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NEW CLASSES OF k-UNIFORMLY CONVEX AND STARLIKE FUNCTIONS

ESSAM AQLAN, JAY M. JAHANGIRI AND S. R. KULKARNI

Abstract. Certain classes of analytic functions are defined which will generalize new, as well as well-known, classes of k-uniformly convex and starlike functions. We provide necessary and sufficent coefficient conditions, distortion bounds, extreme points and radius of starlikeness for these classes.

1. Introduction

Let $\mathcal{A}$ denote the family of functions f that are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$ and consider the subclass $\mathcal{T}$ consisting of functions f in $\mathcal{A}$ which are univalent in Δ and are of the form $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$ where $a_m \geq 0$. The class $\mathcal{T}$ was introduced and studied by Silverman [9]. For $0 \leq \lambda \leq 1$, $0 \leq \beta < 1$ and $k \geq 0$ we let $\mathcal{U}(k, \beta, \lambda)$ consist of functions f in $\mathcal{T}$ satisfying the condition

$$Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)}\right) \geq k\left|\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right| + \beta. \tag{1.1}$$

The family $\mathcal{U}(k, \beta, \lambda)$ is of special interest for it contains many well-known, as well as new, classes of analytic univalent functions. In particular, $\mathcal{U}(0, \beta, 0)$ is the family of functions starlike of order β and $\mathcal{U}(0, \beta, 1)$ is the family of functions convex of order β. For $\mathcal{U}(k, 0, 0)$ and $\mathcal{U}(k, 0, 1)$ we, respectively, obtain the classes of k- uniformly starlike and k-uniformly convex functions. The case for β to be other than zero, i.e. $\beta \in (0, 1)$, is of special interest. For instance, if $\beta \in (0, 1)$ then these classes are generalized to $\mathcal{U}(k, \beta, 0)$ and $\mathcal{U}(k, \beta, 1)$ of k-uniformly starlike functions of order β and k-uniformly convex functions of order β. More generally speaking, as β and λ vary, the family $\mathcal{U}(k, \beta, \lambda)$ provides a transition from the class k-uniformly starlike functions of order β and type λ to the class of k-uniformly convex functions of order β and type λ in Δ. The main feature of the elements of these classes is the fact that they map circular arcs with center at any point ζ in the open unit disk Δ onto convex arcs or arcs starlike with respect to $f(\zeta)$, respectively. We remark that the classes of uniformly convex and uniformly starlike functions were introduced by Goodman [3,4] and later generalized by Kanas et

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al. in [5, 6, 7, 8]. In this paper we provide necessary and sufficent coefficient conditions, distortion bounds, extreme points, radius of starlikeness and convexity, closure theorem for functions in $\mathcal{U}(k,\beta,\lambda)$.

2. Main Results

Our first theorem is on the necessary and sufficient coefficient requirements for functions to be in the class $\mathcal{U}(k,\beta,\lambda)$.

Theorem 1. $f \in \mathcal{U}(k,\beta,\lambda)$ *if and only if*

$$\sum_{m=2}^{\infty}(1+m\lambda-\lambda)(m(1+k)-(k+\beta))a_m \le 1-\beta \tag{2.1}$$

where $0 \le \beta < 1,\ k \ge 2,\ 0 \le \lambda \le 1,$ *and* $-\pi < \theta \le \pi.$

Proof. We have $f \in \mathcal{U}(k,\beta,\lambda)$ if and only if the condition (1.1) is satisfied. Upon using the fact that

$$Re\ \ w > k|w-1|+\beta \Leftrightarrow Re\{w(1+ke^{i\theta})-ke^{i\theta}\} > \beta$$

the condition (1.1) may be written as

$$Re\left(\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda z f'(z)}(1+ke^{i\theta})-ke^{i\theta}\right) \ge \beta$$

or equivalenlty

$$Re\left(\frac{(zf'(z)+\lambda z^2 f''(z))(1+ke^{i\theta})-ke^{i\theta}((1-\lambda)f(z)+\lambda z f'(z))}{(1-\lambda)f(z)+\lambda z f'(z)}\right) \ge \beta. \tag{2.2}$$

Now we let $A(z) = [zf'(z)+\lambda z^2 f''(z)](1+ke^{i\theta})-ke^{i\theta}[(1-\lambda)f(z)+\lambda z f'(z)]$ and let $B(z) = (1-\lambda)f(z)+\lambda z f'(z)$. Then (2.2) is equivalent to

$$|A(z)+(1-\beta)B(z)| \ge |A(z)-(1+\beta)B(z)| \quad \text{for} \quad 0 \le \beta < 1.$$

For $A(z)$ and $B(z)$ as above, we have

$$\begin{aligned}|A(z)+(1-\beta)B(z)| &= |(2-\beta)z - \sum_{m=2}^{\infty}(m+m\lambda(m-1)+(1-\beta)(1-\lambda+m\lambda))a_m z^m \\ &\quad -ke^{i\theta}(\sum_{m=2}^{\infty}\left[(m+m\lambda(m-1)-(1-\lambda+m\lambda)\right]a_m z^m| \\ &\ge (2-\beta)|z| - \sum_{m=2}^{\infty}\left[m+m\lambda(m-1)+(1-\beta)(1-\lambda+m\lambda)\right]a_m|z|^m \\ &\quad -k\sum_{m=2}^{\infty}\left[m+m\lambda(m-2)-1+\lambda\right]a_m|z|^m\end{aligned}$$

and, similarly,

$$\begin{aligned}|A(z)-(1+\beta)B(z)| \leq \beta|z| + \sum_{m=2}^{\infty}[m+m\lambda(m-1)-(1+\beta)(1-\lambda+m\lambda)]a_m|z|^m \\ +k\sum_{m=2}^{\infty}[m+m\lambda(m-1)-(1-\lambda+m\lambda)]a_m|z|^m.\end{aligned}$$

Therefore,

$$\begin{aligned}&|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)| \\ &\geq 2(1-\beta)|z| - \sum_{m=2}^{\infty}[2m+2m\lambda(m-1)-2\beta(1-\lambda+m\lambda)] \\ &\quad -k[2m+2m\lambda(m-1)-2(1-\lambda+m\lambda)]a_m|z|^m \geq 0\end{aligned}$$

or

$$(1-\beta) \geq \sum_{m=2}^{\infty}[m(1+k)+m\lambda(m-1)(1+k)-(1-\lambda+m\lambda)(\beta+k)]a_m$$

which yields

$$(1-\beta) \geq \sum_{m=2}^{\infty}(1-\lambda+m\lambda)(m(1+k)-(k+\beta)]a_m.$$

On the other hand, we must have

$$Re\left(\frac{[zf'(z)+\lambda z^2 f''(z)](1+ke^{i\theta})-ke^{i\theta}[(1-\lambda)f(z)+\lambda zf'(z)]}{(1-\lambda)f(z)+\lambda zf'(z)}\right) \geq \beta.$$

Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, the above inequality reduces to

$$\begin{aligned}Re\Bigg(\Big((1-\beta)-\sum_{m=2}^{\infty}[m-m^2\lambda-\lambda-\beta(1-\lambda+m\lambda)]a_m r^{m-1} \\ -ke^{i\theta}\sum_{m=2}^{\infty}[m+m^2\lambda-\lambda-(1-\lambda+m\lambda)]a_m r^{m-1}\Big) \\ \Big/\Big(1-\sum_{m=2}^{\infty}(1-\lambda+m\lambda)a_m r^{m-1}\Big)\Bigg) \geq 0.\end{aligned}$$

Since $Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to

$$Re\Bigg(\Big((1-\beta)-\sum_{m=2}^{\infty}[m+m^2\lambda-\lambda-\beta(1-\lambda+m\lambda)]a_m r^{m-1}$$

$$-k\sum_{m=2}^{\infty}[m+m^2\lambda-\lambda-(1-\lambda+m\lambda)]a_m r^{m-1}\Big)$$

$$\Big/\Big(1-\sum_{m=2}^{\infty}(1-\lambda+m\lambda)a_m r^{m-1}\Big)\Big)\geq 0.$$

Letting $r\to 1^-$ we get desired conclusion.

Remark. As special cases of Theorem 1, for $\lambda=0$, see [2] and for $k=0$, see [1].

The distortion theorem for the class $\mathcal{U}(k,\beta,\lambda)$ is given next.

Theorem 2. *If $f\in\mathcal{U}(k,\beta,\lambda)$ and $|z|\leq r<1$, then we have the sharp bounds*

$$r-\frac{1-\beta}{(1+\lambda)(2+k-\beta)}r^2\leq|f(z)|\leq r+\frac{1-\beta}{(1+\lambda)(2+k-\beta)}r^2 \tag{2.3}$$

and

$$1-\frac{2(1-\beta)}{(1+\lambda)(2+k-\beta)}r\leq|f'(z)|\leq 1+\frac{2(1-\beta)}{(1+\lambda)(2+k-\beta)}r.$$

Proof. We only prove the right hand side inequality in (2.3) since the other inequalities can be justified using similar arguments. On account of (2.1), we may write

$$\sum_{m=2}^{\infty}(1+\lambda)(2+k-\beta)a_m\leq\sum_{m=2}^{\infty}(1+m\lambda-\lambda)(m(1+k)-(k+\beta))a_m\leq 1-\beta.$$

Hence

$$|f(z)|\leq|z|+|z|^2\sum_{m=2}^{\infty}a_m\leq r+r^2\sum_{m=2}^{\infty}a_m\leq r+\frac{1-\beta}{(1+\lambda)(2+k-\beta)}r^2.$$

The distortion bounds in Theorem 2 are sharp for

$$f(z)=z-\frac{(1-\beta)}{(1+\lambda)(2+k-\beta)}z^2,\ z=\pm r.$$

In the following theorem, we study the properties of extreme points of functions in the family $\mathcal{U}(k,\beta,\lambda)$.

Theorem 3. *Let $f_1(z)=z$ and $f_m(z)=z-\frac{1-\beta}{(1+m\lambda-\lambda)(m(1+k)-(k+\beta))}z^m$ where $\lambda\geq 0$, $0\leq\beta<1$, $k\geq 0$, and $m\geq 2$. Then $f(z)$ is in $\mathcal{U}(k,\beta,\lambda)$ if and only if it can be expressed in the form $f(z)=\sum_{m=1}^{\infty}\gamma_m f_m(z)$ where $\gamma_m\geq 0$ and $\sum_{m=1}^{\infty}\gamma_m=1$.*

Proof. Let $f(z)=\sum_{m=1}^{\infty}\gamma_m f_m(z)$ where $\gamma_m\geq 0$ and $\sum_{m=1}^{\infty}\gamma_m=1$. Letting

$$f(z)=z-\sum_{m=2}^{\infty}\frac{1-\beta}{(1+m\lambda-\lambda)(m(1+k)-(k+\beta))}\gamma_m z^m.$$

we get

$$\sum_{m=2}^{\infty}\left(\frac{(1+m\lambda-\lambda)(m(1+k)-(k+\beta))}{1-\beta}\right)\gamma_m\frac{1-\beta}{(1+m\lambda-\lambda)(m(1+k)-(k+\beta))}$$
$$=\sum_{m=2}\gamma_m=1-\gamma_1\leq 1\quad\text{(by Theorem 1)}.$$

Therefore $f\in\mathcal{U}(k,\beta,\lambda)$. Conversely, suppose that $f\in\mathcal{U}(k,\beta,\lambda)$. Then

$$a_m\leq\frac{1-\beta}{(1+m\lambda-\lambda)(m(1+k)-(k+\beta))},\qquad (m\geq 2).$$

Now, by letting $\gamma_m=\frac{(1+m\lambda-\lambda)(m(1+k)-(k+\beta))}{1-\beta}a_m$ and $\gamma_1=1-\sum_{m=2}^{\infty}\gamma_m$ we conclude the theorem, since $f(z)=\sum_{m=1}^{\infty}\gamma_m f_m=\gamma_1 f_1(z)+\sum_{m=2}^{\infty}\gamma_m f_m(z)$.

Remark. For $\lambda=0$, we obtain the extreme points given earlier in [2].

Finally, we discuss the radius of starlikeness of the functions in $\mathcal{U}(k,\beta,\lambda)$.

Theorem 5. Let the f be in the class $\mathcal{U}(k,\beta,\lambda)$. Then f is starlike of order $\delta(0\leq\delta<1)$ in $|z|<r_2(\beta,\lambda,k,\delta)$, where

$$r_2(\beta,\lambda,k,\delta)=\inf_m\left(\frac{(1-\delta)(1+m\lambda-\lambda)(m(1+k)-(k+\beta))}{(m-\delta)(1-\beta)}\right)^{\frac{1}{m-1}},\quad m\geq 2. \tag{2.4}$$

Proof. It suffices to show that $\left|\frac{zf'(z)}{f(z)}-1\right|\leq 1-\delta$ for $|z|<r_2(\beta,\lambda,k,\delta)$. Note that

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq\frac{\sum_m^{\infty}(m-1)a_m|z|^{m-1}}{1-\sum_m^{\infty}a_m|z|^{m-1}}.$$

Now $\left|\frac{zf'(z)}{f(z)}-1\right|\leq 1-\delta$ if we have the condition

$$\sum_{m=2}^{\infty}\frac{(m-\delta)a_m|z|^{m-1}}{(1-\delta)}\leq 1. \tag{2.5}$$

Considering the coefficient conditions required by Theorem 1, the above inequality (2.5) is true if

$$\frac{m-\delta}{1-\delta}|z|^{m-1}\leq\frac{(1+m\lambda-\lambda)(m(1+k)-(k+\beta))}{(1-\beta)}$$

or if

$$|z|\leq\left\{\frac{(1-\delta)(1+m\lambda-\lambda)(m(1+k)-(k+\beta))}{(m-\delta)(1-\beta)}\right\}^{\frac{1}{m-1}},\quad m\geq 2.$$

This last expression yields the bound required by the above theorem.

References

[1] O. Altintas, *On a subclass of certain starlike functions with negative coefficient*, Math. Japon. **36** (1991), 489-495.

[2] R. Bharati, R. Parvatham and A. Swaminathan, *On subclasses of uniformaly convex functions and correspondding class of starlike functions*, Tamkang J. Math. **28** (1997), 17-32.

[3] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. **56** (1991), 87-92.

[4] A. W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl. **155** (1991), 364-370.

[5] S. Kanas and A. Wisniowska, *Conic regions and k-uniform convexity*, J. Comput. Appl. Math. **105** (1999), 327-336.

[6] S. Kanas and A. Wisniowska, *Conic region and k-uniform convexity*, II, Zeszyty. Nauk. Politech. Rzes. Mat. **22** (1998), 65-78.

[7] S. Kanas and A. Wisniowska, *Conic domains and starlike functions*, Rev. Roumaine Math. Pures Appl. **45** (2000), 647-657.

[8] S. Kanas and H. M. Srivastava, *Linear operators associated with k-uniformly convex functions*, Integral Transform Spec. Funct. **9** (2000), 121-132.

[9] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109-116.

Department of Mathematics, Fergusson College, Pune - 411 004, India.

Mathematical Sciences, Kent State University, 14111 Claridon Troy Road, Burton, Ohio 44021-9500, U.S.A.

E-mail: jay@geauga.kent.edu

Department of Mathematics, Fergusson College, Pune - 411 004, India.

E-mail: Kulkarni_ferg@yahoo.com