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THE MODULUS OF OPERATORS ON GROUP ALGEBRAS

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Abstract. Let G be a locally compact group. In this paper, we study the modulus of right multipliers on second dual of group algebras and modulus of operators on $L^\infty(G)$ which commute with convolutions.

1. Preliminaries and Notations

Let G be a locally compact group. For $f \in L^\infty(G)$ and $\mu \in L^1(G)$, let the functional $f\mu \in L^\infty(G)$ be defined by $\langle f\mu, \nu\rangle = \langle f, \mu * \nu\rangle$ where $\nu \in L^1(G)$. Also for $F \in L^1(G)^{**}$, let $Ff \in L^\infty(G)$ be defined by $\langle Ff, \mu\rangle = \langle F, f\mu\rangle$. Finally for F, $G \in L^1(G)^{**}$, let $FG \in L^1(G)^{**}$ be defined by $\langle FG, f\rangle = \langle F, Gf\rangle$. We know that $L^1(G)^{**}$ with the first Arens product defined as above is a Banach algebra. Also, we can define the first Arens product on $LUC(G)^*$ by symmetry. Of course, it is well known that $LUC(G) = L^\infty(G)L^1(G)$ and $RUC(G) = L^1(G)L^\infty(G)$. If $\pi : L^1(G)^{**} \to LUC(G)^*$ is the adjoint of embedding of $LUC(G)$ in $L^\infty(G)$, then for F, $G \in L^1(G)^{**}$ and $f \in L^\infty(G)$, we have $FG = F\pi(G)$ and $Ff = \pi(F)f$ [6].

For a Banach lattice X and an operator T on X, the modulus $|T|$ of T is defined by $|T|(x) = \sup\{|T(y)|; |y| \le x\}$ for all $x \ge 0$, provided that the supremum exists ([1], [3]). Most of our notation in this paper is taken from ([3], [6]).

We say that a bounded linear map $T : L^\infty(G) \to L^\infty(G)$ commutes with convolutions if $T(f\mu) = T(f)\mu$, for $f \in L^\infty(G)$ and $\mu \in L^1(G)$. Lau and Pym in [6] have studied the operators on $L^\infty(G)$ which commute with convolutions. Ghahramani and Lau studied the modulus of left multipliers on $L^1(G)^{**}$ [3]. We prove, among other things, that if $n = \Gamma_E(\mu) =$ weak*-limit $e_\alpha * \mu$ [2] where $\mu \in M(G)$ and E is a weak*-limit of (e_α) ((e_α) is a bounded approximate identity in $L^1(G)$), then $|\rho_n|(\nu) = \nu|n|$ for all $\nu \in L^1(G)$ (see below). Moreover, we know that if T is an operator on $L^\infty(G)$ which commute with convolutions, then there exists $n \in L^1(G)^{**}$ (or $n \in LUC(G)^*$) such that $T = T_n$ where T_n is defined by $T_n(f) = nf$ for all $f \in L^\infty(G)$ [6]. We show that if $\mu \in M(G)$ and E is a weak*-limit of a bounded approximate identity in $L^1(G)$, then $|T_n| = T_{|n|}$ where $n = \Gamma_E(\mu)$.

Finally, we recall that for $\mu \in M(G)$ the functional $\mu : LUC(G) \to \mathbb{C}$ given by $\langle \mu, f\nu\rangle = \langle f, \nu * \mu\rangle$ is a member of $LUC(G)^*$.

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2. The Modulus of Operators

Let H be a topologically left invariant subspace of $L^\infty(G)$, i.e. $f\mu \in H$ for all $f \in H$ and $\mu \in L^1(G)$. We define $M(L^\infty(G), H) = \{T, T : L^\infty(G) \to H$ is a bounded linear map and $T(f\mu) = T(f)\mu$ for $f \in L^\infty(G)$, $\mu \in L^1(G)\}$. It is known that $M(L^\infty(G), H)$ can be identified with a subspace of $LUC(G)^*$ [6]. Lau and Pym have characterized $M(L^\infty(G), LUC(G))$ and $M(L^\infty(G), C(G))$ [6]. For the subspace $H = \{f \in L^\infty(G) :$ $x \to \delta_x f$ is weak continuous and for all $F \in L^1(G)^{**}$ and $\mu \in L^1(G)$, $\int \langle F, \delta_x f\rangle d\mu(x) = \langle F\mu, f\rangle\}$ of $L^\infty(G)$, we characterize $M(L^\infty(G), H)$.

We recall that the map π is the identity on $L^1(G)$, so $\pi(L^1(G)L^1(G)^{**}) = L^1(G)$ $L^1(G)^{**}$. Indeed, the map π is an isometric isomorphism of $L^1(G)L^1(G)^{**}$ onto $\pi(L^1(G)$ $L^1(G)^{**})$. Let τ be a topology on $LUC(G)^*$ such that $n_\alpha \to n$ in the τ-topology if and only if for all $F \in L^1(G)^{**}$ and $f \in L^\infty(G)$, $\langle F, \delta_x n_\alpha f\rangle \to \langle F, \delta_x n f\rangle$ in the uniform topology on compacta. In the following Theorem, we will show that $cl(L^1(G)L^1(G)^{**}) = M(L^\infty(G), H)$ where the closure is taken in the τ-topology.

Theorem 2.1. *Let H and τ be given as above. Then*

$$cl(L^1(G)L^1(G)^{**}) = M(L^\infty(G), H).$$

Proof. It is easy to see that $L^1(G) \subseteq M(L^\infty(G), H)$, so by ([6], Lemma 3.1),

$$L^1(G)L^1(G)^{**} \subseteq M(L^\infty(G), H).$$

We prove that $M(L^\infty(G), H)$ is τ-closed. Let (n_α) be a net in $M(L^\infty(G), H)$ such that $n_\alpha \to n$ ($n \in LUC(G)^*$) in the τ-topology. Now if $f \in L^\infty(G)$, $F \in L^1(G)^{**}$ and $x_0 \in G$, then for $\varepsilon > 0$, there is a relatively compact neighbourhood U of x_0 and a α_0 such that for all $x \in U$, $|\langle F, \delta_x n_{\alpha_0} f\rangle - \langle F, \delta_x n f\rangle| < \varepsilon/3$. Hence, there is a neighbourhood $V \subseteq U$ containing x_0 such that for all $x \in V$, $|\langle F, \delta_x n_{\alpha_0} f\rangle - \langle F, \delta_{x_0} n_{\alpha_0} f\rangle| \leq \varepsilon/3$. Consequently, for $x \in V$ we have

$$\begin{aligned}|\langle F, \delta_x n f\rangle - \langle F, \delta_{x_0} n f\rangle| &\leq |\langle F, \delta_x n f\rangle - \langle F, \delta_x n_{\alpha_0} f\rangle| + |\langle F, \delta_x n_{\alpha_0} f\rangle - \langle F, \delta_{x_0} n_{\alpha_0} f\rangle| \\ &\quad + |\langle F, \delta_{x_0} n_{\alpha_0} f\rangle - \langle F, \delta_{x_0} n f\rangle| < \varepsilon.\end{aligned}$$

On the other hand, for any α and $\mu \in L^1(G)$, $\int \langle F, \delta_x n_\alpha f\rangle d\mu(x) = \langle F\mu, n_\alpha f\rangle$. Since $n_\alpha \to n$ in the τ-topology, hence $\int \langle F, \delta_x n f\rangle d\mu(x) = \langle F\mu, n f\rangle$. It follows that $nf \in H$. So $cl(L^1(G)L^1(G)^{**}) \subseteq M(L^\infty(G), H)$.

To prove the reverse inclusion, let $n \in M(L^\infty(G), H)$ and U be a compact neighbourhood of e in G. For a bounded approximate identity (e_α) in $L^1(G)$ with $\operatorname{supp} e_\alpha \subseteq U$, we will prove that $e_\alpha n \to n$ in the τ-topology. If $F \in L^1(G)^{**}$, $f \in L^\infty(G)$ and K is a compact subset of G, then we can take $\psi \in C_0(G)$ such that $\psi(KU) = 1$. For $\varphi(x) = \langle F, \delta_x n f\rangle$ and $\varphi_\alpha(x) = \langle F, \delta_x * e_\alpha n f\rangle$, we have

$$\begin{aligned}\langle \varphi, \delta_x * e_\alpha\rangle &= \int \varphi(t) d\delta_x * e_\alpha(t) = \int \langle F, \delta_t n f\rangle d\delta_x * e_\alpha(t) \\ &= \langle F, \delta_x * e_\alpha n f\rangle = \varphi_\alpha(x).\end{aligned}$$

But for $x \in K$, we have $\langle \varphi\psi, \delta_x * e_\alpha \rangle = \langle \varphi, \delta_x * e_\alpha \rangle$. On the other hand, $\varphi\psi$ is of the form $\nu g = \varphi\psi$ for some $\nu \in L^1(G)$ and $g \in L^\infty(G)$. So for $x \in K$ we have

$$\varphi_\alpha(x) = \langle \varphi, \delta_x * e_\alpha \rangle = \langle \varphi\psi, \delta_x * e_\alpha \rangle = \langle \nu g, \delta_x * e_\alpha \rangle.$$

Sicne (e_α) is an approximate identity, hence $\varphi_\alpha \to \varphi$ uniformly on K. Consequently $cl(L^1(G)L^1(G)^{**}) = M(L^\infty(G), H)$.

By above Theorem, the set of all operators $T : L^\infty(G) \to H$ which commute with convolutions is identified with $cl(L^1(G)L^1(G)^{**})$ (the closure is taken in the τ-topology). By [6] we know that $T : L^\infty(G) \to L^\infty(G)$ commute with convolutions if and only if for some $n \in LUC(G)^*$, $T = T_n$ where $T_n(f) = nf$ for $f \in L^\infty(G)$. In the following Proposition, we show that if $\mu \in M(G)$ and E is a weak*-limit of a bounded approximate identity in $L^1(G)$, then for $n = \Gamma_E(\mu)$ we have $|T_n| = T_{|n|}$. Of course, we recall that for $1 \le p \le \infty$, the space $L^p(G)$ is a complete Banach lattice with positive cone

$$C = \{f \in L^p(G) : f \ge 0 \quad \text{almost every where}\}.$$

Moreover for $\mu \in M(G)$, we take ρ_μ as a right multiplier on $L^1(G)$, i.e. $\rho_\mu(\nu) = \nu * \mu$ for all $\nu \in L^1(G)$.

Proposition 2.2. *The following statements hold:*

(1) *For $\mu \in M(G)$, $|\rho^*_\mu| = \rho_{|\mu|}{}^*$.*
(2) *If $n = \Gamma_E(\mu)$, then $|T_n| = T_{|n|}$.*
(3) *If $n \in LUC(G)^\perp$ and $n \neq 0$, then $|T_n| \neq T_{|n|}$ where $LUC(G)^\perp = \{F \in L^1(G)^{**}; \langle F, f\rangle = 0,\ f \in LUC(G)\}$.*

Proof. Let $\mu \in M(G)$ and $\operatorname{supp}\mu$ be compact. If Δ is a modular function of G, we define $\hat{\mu} \in M(G)$ by $\langle \hat{\mu}, f\rangle = \langle \mu, (f\Delta)^\star\rangle$, where $f \in C_0(G)$, and for $f \in L^\infty(G)$, $f^\star(x) = f(x^{-1})$ $(x \in G)$. Now for $f \in C_0(G)^+$, $x \in G$, we have

$$\begin{aligned}
\langle |\mu|, L_x f\rangle &= \sup\{|\langle \mu, h\rangle|; |h| \le L_x f\} \\
&= \sup\{|\langle \mu, (h\Delta)^\star\rangle|; |(h\Delta)^\star| \le L_x f\} \\
&= \sup\{|\langle \mu, (h\Delta)^\star\rangle|; |h\Delta| \le (L_x f)^\star\} \\
&= \sup\{|\langle \hat{\mu}, h\rangle|; |h| \le (L_x f\Delta)^\star\} \\
&= \langle |\hat{\mu}|, (L_x f\Delta)^\star\rangle.
\end{aligned}$$

Consequently,

$$\int f(xy)d|\mu|(y) = \langle |\mu|, L_x y\rangle = \langle |\hat{\mu}|, (L_x f\Delta)^\star\rangle = \int f(xy^{-1})\Delta(y^{-1})d|\hat{\mu}|(y).$$

Hence for all $\nu \in L^1(G)$, we have $\langle |\mu| f, \nu\rangle = \langle f * |\hat{\mu}|, \nu\rangle$, i.e. $|\mu| f = f * |\hat{\mu}|$.

It is easy to see that for all $f \in L^\infty(G)^+$ with compact support $\langle |\mu| f, \nu\rangle = \langle f * |\hat{\mu}|, \nu\rangle$ $(\nu \in L^1(G))$. Also, it is obvious that for $g \in L^\infty(G)$ with compact support $\mu g = g * \hat{\mu}$.

Now by an argument similar to the proof in ([3], Theorem 3.5), we have $|\rho_\mu^*| = \rho_{|\mu|^*}$ for all $\mu \in M(G)$.

2) For $f \in L^\infty(G)^+$,

$$|T_n|(f) = \sup\{|ng|; g \in L^\infty(G), |g| \leq f\} = \sup\{|E\mu g|; g \in L^\infty(G), |g| \leq f\}.$$

But for $g \in L^\infty(G)$ and $\nu \in L^1(G)^+$, we have

$$\begin{aligned}\langle |E\mu g|, \nu\rangle &= \sup\{|\langle E\mu g, \eta\rangle|; \eta \in L^1(G), |\eta| \leq \nu\} \\ &= \sup\{|\langle g, \eta * \mu\rangle|; \eta \in L^1(G), |\eta| \leq \nu\} = \langle |\rho_\mu^*(g)|, \nu\rangle.\end{aligned}$$

So, $|E\mu g| = |\rho_\mu^*(g)|$. Consequently,

$$|T_n|(f) = \sup\{|\rho_\mu^*(g)|; g \in L^\infty(G), |g| \leq f\} = |\rho_\mu^*|(f).$$

But by (1), $|\rho_\mu^*|(f) = \rho_{|\mu|}{}^*(f) = |\mu|f$, so $|T_n|(f) = |\mu|f$. On the other hand, $\pi(|n|) = |\mu|$. Indeed, for $f \in L^\infty(G)^+$ and $\nu \in L^1(G)^+$,

$$\langle \nu\pi(|n|), f\rangle = \langle \nu|n|, f\rangle = \langle |n|, f\nu\rangle = \sup\{|\langle n, g\rangle|; g \in L^\infty(G), |g| \leq f\nu\}.$$

Now, if $g \in L^\infty(G)$ and $|g| \leq f\nu$, we have $|\langle n, g\rangle| = |\lim\langle e_\alpha * \mu, g\rangle| \leq \lim|\langle e_\alpha * \mu, g\rangle| \leq \langle |\mu|, f\nu\rangle = \langle \nu * |\mu|, f\rangle$. Consequently, $\nu\pi(|n|) \leq \nu * |\mu|$. But by ([3], Theorem 3.1) $\rho_{|\mu|} = |\rho_\mu|$, so

$$\begin{aligned}\nu * |\mu| = \rho_{|\mu|}(\nu) &= |\rho_\mu|(\nu) = \sup\{|\eta * \mu|; \eta \in L^1(G), |\eta| \leq \nu\} \\ &= \sup\{|\eta n|; \eta \in L^1(G), |\eta| \leq \nu\} \leq \nu|n| = \nu\pi(|n|).\end{aligned}$$

It follows that for all $\nu \in L^1(G)$ and $\nu \geq 0$, we have $\nu|n| = \nu * |\mu|$. Therefore $\pi(|n|) = |\mu|$.

3) Let $n \in LUC(G)^\perp$ and $n \neq 0$. For $f \in L^\infty(G)$ with $f \geq 0$, we have $|T_n|(f) = \sup\{|ng|; g \in L^\infty(G), |g| \leq f\} = 0$. On the other hand, since $n \neq 0$, there exists $g \in L^\infty(G)$ such that $\langle n, g\rangle \neq 0$. Now we take $k \in \mathbb{N}$ such that $|g| \leq k1$. For $\mu \in L^1(G)^+$, we have

$$\begin{aligned}\mu(G)|\langle n, g\rangle| &\leq \mu(G)\sup\{|\langle n, h\rangle|; h \in L^\infty(G), |h| \leq k1\} \\ &= \mu(G)k\langle |n|, 1\rangle = k\langle |n|1, \mu\rangle.\end{aligned}$$

Consequently $|n|1 \neq 0$, i.e. $T_{|n|} \neq 0$.

For $n \in L^1(G)^{**}$, we define $\rho_n : L^1(G)^{**} \to L^1(G)^{**}$ by $\rho_n(F) = Fn$. The operator ρ_n is called a right multiplier on $L^1(G)^{**}$.

Theorem 2.3. *Let $n = \Gamma_E(\mu)$, where E is a weak*-limit a bounded approximate identity in $L^1(G)$ and $\mu \in M(G)$. The following statements hold:*

(1) *$|\rho_n|(\nu) = \rho_{|n|}(\nu)$, for all $\nu \in L^1(G)$.*
(2) *If $|\rho_n|$ is weak*-weak* continuous, then $|\rho_n| = \rho_{|n|}$.*
(3) *If $m \in LUC(G)^\perp$ and $m \neq 0$, then $|\rho_m| \neq \rho_{|m|}$.*

Proof. Since $\mu \in M(G)$, for all $\nu \in L^1(G)$, we have $\nu n \in L^1(G)$. Hence there exists a measure $\eta \in M(G)$ such that $\nu n = \nu * \eta$ for all $\nu \in L^1(G)$ (since $\nu \to \nu n$ is a right multiplier on $L^1(G)$). It is easy to see that $\mu = \eta$.

Now for all $\nu \in L^1(G)^+$, we can write $|\rho_n|(\nu) = \sup\{|Fn|; F \in L^1(G)^{**}, |F| \leq \nu\}$. But $L^1(G)$ is a solid sublattice of $L^1(G)^{**}$ ([5], p.234), hence

$$\begin{aligned}|\rho_n|(\nu) &= \sup\{|\eta_1 n|; \eta_1 \in L^1(G), |\eta_1| \leq \nu\}\\ &= \sup\{|\eta_1 * \mu|; \eta_1 \in L^1(G), |\eta_1| \leq \nu\} = |\rho_\mu|(\nu) = \nu * |\mu|.\end{aligned}$$

On the other hand, $\nu * |\mu| = \nu\pi(|n|) = \nu|n|$. Consequently, for all $\nu \in L^1(G)$, $|\rho_n|(\nu) = \rho_{|n|}(\nu)$.

2) By (1) and the Goldestines theorem, we have $|\rho_n| = \rho_{|n|}$.

3) If $m \in LUC(G)^\perp$, then for all $\nu \in L^1(G)$, $\nu m = 0$. So for $\mu \in L^1(G)^+$, we have $|\rho_m|(\mu) = \sup\{|\nu m|; \nu \in L^1(G), |\nu| \leq \mu\} = 0$. By a similar argument as given in part (3) of Proposition 2.2, for all $\mu \in L^1(G)^+$, we have $\mu|m| \neq 0$, i.e. $\rho_{|m|}(\mu) \neq 0$. Consequently $|\rho_m| \neq \rho_{|m|}$.

Theorem 2.4. *Let G be a compact group and $\mu \in L^1(G)$. The following statements hold:*

(1) $\{|P\mu|; P \in L^1(G)^{**}, |P| \leq F\} = \{|P\mu|; P \in L^1(G)^{**}, |P| \leq EF\}$ *where E is a weak* limit positive approximate identity with norm one in $L^1(G)$ and $F \in L^1(G)^{**}$ with $F \geq 0$.*

(2) $|\rho_\mu| = \rho_{|\mu|}$.

Proof. Let $F \in L^1(G)^{**}$ and $F \geq 0$. If $P \in L^1(G)^{**}$ and $|P| \leq F$, then $|EP| \leq EF$ and $EP\mu = P\mu$. Indeed, since G is compact, $L^1(G)$ is an ideal in $L^1(G)^{**}$ [4], hence $EP_\mu = P_\mu$. Consequently

$$\{|P\mu|; P \in L^1(G)^{**}, |P| \leq F\} \subseteq \{|P\mu|; P \in L^1(G)^{**}, |P| \leq EF\}.$$

To prove the reverse inclusion, let $P \in L^1(G)^{**}$ and $|P| \leq EF$. Since G is compact, $\pi(P)$ and $\pi(F)$ are measures in $M(G)$. If $\pi(P) = \nu$ and $\pi(F) = \eta$, then for $f \in C(G)$, $|\langle \nu, f\rangle| \leq \langle \eta, |f|\rangle = \langle F, |f|\rangle$. So we can choose a $P_1 \in L^1(G)^{**}$ such that $|P_1| \leq F$ and $\langle \nu, f\rangle = \langle P_1, f\rangle$ for all $f \in C(G)$. Hence for $f \in C(G)$, we have $\langle P_1\mu, f\rangle = \langle P_1, \mu f\rangle = \langle \nu, \mu f\rangle = \langle P, \mu f\rangle = \langle P\mu, f\rangle$, i.e. $P\mu = P_1\mu$. Consequently,

$$\{|P\mu|; P \in L^1(G)^{**}, |P| \leq F\} = \{|P\mu|; P \in L^1(G)^{**}, |P| \leq EF\}.$$

2) Let $F \in L^1(G)^{**}$, $F \geq 0$ and $\pi(F) = \eta$. It is easy to see that

$$\{|P\mu|; P \in L^1(G)^{**}, |P| \leq EF\} = \{|\nu * \mu|; \nu \in M(G), |\nu| \leq \eta\}.$$

So by (1), $|\rho_\mu|(F) = \eta * |\mu| = F|\mu|$. Indeed, since G is compact, $F|\mu| \leq L^1(G)$, and any $f \in C(G)$ is of the form $f = g\nu$ for some $g \in L^\infty(G)$ and $\nu \in L^1(G)$. Hence

$$\langle F|\mu|, f\rangle = \langle F|\mu|, g\nu\rangle = \langle \nu F|\mu|, g\rangle = \langle \nu * \eta * |\mu|, g\rangle = \langle \eta * |\mu|, g\nu\rangle = \langle \eta * |\mu|, f\rangle.$$

Consequently $\rho_{|\mu|} = |\rho_\mu|$.

For $n \in L^1(G)^{**}$, we denote λ_n as a left multiplier on $L^1(G)^{**}$. We know that $|\lambda_n|$ is a left multiplier if and only if $|\lambda_n| = \lambda_{|n|}$ ([3], Lemma 3.6). For $n \in LUC(G)^{\perp}$, $|\rho_n|(\nu) = 0$ ($\nu \in L^1(G)$). If $|\rho_n|$ is a right multiplier, then $|\rho_n|$ is weak*-weak* continuous, so $|\rho_n| = 0$. Moreover, if $n \neq 0$, then $\rho_{|n|} \neq 0$, i.e. $|\rho_n| \neq \rho_{|n|}$. Also, it is not known whether for any $\mu \in M(G)$, we have $|\rho_\mu^{**}| = \rho_{|\mu|}{}^{**}$. In the following Corollary we give some cases where the equality holds.

Corollary 2.5. *For $\mu \in L^1(G)$, $|\rho_\mu^{**}| = \rho_{|\mu|}{}^{**}$ whenever one of the following conditions holds:*

(1) *G is a compact group.*
(2) *$|\rho_\mu^{**}|$ is compact.*
(3) *$|\rho_\mu^{**}|$ is weak*-weak* continuous.*

Proof. Assume that (1) holds. By Theorem 2.4, $|\rho_\mu^{**}| = \rho_{|\mu|}{}^{**}$. If (2) holds, since $|\rho_\mu^{**}|$ is compact, so $|\rho_\mu| : L^1(G) \to L^1(G)$ is compact. Consequently G is compact [7]. The statement follows from (1). Suppose (3) holds. By Theorem 2.3, $|\rho_\mu^{**}| = \rho_{|\mu|}{}^{**}$.

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