|A(z) + (1 - \beta)B(z)| - |A(z) - (1 + \beta)B(z)|
\]

\[
\geq 2(1 - \beta)|z| - \sum_{m=2}^{\infty} [2m + 2m\lambda(m - 1) - 2\beta(1 - \lambda + m\lambda)]
\]

\[-k[2m + 2m\lambda(m - 1) - 2(1 - \lambda + m\lambda)]a_m|z|^m \geq 0
\]

or

\[
(1 - \beta) \geq \sum_{m=2}^{\infty} [m(1 + k) + m\lambda(m - 1)(1 + k) - (1 - \lambda + m\lambda)(\beta + k)]a_m
\]

which yields

\[
(1 - \beta) \geq \sum_{m=2}^{\infty} (1 - \lambda + m\lambda)(m(1 + k) - (k + \beta)]a_m.
\]

On the other hand, we must have

\[
Re \left( \frac{|zf'(z) + \lambda z^2 f''(z)|(1 + ke^{i\theta}) - ke^{i\theta}[1 - (1 - \lambda)f(z) + \lambda zf'(z)]}{(1 - \lambda)f(z) + \lambda zf'(z)} \right) \geq \beta.
\]

Upon choosing the values of \(z\) on the positive real axis where \(0 \leq z = r < 1\), the above inequality reduces to

\[
Re \left( (1 - \beta) - \sum_{m=2}^{\infty} [m - m^2\lambda - \lambda - \beta(1 - \lambda + m\lambda)]a_m r^{m-1}
\]

\[-ke^{i\theta} \sum_{m=2}^{\infty} [m + m^2\lambda - \lambda - (1 - \lambda + m\lambda)]a_m r^{m-1}
\]

\[
\leq \left( 1 - \sum_{m=2}^{\infty} (1 - \lambda + m\lambda)a_m r^{m-1} \right) \geq 0.
\]

Since \(Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1\), the above inequality reduces to

\[
Re \left( (1 - \beta) - \sum_{m=2}^{\infty} [m + m^2\lambda - \lambda - \beta(1 - \lambda + m\lambda)]a_m r^{m-1}
\]
\[ -k \sum_{m=2}^{\infty} [m + m^2 \lambda - \lambda - (1 - \lambda + m \lambda)] a_m r^{m-1} \]

\[ \left/ \left(1 - \sum_{m=2}^{\infty} (1 - \lambda + m \lambda) a_m r^{m-1} \right) \right. \geq 0. \]

Letting \( r \to 1^- \) we get desired conclusion.

**Remark.** As special cases of Theorem 1, for \( \lambda = 0 \), see [2] and for \( k = 0 \), see [1].

The distortion theorem for the class \( \mathcal{U}(k, \beta, \lambda) \) is given next.

**Theorem 2.** If \( f \in \mathcal{U}(k, \beta, \lambda) \) and \( |z| \leq r < 1 \), then we have the sharp bounds

\[ r - \frac{1 - \beta}{(1 + \lambda)(2 + k - \beta)} r^2 \leq |f(z)| \leq r + \frac{1 - \beta}{(1 + \lambda)(2 + k - \beta)} r^2 \] 

and

\[ 1 - \frac{2(1 - \beta)}{(1 + \lambda)(2 + k - \beta)} r \leq |f'(z)| \leq 1 + \frac{2(1 - \beta)}{(1 + \lambda)(2 + k - \beta)} r. \]

**Proof.** We only prove the right hand side inequality in (2.3) since the other inequalities can be justified using similar arguments. On account of (2.1), we may write

\[ \sum_{m=2}^{\infty} (1 + \lambda)(2 + k - \beta) a_m \leq \sum_{m=2}^{\infty} (1 + m \lambda - \lambda)(m(1 + k) - (k + \beta)) a_m \leq 1 - \beta. \]

Hence

\[ |f(z)| \leq |z| + |z|^2 \sum_{m=2}^{\infty} a_m \leq r + r^2 \sum_{m=2}^{\infty} a_m \leq r + \frac{1 - \beta}{(1 + \lambda)(2 + k - \beta)} r^2. \]

The distortion bounds in Theorem 2 are sharp for

\[ f(z) = z - \frac{(1 - \beta)}{(1 + \lambda)(2 + k - \beta)} z^2, \quad z = \pm r. \]

In the following theorem, we study the properties of extreme points of functions in the family \( \mathcal{U}(k, \beta, \lambda) \).

**Theorem 3.** Let \( f_1(z) = z \) and \( f_m(z) = z - \frac{1 - \beta}{(1 + m \lambda - \lambda)(m(1 + k) - (k + \beta))} z^m \) where \( \lambda \geq 0, 0 \leq \beta < 1, k \geq 0, \) and \( m \geq 2 \). Then \( f(z) \) is in \( \mathcal{U}(k, \beta, \lambda) \) if and only if it can be expressed in the form \( f(z) = \sum_{m=1}^{\infty} \gamma_m f_m(z) \) where \( \gamma_m \geq 0 \) and \( \sum_{m=1}^{\infty} \gamma_m = 1 \).

**Proof.** Let \( f(z) = \sum_{m=1}^{\infty} \gamma_m f_m(z) \) where \( \gamma_m \geq 0 \) and \( \sum_{m=1}^{\infty} \gamma_m = 1 \). Letting

\[ f(z) = z - \sum_{m=2}^{\infty} \frac{1 - \beta}{(1 + m \lambda - \lambda)(m(1 + k) - (k + \beta))} \gamma_m z^m. \]
we get
\[
\sum_{m=2}^{\infty} \left( \frac{(1 + m\lambda - \lambda)(m(1 + k) - (k + \beta))}{1 - \beta} \right) \gamma_m \frac{1 - \beta}{(1 + m\lambda - \lambda)(m(1 + k) - (k + \beta))} = \sum_{m=2}^{\infty} \gamma_m = 1 - \gamma_1 \leq 1 \quad (\text{by Theorem 1})
\]

Therefore \( f \in \mathcal{U}(k, \beta, \lambda) \). Conversely, suppose that \( f \in \mathcal{U}(k, \beta, \lambda) \). Then
\[
a_m \leq \frac{1 - \beta}{(1 + m\lambda - \lambda)(m(1 + k) - (k + \beta))}, \quad (m \geq 2).
\]

Now, by letting \( \gamma_m = \frac{(1 + m\lambda - \lambda)(m(1 + k) - (k + \beta))}{1 - \beta} a_m \) and \( \gamma_1 = 1 - \sum_{m=2}^{\infty} \gamma_m \) we conclude the theorem, since \( f(z) = \sum_{m=1}^{\infty} \gamma_m f_m = \gamma_1 f_1(z) + \sum_{m=2}^{\infty} \gamma_m f_m(z) \).

**Remark.** For \( \lambda = 0 \), we obtain the extreme points given earlier in [2].

Finally, we discuss the radius of starlikeness of the functions in \( \mathcal{U}(k, \beta, \lambda) \).

**Theorem 5.** Let the \( f \) be in the class \( \mathcal{U}(k, \beta, \lambda) \). Then \( f \) is starlike of order \( \delta (0 \leq \delta < 1) \) in \( |z| < r_2(\beta, \lambda, k, \delta) \), where
\[
r_2(\beta, \lambda, k, \delta) = \inf_m \left( \frac{(1 - \delta)(1 + m\lambda - \lambda)(m(1 + k) - (k + \beta))}{(m - \delta)(1 - \beta)} \right)^{\frac{1}{m-1}}, \quad m \geq 2. \quad (2.4)
\]

**Proof.** It suffices to show that \( \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \delta \) for \( |z| < r_2(\beta, \lambda, k, \delta) \). Note that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \sum_{m=1}^{\infty} \frac{(m-1)a_m |z|^{m-1}}{1 - \sum_{m=2}^{\infty} a_m |z|^{m-1}}.
\]

Now \( \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \delta \) if we have the condition
\[
\sum_{m=2}^{\infty} \frac{(m-\delta)a_m |z|^{m-1}}{(1-\delta)} \leq 1. \quad (2.5)
\]

Considering the coefficient conditions required by Theorem 1, the above inequality (2.5) is true if
\[
\frac{m-\delta}{1-\delta} |z|^{m-1} \leq \frac{(1 + m\lambda - \lambda)(m(1 + k) - (k + \beta))}{(1 - \beta)}
\]
or if
\[
|z| \leq \left\{ \frac{(1 - \delta)(1 + m\lambda - \lambda)(m(1 + k) - (k + \beta))}{(m - \delta)(1 - \beta)} \right\}^{\frac{1}{m-1}}, \quad m \geq 2.
\]

This last expression yields the bound required by the above theorem.
References


Department of Mathematics, Fergusson College, Pune - 411 004, India.

Mathematical Sciences, Kent State University, 14111 Claridon Troy Road, Burton, Ohio 44021-9500, U.S.A.

E-mail: jay@geauga.kent.edu

Department of Mathematics, Fergusson College, Pune - 411 004, India.

E-mail: Kulkarniergy@yahoo.com