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DOUBLE TRIGONOMETRIC SERIES WITH COEFFICIENTS OF BOUNDED VARIATION OF HIGHER ORDER

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Abstract. In this paper the following convergence properties are established for the rectangular partial sums of the double trigonometric series, whose coefficients form a null sequence of bounded variation of order $(p,0)$, $(0,p)$ and (p,p), for some $p \geq 1$: (a) pointwise convergence; (b) uniform convergence; (c) L^r-integrability and L^r-metric convergence for $0 < r < \frac{1}{p}$. Our results extend those of Chen [2, 4, 5] and Móricz [7, 8, 9] and Stanojevic [10].

1. Introduction

We consider the double trigonometric series

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} e^{i(jx+ky)} \tag{1.1}$$

on two-dimensional torus $T^2 = \{(x,y); 0 \leq x, y < 2\pi\}$.

The rectangular partial sums $S_{mn}(f;x,y)$ and the Cesàro means $\sigma_{mn}(x,y)$ of the series (1.1) are defined as

$$S_{mn}(f,x,y) = \sum_{|j|\leq m} \sum_{|k|\leq n} c_{jk} e^{i(jx+ky)},$$

$$\sigma_{mn}(f,x,y) = \frac{1}{(m+1)(n+1)} \sum_{j=0}^{m} \sum_{k=0}^{n} S_{jk}(x,y)$$

where $m,\ n \geq 0$. If $\{c_{jk}\}$ are the Fourier coefficients of some $f \in L^1(T^2)$, then the symbols $S_{mn}(f)$ and $S_{mn}(f,x,y)$ will have the same meaning as $S_{mn}(f)$.

Similarly $\sigma_{mn}(f) = \sigma_{mn}(f,x,y) = \sigma_{mn}$.

Let the coefficients $\{c_{jk}\}$ satisfies the following conditions for some positive integer p:

$$c_{jk} \to 0 \quad \text{as} \quad \max\{|j|,|k|\} \to \infty, \tag{1.2}$$

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$$\lim_{|k|\to\infty} \sum_{j=-\infty}^{\infty} |\Delta_{p0} c_{jk}| = 0, \tag{1.3}$$

$$\lim_{|j|\to\infty} \sum_{k=-\infty}^{\infty} |\Delta_{0p} c_{jk}| = 0, \tag{1.4}$$

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{pp} c_{jk}| < \infty. \tag{1.5}$$

The finite order differences $\Delta_{pp} c_{jk}$ are defined by

$$\begin{aligned} &\Delta_{00} c_{jk} = c_{jk}; \\ &\Delta_{pq} c_{jk} = \Delta_{p-1,q} c_{jk} - \Delta_{p-1,q} C_{\tau(j),k} \quad (p \geq 1), \\ &\Delta_{pq} c_{jk} = \Delta_{p,q-1} c_{jk} - \Delta_{p,q-1} C_{j,\tau(k)} \quad (q \geq 1). \end{aligned}$$

Here the function $\tau(j)$ is defined by $\tau(j) = j+1$ for $j \geq 1$, and $\tau(j) = j-1$ for $j \leq -1$.

We mention that a double induction argument gives

$$\Delta_{pq} c_{jk} = \sum_{s=0}^{p} \sum_{t=0}^{q} (-1)^{s+t} \binom{p}{s} \binom{q}{t} c_{j+s,k+t}.$$

Conditions (1.3)-(1.5) are known as conditions of bounded variation of order $(p,0)$, $(0,p)$ and (p,p) respectively. For $p=1$, conditons (1.3) and (1.4) are excessive, as they can be derived from (1.2) and (1.5). Obviously, conditions (1.3)-(1.5) generalize the concept of monotone sequences.

The pointwise convergence of the series (1.1) is usually defined in Pring-sheim's sense ([11], vol. 2, Ch. 17). This means that we form the rectangular partial sums

$$S_{MN}(x,y) = \sum_{j=-M}^{M} \sum_{k=-N}^{N} c_{jk} e^{i(jx+ky)} \quad (M,\ N \geq 0),$$

and then let both M and N tend to ∞, *independently of one another, and assign the limit* $f(x,y)$ (if exists) to series (1.1) as its sum. For $E \subset T^2$, we say S_{mn} that converges uniformly on E to $f(x,y)$ if $S_{mn}(f) \to f(x,y)$ uniformly on E as $\min(m,n) \to \infty$.

We shall study the convergence of the series (1.1) in $L^r(T^2)$-norm. Thus we agree in the notation defined by

$$\|g\|_r = \left[\int_0^{2\pi} \int_0^{2\pi} |g(x,y)|^r dxdy\right]^{1/r}.$$

In this paper the following convergence properties are established for the rectangular partial sums of the double trigonometric series, whose coefficients form a null sequence of bounded variation of order $(p,0)$, $(0,p)$ and (p,p), for some p, $q \geq 1$:

$$S_{mn}(x,y) \text{ converges pointwise to } f(x,y) \text{ for every } (x,y) \in T^2, \tag{1.6}$$

$$S_{mn}(x,y) \text{ converges uniformly to } f(x,y) \text{ on } T^2, \tag{1.7}$$

$$f \in L^r(T^2) \text{ uniformly, and } \|S_{mn}(f) - f\|_r = o(1) \text{ as } \min(m,n) \to \infty. \tag{1.8}$$

These problems have been investigated by a number of authors [2, 4, 5, 6, 7, 8, 9, 10] for single and higher dimensions. Our goal is to extend the above results from $p = 1$ to general cases for double trigonometric series.

In the sequel we set $\lambda_n = [\lambda n]$ where n is positive integer, $\lambda > 1$ is a real number, and [.] means the greatest integeral part.

2. Lemmas

The following Lemmas will be useful for the proof of our result:

Lemma 2.1. *For $M_1 < M_2$, $N_1 < N_2$, we prove the following Lemma:*

$$w^p w'^p \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} c_{jk} e^{i(jx+ky)}$$

$$\begin{aligned}
= &\sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} \Delta_{pp} c_{jk} e^{i(jx+ky)} \\
&+ \sum_{j=M_1}^{M_2} \sum_{t=0}^{p-1} w'^{p-1-t} \Delta_{pt} c_{j,N_2+1} e^{i(jx+N_2 y)} - \sum_{j=M_1}^{M_2} \sum_{t=0}^{p-1} {w'}^{p-1-t} \Delta_{pt} c_{j,N_1} e^{i(jx+(N_1-1)y)} \\
&+ \sum_{k=N_1}^{N_2} \sum_{s=0}^{p-1} {w'}^{p-1-s} \Delta_{sp} c_{M_2+1,k} e^{i(M_2 x+ky)} - \sum_{k=N_1}^{N_2} \sum_{s=0}^{p-1} w^{p-1-s} \Delta_{sp} c_{M_1,k} e^{i((M_1-1)x+ky)} \\
&+ \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} w^{p-1-s} {w'}^{p-1-t} \Delta_{st} c_{M_2+1,N_2+1} e^{i(M_2 x+N_2 y)} \\
&- \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} w^{p-1-s} {w'}^{p-1-t} \Delta_{st} c_{M_1,N_{2+1}} e^{i((M_1-1)x+N_2 y)} \\
&- \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} w^{p-1-s} {w'}^{p-1-t} \Delta_{st} c_{M_2+1,N_1} e^{i(M_2 x+(N_1-1)y)} \\
&+ \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} w^{p-1-s} {w'}^{p-1-t} \Delta_{st} c_{M_1,N_1} e^{i((M_1-1)x+(N_1-1)y)}
\end{aligned}$$

where $w(x) = w = (1 - e^{-ix})$, $w'(y) = w' = (1 - e^{-iy})$ and

$$|w| = 2\sin\frac{x}{2}, \quad |w| = 2\sin\frac{y}{2} \quad \textit{for} \quad 0 \leq x,\ y < 2\pi.$$

The corresponding result for one dimension case is:

$$w^p \sum_{j=M_1}^{M_2} c_{jk} e^{ijx} = \sum_{j=M_1}^{M_2} \Delta^p c_j e^{ijx} + \sum_{s=0}^{p-1} w^{p-1-s} \Delta^s c_{M_2+1} e^{iM_2 x}$$
$$- \sum_{s=0}^{p-1} w^{p-1-s} \Delta^s c_{M_1} e^{i(M_1-1)x}$$

Proof.

$$w^p w'^p \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} c_{jk} e^{i(jx+ky)}$$
$$= w'^p \sum_{k=N_1}^{N_2} e^{iky} \left[w^p \sum_{j=M_1}^{M_2} c_{jk} e^{ikx} \right]$$
$$= w'^p \sum_{k=N_1}^{N_2} e^{iky} \left[\sum_{j=M_1}^{M_2} \Delta_{p0} c_{jk} e^{ijx} + \sum_{s=0}^{p-1} w^{p-1-s} \Delta_{s0} c_{M_2+1,k} e^{iM_2 x} \right.$$
$$\left. - \sum_{s=0}^{p-1} w^{p-1-s} \Delta_{s0} c_{M_1,k} e^{i(M_1-1)x} \right]$$

Now

$$\sum_{j=M_1}^{M_2} \left[w'^p \sum_{k=N_1}^{N_2} \Delta_{p0} c_{jk} e^{iky} \right] e^{ijx}$$
$$= \sum_{j=M_1}^{M_2} \left[\sum_{k=N_1}^{N_2} \Delta_{pp} c_{jk} e^{iky} + \sum_{t=0}^{p-1} w'^{p-1-t} \Delta_{pt} c_{j,N_2+1} e^{iN_2 y} \right.$$
$$\left. - \sum_{t=0}^{p-1} w^{p-1-t} \Delta_{pt} c_{j,N_1} e^{i(N_1-1)y} \right] e^{ijx}$$
$$= \sum_{j=M_1}^{M_2} \sum_{k=N_1}^{N_2} \Delta_{pp} c_{jk} e^{i(jk+ky)} + \sum_{j=M_1}^{M_2} \sum_{t=0}^{p-1} w'^{p-1-t} \Delta_{pt} c_{j,N_2+1} e^{i(jx+N_2 y)}$$
$$- \sum_{j=M_1}^{M_2} \sum_{s=0}^{P-1} w'^{p-1-s} \Delta_{pt} c_{j,N_1,k} e^{i(jx+(N_1-1)y)}$$

Also

$$\sum_{s=0}^{p-1} w^{p-1-s} \left[w'^p \sum_{k=N_1}^{N_2} \Delta_{s0} c_{M_2+1,k} e^{iky} \right] e^{iM_2 x}$$
$$= \sum_{s=0}^{p-1} w^{p-1-s} \left[\sum_{k=N_1}^{N_2} \Delta_{sp} c_{M_2+1,k} e^{iky} + \sum_{t=0}^{p-1} w'^{p-1-t} \Delta_{st} c_{M_2+1,N_2+1} e^{iN_2 y} \right.$$

$$
\left.-\sum_{t=0}^{p-1} w'^{p-1-t}\Delta_{st}c_{M_2+1,N_1}e^{i(N_1-1)y}\right] e^{iM_2x}
$$

$$
\begin{aligned}
&= \sum_{k=N_1}^{N_2}\sum_{s=0}^{p-1} w^{p-1-s}\Delta_{sp}c_{M_2+1,k}e^{i(M_2x+ky)}\\
&\quad+\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} w^{p-1-s}w'^{p-1-t}\Delta_{st}c_{M_2+1,N_2+1}e^{i(M_2x+N_2y)}\\
&\quad-\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} w^{p-1-s}w'^{p-1-s}\Delta_{st}c_{M_2+1,N_1}e^{i(M_2x+(N_1-1)y)}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\sum_{s=0}^{p-1} w^{p-1-s}\left[w'^{p}\sum_{k=N_1}^{N_2}\Delta_{s0}c_{M_1,k}e^{iky}\right]e^{iM_2x}\\
&=\sum_{s=0}^{p-1} w^{p-1-s}\left[\sum_{k=N_1}^{N_2}\Delta_{sp}c_{M,k}e^{iky}+\sum_{t=0}^{p-1}w'^{p-1-t}\Delta_{st}c_{M_1,N_2+1}e^{iN_2y}\right.\\
&\qquad\qquad\left.-\sum_{t=0}^{p-1}w'^{p-1-t}\Delta_{st}c_{M_1,N_1}e^{i(N_1-1)y}\right]e^{iM_2x}\\
&= \sum_{k=N_1}^{N_2}\sum_{s=0}^{p-1} w^{p-1-s}\Delta_{sp}c_{M_1,k}e^{i((M_1-1)x+ky)}\\
&\quad+\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} w^{p-1-s}w'^{p-1-t}\Delta_{st}c_{M_1,N_2+1}e^{i((M_1-1)x+N_2y)}\\
&\quad-\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} w^{p-1-s}w'^{p-1-s}\Delta_{st}c_{M_1,N_1}e^{i((M_1-1)x+(N_1-1)y)}.
\end{aligned}
$$

Combining all above, we have the required result.

Lemma 2.2. [3] *For m, $n \geq 0$ and $\lambda > 1$, the following representation holds:*

$$
\begin{aligned}
S_{mn}-\sigma_{mn} &= \frac{\lambda_m+1}{\lambda_m-m}\frac{\lambda_n+1}{\lambda_n-n}(\sigma_{\lambda_m,\lambda_n}-\sigma_{\lambda_m,n}-\sigma_{m,\lambda_n}+\sigma_{mn})+\frac{\lambda_m+1}{\lambda_m-m}(\sigma_{\lambda_m,n}-\sigma_{mn})\\
&\quad+\frac{\lambda_n+1}{\lambda_n-n}(\sigma_{m,\lambda_n}-\sigma_{mn})-\Sigma_{10}^{\lambda}(m,n;x,y)-\Sigma_{01}^{\lambda}(m,n;x,y)-\Sigma_{11}^{\lambda}(m,n;x,y)
\end{aligned}
$$

where

$$
\Sigma_{01}^{\lambda}(m,n;x,y)=\sum_{|j|\leq m}\sum_{|k|=n+1}^{\lambda_n}\frac{\lambda_n+1-|k|}{\lambda_n-m}c_{jk}e^{i(jx+ky)}
$$

$$\Sigma_{10}^{\lambda}(m,n;x,y)=\sum_{|j|=m+1}^{\lambda_m}\sum_{|k|\le n}\frac{\lambda_m+1-|j|}{\lambda_m-m}c_{jk}e^{i(jx+ky)}$$

$$\Sigma_{11}^{\lambda}(m,n;x,y)=\sum_{|j|=m+1}^{\lambda_m}\sum_{|k|=n+1}^{\lambda_n}\frac{\lambda_m+1-|j|}{\lambda_m-m}\frac{\lambda_n+1-|k|}{\lambda_n-n}c_{jk}e^{i(jx+ky)}$$

3. Main Results

We will prove the following results:

Theorem 3.1. *Let* $\{c_{jk}\}_{|j|,|k|<\infty}$ *satisfies the conditions* (1.2)-(1.5) *for some* $p\ge 1$. *Then the series* (1.1)
(i) *converges pointwise to some function* $f(x,y)$ *for every* $(x,y)\in T^2$.
(ii) *converges in the* $L^r(T^2)$*-metric to* f *for all* $0<r<1/p$.

Theorem 3.2. (i) *Let* $E\subset T^2$. *Assume that the following conditions are satisfied:*

$$\lim_{\lambda\downarrow 1}\lim_{m,n\to\infty}\sup\left(\sup_{(x,y)\in E}\left|\Sigma_{10}^{\lambda}(m,n;x,y)\right|\right)=0, \tag{3.1}$$

$$\lim_{\lambda\downarrow 1}\lim_{m,n\to\infty}\sup\left(\sup_{(x,y)\in E}\left|\Sigma_{01}^{\lambda}(m,n;x,y)\right|\right)=0, \tag{3.2}$$

If $\sigma_{mn}(x,y)$ *converges uniformly on* E *to* $f(x,y)$, *then so does* S_{mn}.
(ii) *Assume that the following conditions are satisfied for some* $r<1$:

$$\lim_{\lambda\downarrow 1}\lim_{m,n\to\infty}\sup\left(\left\|\Sigma_{10}^{\lambda}(m,n;x,y)\right\|_r\right)=0, \tag{3.3}$$

$$\lim_{\lambda\downarrow 1}\lim_{m,n\to\infty}\sup\left(\left\|\Sigma_{01}^{\lambda}(m,n;x,y)\right\|_r\right)=0. \tag{3.4}$$

If $\|\sigma_{mn}-f\|_r\to 0$ *unrestrictedly, then* $\|S_{mn}-f\|_r\to 0$ *as* $\min(m,n)\to\infty$.
Here the limit superior of a double sequence $\{d_{jk}:-\infty<j,k<\infty\}$ *of extended real numbers is known as*

$$\lim_{m,n\to\infty}\sup d_{mn}=\inf_{m,n\ge 1}(\sup d_{jk})=\lim_{m,n\to\infty}\left(\sup_{j\ge m,k\ge n}d_{jk}\right).$$

Proof of Theorem 3.2. We have

$$\Sigma_{11}^{\lambda}(m,n;x,y)=\frac{1}{\lambda_m-m}\sum_{u=m+1}^{\lambda_m}\left(\Sigma_{01}^{\lambda}(u,n;x,y)-\Sigma_{01}^{\lambda}(m,n;x,y)\right)$$

$$=\frac{1}{\lambda_n-n}\sum_{v=n+1}^{\lambda_n}\left(\Sigma_{10}^{\lambda}(m,v;x,y)-\Sigma_{01}^{\lambda}(m,n;x,y)\right)$$

This implies

$$\left|\Sigma_{11}^{\lambda}(m,n;x,y)\right| \le 2 \sup_{m\le u\le\lambda_m} \left(\left|\Sigma_{01}^{\lambda}(u,n;x,y)\right|\right) 2 \sup_{n\le u\le\lambda_n} \left(\left|\Sigma_{10}^{\lambda}(m,v;x,y)\right|\right) \tag{3.5}$$

Using the above relation, we find that (3.1) implies that

$$\lim_{\lambda\downarrow 1} \lim_{m,n\to\infty} \sup \left(\sup_{(x,y)\in E} \left|\Sigma_{11}^{\lambda}(m,n;x,y)\right| \right) = 0, \tag{3.6}$$

Assume that $\sigma_{mn}(x,y)$ converges uniformly on E to $f(x,y)$. Then by Lemma 2.2, we get

$$\begin{aligned}
&\lim_{m,n\to\infty} \sup \left(\left| \sup_{(x,y)\in E} S_{mn}(x,y) - \sigma_{mn}(x,y) \right| \right) \\
&\le \lim_{m,n\to\infty} \sup \left(\sup_{(x,y)\in E} \left|\Sigma_{10}^{\lambda}(m,n;x,y)\right| \right) + \lim_{m,n\to\infty} \sup \left(\sup_{(x,y)\in E} \left|\Sigma_{01}^{\lambda}(m,n;x,y)\right| \right) \\
&\quad + \lim_{m,n\to\infty} \sup \left(\sup_{(x,y)\in E} \left|\Sigma_{11}^{\lambda}(m,n;x,y)\right| \right)
\end{aligned}$$

After taking $\lambda \downarrow 1$ the first part of Theorem 3.2 follows from (3.1)-(3.2) and (3.6). For (ii), by (3.5) we have

$$\begin{aligned}
\left\|\Sigma_{11}^{\lambda}(m,n;x,y)\right\|_r &= \frac{1}{\lambda_m - m} \sum_{u=m+1}^{\lambda_m} \left(\left\|\Sigma_{01}^{\lambda}(u,n;x,y)\right\|_r + \left\|\Sigma_{01}^{\lambda}(m,n;x,y)\right\|_r \right) \\
&\le 2 \left(\sup_{m\le u\le\lambda_m} \left(\left\|\Sigma_{01}^{\lambda}(u,n;x,y)\right\|_r \right) \right)
\end{aligned}$$

Thus, (3.4) implies

$$\lim_{\lambda\downarrow 1} \lim_{m,n\to\infty} \sup \left\|\Sigma_{11}^{\lambda}(m,n;x,y)\right\|_r = 0.$$

Therefore the result of Theorem 3.2 follows.

The following result follows from Theorem 3.2.

Theorem 3.3. *Assume that conditions* (1.2)-(1.4) *are satisfied for some* $p \ge 1$.

$$\lim_{\lambda\downarrow 1} \lim_{n\to\infty} \sup \sum_{j=-\infty}^{\infty} \sum_{|k|=n+1}^{\lambda_n} \frac{\lambda_n + 1 - |k|}{\lambda_n - n} |\Delta_{pp} c_{jk}| = 0, \tag{3.7}$$

$$\lim_{\lambda\downarrow 1} \lim_{m\to\infty} \sup \sum_{|j|=m+1}^{\lambda_m} \sum_{k=-\infty}^{\infty} \frac{\lambda_m + 1 - |j|}{\lambda_m - m} |\Delta_{pp} c_{jk}| = 0, \tag{3.8}$$

Then the following statements are true.

(i) *If* $\sigma_{mn}(x,y)$ *converges uniformly on* E *to* $f(x,y)$ *then so does* S_{mn}.
(ii) *If* $\|\sigma_{mn}-f\|_r \to 0$ *unrestrictedly for some* r *with* $0<r<1/p$, *then*

$$\|S_{mn}-f\|_r \to 0 \quad as \quad \min(m,n)\to\infty.$$

Obviously, condition (1.5) *implies any of the conditions* (3.7)-(3.8). *These conditions have appeared in many places and were originally taken into consideration in the development of the point-wise convergence of single and double trigonometric series* [2, 4, 5].

Proof of Theorem 3.1. Setting $M_1=-m$, $M_2=m$, $N_1=-n$ and $N_2=n$ in Lemma 2.1, we have

$$\begin{aligned}
S_{mn} &= \sum_{|j|\le m}\sum_{|k|\le n} c_{jk}e^{i(jx+ky)}\\
&= \frac{1}{w^p w'^p}\Bigg[\sum_{|j|\le m}\sum_{|k|\le n}\Delta_{pp}c_{jk}e^{i(jx+ky)} + \sum_{|j|\le m}\sum_{t=0}^{p-1} {w'}^{p-1-t}\Delta_{pt}c_{j,\tau(n)}e^{i(jx+ny)}\\
&\quad - \sum_{|j|\le m}\sum_{t=0}^{p-1} {w'}^{p-1-t}\Delta_{pt}c_{j,-n}e^{i(jx+(-n-1)y)} + \sum_{|k|\le n}\sum_{s=0}^{p-1} w^{p-1-s}\Delta_{sp}c_{\tau(m),k}e^{i(mx+ky)}\\
&\quad - \sum_{|k|\le n}\sum_{s=0}^{p-1} w^{p-1-s}\Delta_{sp}c_{-m,k}e^{i((-m-1)x+ky)}\\
&\quad + \sum_{s=0}^{p-1}\sum_{t=0}^{p-1} w^{p-1-s}{w'}^{p-1-t}\Delta_{st}c_{\tau(m),\tau(n)}e^{i(mx+ny)}\\
&\quad \sum_{s=0}^{p-1}\sum_{t=0}^{p-1} w^{p-1-s}{w'}^{p-1-t}\Delta_{st}c_{-m,\tau(n)}e^{i((-m-1)x+ny)}\\
&\quad + \sum_{s=0}^{p-1}\sum_{t=0}^{p-1} w^{p-1-s}{w'}^{p-1-t}\Delta_{st}c_{\tau(m),-n}e^{i(mx+(-n-1)y)}\\
&\quad \sum_{s=0}^{p-1}\sum_{t=0}^{p-1} w^{p-1-s}{w'}^{p-1-t}\Delta_{st}c_{-m,-n}e^{i((-m-1)x+(-n-1)y)}\Bigg]
\end{aligned}$$

Now

$$\begin{aligned}
&\left|\sum_{|j|\le m}\sum_{t=0}^{p-1} {w'}^{p-1-t}\Delta_{pt}c_{j,\tau(n)}e^{i(jx+ny)} + \sum_{|j|\le m}\sum_{t=0}^{p-1} {w'}^{p-1-t}\Delta_{pt}c_{j,-n}e^{i(jx+(-n-1)y)}\right|\\
&\le 2^{p-1}\sum_{t=0}^{p-1}\sum_{v=0}^{t}|\Delta_{pt}c_{j,\tau(k)}|
\end{aligned}$$

$$= 2^{p-1}\sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\sum_{|j|\le m}\sum_{|k|=n+v+1}|\Delta_{p0}c_{jk}|$$

$$\le C_p \sup_{n<|k|\le n+p}\sum_{|j|\le m}|\Delta_{p0}c_{jk}|$$

Similarly, we have

$$\left|\sum_{|k|\le n}\sum_{s=0}^{p-1}w^{p-1-t}\Delta_{sp}c_{\tau(m),k}e^{i(mx+ky)} + \sum_{|k|\le n}\sum_{s=0}^{p-1}w^{p-1-s}\Delta_{sp}c_{-m,k}e^{i((-m+1)x+ky)}\right|$$

$$\le 2^{p-1}\sum_{s=0}^{p-1}\sum_{|j|=m}^{t}\sum_{|k|\le n}|\Delta_{sp}c_{\tau(j),k}|$$

$$\le 2^{p-1}\sum_{s=0}^{p-1}\sum_{u=0}^{s}\binom{s}{u}\sum_{|j|=m+u+1}\sum_{|k|\le n}|\Delta_{0p}c_{jk}|$$

$$\le C_p \sup_{m<|j|\le m+p}\sum_{|k|\le n}|\Delta_{p0}c_{jk}|$$

and

$$\Bigg|\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}w^{p-1-s}{w'}^{p-1-t}\Delta_{st}c_{\tau(m),\tau(n)}e^{i(mx+ny)}$$
$$-\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}w^{p-1-s}{w'}^{p-1-t}\Delta_{st}c_{-m,\tau(n)}e^{i((-m-1)x+ny)}$$
$$-\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}w^{p-1-s}{w'}^{p-1-t}\Delta_{st}c_{\tau(m),-n}e^{i((mx+)(-n-1)y)}$$
$$+\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}w^{p-1-s}{w'}^{p-1-t}\Delta_{st}c_{-m,-n}e^{i((-m-1)+(-n-1)y)}\Bigg|$$

$$\le 4^{p-1}\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\sum_{u=0}^{s}\sum_{v=0}^{t}\binom{s}{u}\binom{t}{v}\sum_{|j|=m+u+1}\sum_{|k|=n+v+1}|\Delta_{00}c_{jk}|$$

$$\le C_p \sup_{|j|>m,|k|>n}|C_{jk}|$$

where C_p is an absolute constant not necessarily the same at each occurrence.

Making use of (1.2)-(1.5), we can see that each term on the right-hand side tends to zero as $\min(|m|,|n|)\to\infty$. Thus, the sum $f(x,y)$ of the series (1.1) exists for all $0<x$, $y\le 2\pi$.

For the proof of part (ii),

Let R_{mn} consist of all (j,k) with $|j|>m$ or $|k|>n$.

$$\left(\int_0^{2\pi}\int_0^{2\pi}|f(x,y)-S_{mn}(x,y)|^r dxdy\right)^{1/r}$$

$$= \frac{1}{w^p w'^p}\left[\sum_{R_{mn}}|\Delta_{pp}c_{jk}|\right] + 2^{p-1}\left(\sum_{|j|\le m}\sum_{t=0}^{p-1}|\Delta_{pt}c_{j,\tau(n)}| + |\Delta_{pt}c_{j,-n}|\right)$$
$$+2^{p-1}\left(\sum_{|k|\le n}\sum_{s=0}^{p-1}|\Delta_{sp}c_{\tau(m),k}| + |\Delta_{sp}c_{-m,k}|\right)$$
$$+4^{p-1}\left(\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}(|\Delta_{st}c_{\tau(m),\tau(n)}| + |\Delta_{st}c_{-m,\tau(n)}| + |\Delta_{st}c_{\tau(m),-n}| + |\Delta_{st}c_{-m,-n}|\right)$$

Since for $pr < 1$

$$\int_0^{2\pi}\int_0^{2\pi}\frac{1}{|w(x)w'(y)|^{pr}}dxdy \le K, \quad \text{where } K \text{ is an absolute constant.}$$

Therefore

$$\left(\int_0^{2\pi}\int_0^{2\pi}|f(x,y) - S_{mn}(x,y)|^r dxdy\right)^{1/r}$$
$$\le K\left(\sum_{R_{mn}}|\Delta_{pq}c_{jk}|\right) + C_p\left(\sup_{n<|k|\le m+p}\sum_{|j|\le m}|\Delta_{p0}c_{jk}|\right)$$
$$+C_p\left(\sup_{m<|j|\le m+p}\sum_{|k|\le n}|\Delta_{0p}c_{jk}|\right) + C_p\left(\sup_{|j|>m,|k|>n}|c_{jk}|\right)$$
$$\to 0 \quad \text{as} \quad \min(m,n)\to\infty \quad \text{by (1.2)-(1.5).}$$

This concludes the proof of Theorem 3.1.

Proof of Theorem 3.3. Using summation by parts, we have

$$\Sigma_{10}^{\lambda}(m,n;x,y) = \sum_{|j|=m+1}^{\lambda_m}\sum_{|k|\le n}\frac{\lambda_m+1-|j|}{\lambda_m - m}c_{jk}e^{i(jx+ky)}$$
$$= \frac{1}{w^p w'^p}\left[\sum_{|j|=m+1}^{\lambda_m}\sum_{|k|\le n}\frac{\lambda_m+1-|j|}{\lambda_m-m}\Delta_{pp}c_{jk}e^{i(jx+ky)}\right.$$
$$+\frac{1}{\lambda_m - m}\sum_{|j|=m+1}^{\lambda_m}\sum_{|k|\le n}\sum_{s=0}^{p-1}w^{p-1-s}\Delta_{sp}c_{\tau(j),k}e^{i(jx+ky)}$$
$$-\sum_{|j|=m}\sum_{|k|\le n}\sum_{s=0}^{p-1}w^{p-1-s}\Delta_{sp}c_{\tau(j),k}e^{i(jx+ky)}$$
$$-\sum_{|j|=m+1}^{\lambda_m}\sum_{t=0}^{p-1}{w'}^{p-1-t}\frac{\lambda_m+1-|j|}{\lambda_m-m}\Delta_{pt}c_{j,\tau(n)}e^{i(jx+ny)}$$

$$
\begin{aligned}
&- \sum_{|j|=m+1}^{\lambda_m} \sum_{t=0}^{p-1} {w'}^{p-1-t} \frac{\lambda_m + 1 - |j|}{\lambda_m - m} \Delta_{pt} c_{j,-n} e^{i(jx+(-n-1)y)} \\
&- \sum_{|j|=m} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} w^{p-1-s} {w'}^{p-1-t} \Delta_{st} c_{\tau(j),\tau(n)} e^{i(jx+ny)} \\
&+ \sum_{|j|=m} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} w^{p-1-s} {w'}^{p-1-t} \Delta_{st} c_{\tau(j),-n} e^{i(jx+(-n-1)y)} \\
&- \frac{1}{\lambda_m - m} \sum_{|j|=m+1} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} w^{p-1-s} {w'}^{p-1-t} \Delta_{st} c_{\tau(j),\tau(n)} e^{i(jx+ny)} \\
&- \frac{1}{\lambda_m - m} \sum_{|j|=m+1} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} w^{p-1-s} {w'}^{p-1-t} \Delta_{st} c_{\tau(j),-n} e^{i(jx+(-n-1)y)} \Bigg] \\
= {} & \frac{1}{w^p {w'}^p} [I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9].
\end{aligned}
$$

Now

$$
\begin{aligned}
|I_1| &= \left| \sum_{|j|=m+1}^{\lambda_m} \sum_{|k|\le n} \frac{\lambda_m + 1 - |j|}{\lambda_m - m} \Delta_{pp} c_{jk} e^{i(jx+ky)} \right| \\
&\le \sum_{|j|=m+1}^{\lambda_m} \sum_{|k|\le n} \frac{\lambda_m + 1 - |j|}{\lambda_m - m} |\Delta_{pp} c_{jk}| \\
|I_2| &= \left| \frac{1}{\lambda_m - m} \sum_{|j|=m+1}^{\lambda_m} \sum_{|k|\le n} \sum_{s=0}^{p-1} w^{p-1-s} \Delta_{sp} c_{\tau(j),k} e^{i(jx+ky)} \right| \\
&\le 2^{p-1} \sup_{m<|j|\le \lambda_m} \sum_{s=0}^{p-1} \sum_{|k|\le n} |\Delta_{sp} c_{\tau(j),k}| \\
&\le 2^{p-1} \sup_{m<|j|\le \lambda_m} \sum_{s=0}^{p-1} \sum_{u=0}^{s} \binom{s}{u} \sum_{|j|=m+u+1} \sum_{|k|\le n} |\Delta_{0p} c_{jk}| \\
&\le C_p \left(\sup_{m<|j|\le \lambda_m + p} \sum_{|k|\le n} |\Delta_{0p} c_{jk}| \right).
\end{aligned}
$$

Similarly

$$
|I_3| \le C_p \left(\sup_{m<|j|\le m+p} \sum_{|k|\le n} |\Delta_{0p} c_{jk}| \right)
$$

$$\begin{aligned}|I_4+I_5| &= \Bigg| \sum_{|j|=m+1}^{\lambda_m} \sum_{t=0}^{p-1} {w'}^{p-1-t} \frac{\lambda_m+1-|j|}{\lambda_m-m} \Delta_{pt} c_{j,\tau(n)} e^{i(jx+ny)} \\ &\quad - \sum_{|j|=m+1}^{\lambda_m} \sum_{t=0}^{p-1} {w'}^{p-1-t} \frac{\lambda_m+1-|j|}{\lambda_m-m} \Delta_{pt} c_{j,-n} e^{i(jx+(-n-1)y)} \Bigg| \\ &\le 2^{p-1} \sum_{|j|=m+1}^{\lambda_m} \sum_{|k|=n} \sum_{t=0}^{p-1} \frac{\lambda_m+1-|j|}{\lambda_m-m} \Delta_{pt} c_{j,\tau(k)} \\ &\le C_p \sum_{t=0}^{p-1} \sum_{v=0}^{t} \binom{t}{v} \sum_{|j|=m+1}^{\lambda_m} \sum_{|k|=n+v+1} |\Delta_{p0} c_{jk}| \\ &\le C_p \left(\sup_{n<|k|\le n+p} \sum_{|j|=m+1}^{\lambda_m} |\Delta_{p0} c_{jk}| \right)\end{aligned}$$

and

$$\begin{aligned}|I_4+I_5| &= \sum_{|j|=m} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} w^{p-1-s} {w'}^{p-1-t} \Delta_{st} c_{\tau(j),\tau(n)} e^{i(jx+ny)} \\ &\quad - \sum_{|j|=m} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} w^{p-1-s} {w'}^{p-1-t} \Delta_{st} c_{\tau(j),-n} e^{i(jx+(-n-1)y)} \\ &\le 4^{p-1} \sup_{|j|=m} \sum_{|k|=n} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} |\Delta_{st} c_{\tau(j),\tau(k)}| \\ &\le C_p \left(\sup_{|j|\ge m, |k|\ge n} |c_{jk}| \right).\end{aligned}$$

Similarly

$$|I_8+I_9| \le C_p \left(\sup_{|j|\ge m, |k|\ge n} |c_{jk}| \right).$$

Combining these with (1.2)-(1.4) and (3.8), we get (3.1).
Similarly (1.2)-(1.4) and (3.7), results in (3.2).
Thus, (i) follows from (i) of Theorem 3.2.

For proof of (ii)
Assume that $\|\sigma_{mn} - f\|_r \to 0$ unrestrictedly for some r with $0 < r < \frac{1}{p}$, we have

$$\|S_{mn} - f\|_r^r \le \|S_{mn} - \sigma_{mn}\|_r^r + \|\sigma_{mn} - f\|_r^r$$

So it is sufficient to show that

$$\|S_{mn} - \sigma_{mn}\|_r^r \to 0 \quad \text{as} \quad \min(m,n) \to \infty.$$

By Lemma 2.2, we have

$$\begin{aligned}\|S_{mn}-\sigma_{mn}\|_r^r \le &\left(\frac{\lambda_m+1}{\lambda_m-m}\right)^r\left(\frac{\lambda_n+1}{\lambda_n-n}\right)^r\|\sigma_{\lambda_m,\lambda_n}-\sigma_{\lambda_m,n}-\sigma_{m,\lambda_n}+\sigma_{mn}\|_r^r\\ &+\left(\frac{\lambda_m+1}{\lambda_m-m}\right)^r\|\sigma_{\lambda_m,n}-\sigma_{mn}\|_r^r+\left(\frac{\lambda_n+1}{\lambda_n-n}\right)^r\|\sigma_{m,\lambda_n}-\sigma_{mn}\|_r^r\\ &+\left\|\Sigma_{10}^{\lambda}(m,n;x,y)\right\|_r^r+\left\|\Sigma_{01}^{\lambda}(m,n;x,y)\right\|_r^r+\left\|\Sigma_{11}^{\lambda}(m,n;x,y)\right\|_r^r\end{aligned}$$

By hypothesis the first three terms of the above inequality tend to zero as $\min(m,n)\to\infty$. We have

$$\begin{aligned}\left\|\Sigma_{10}^{\lambda}(m,n;x,y)\right\|_r^r \le &\left(\sum_{|j|=m+1}^{\lambda_m}\sum_{|k|\le n}\frac{\lambda_m+1-|j|}{\lambda_m-m}|\Delta_{pp}c_{jk}|\right)^r\\ &+C_p\left(\sup_{n<|k|\le n+p}\sum_{|j|=m+1}^{\lambda_m}|\Delta_{p0}c_{jk}|\right)^r\\ &+C_p\left(\sup_{m<|j|\le m+p}\sum_{|k|\le n}|\Delta_{0p}c_{jk}|\right)^r\\ &+C_p\left(\sup_{m<|j|\le \lambda_m+p}\sum_{|k|\le n}|\Delta_{0p}c_{j,k}|\right)^r+2C_p\left(\sup_{|j|\ge m,|k|\ge n}|c_{jk}|\right)^r.\end{aligned}$$

By (1.2)-(1.4) and (3.8), we conclude that

$$\lim_{\lambda\downarrow 1}\lim_{m,n\to\infty}\sup\left(\left\|\Sigma_{10}^{\lambda}(m,n;x,y)\right|_r\right)$$

Similarly conditions (1.2)-(1.4) and (3.7),

$$\lim_{\lambda\downarrow 1}\lim_{m,n\to\infty}\sup\left(\left\|\Sigma_{01}^{\lambda}(m,n;x,y)\right\|_r\right)$$

By (1.2)-(1.4) and (3.5), we infer that

$$\lim_{\lambda\downarrow 1}\lim_{m,n\to\infty}\sup\left(\left\|\Sigma_{11}^{\lambda}(m,n;x,y)\right\|_r\right)$$

Therefore

$$\|S_{mn}-\sigma_{mn}\|_r^r\to 0 \quad \text{as} \quad \min(m,n)\to\infty.$$

Hence we have the desired result.

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