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THE LANDAU PROBLEM FOR UNIVALENT BOUNDED NONVANISHING FUNCTIONS

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Abstract. This paper is to inverstigate the problem of finding $\sup|a_0 + a_1 + \cdots + a_n|$ for univalent holomorphic nonvanishing functions $f(z) = a_0 + a_1z + \cdots$ in the unit disk $|z| < 1$.

1. Introduction

Consider following families of the functions:

$$\begin{aligned} B &= \{f \in H(D) : f(z) = a_0 + a_1z + \cdots, |f(z)| < 1, z \in D\}, \\ B_0 &= \{f \in B : f(z) \neq 0, z \in D\}, \\ B^s &= \{f \in B_0 : f(z) \neq 0, z \in D \text{ and } f(z) \text{ is univalent function}\}, \\ \Omega &= \{w \in B : w(z) = c_1z + c_2z^2 + \cdots, z \in D\}, \end{aligned}$$

where $H(D)$ denotes the set of holomorphic functions in the unit disk $D = \{z \in C : |z| < 1\}$. With no loss of generality we may assume that for $f \in B_0$ we have the normalization $a_0 = e^{-t}$, $t > 0$.

The function

$$F(t, z) = \exp\left(-t\frac{1+z}{1-z}\right) = e^{-t} + \sum_{n=1}^{\infty} A_n(t)z^n$$

plays an important role in the problem of determining $\max_{f\in B_0} |a_n|$. Landau proved

$$\sup_{f\in B} |a_0 + a_1 + \cdots + a_n| = 1 + \sum_{v=1}^{n} \left(\frac{1 \cdot 3 \cdots (2v-1)}{2 \cdot 4 \cdots (2v)}\right)^2 := G_n, \quad n \in N.$$

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If f is univalent (say $f \in B^s$), then for every $n \in N$, we have

$$\sup_{f \in B^s} |a_0 + a_1 + \cdots + a_n| < k \approx 1.616 \cdots.$$

However the result is not sharp and Lewandowski and Szynal[4] (1999) proved the following lemma to improve the result,

Lemma. *If $f \in B_0$, then*

$$|a_0 + a_1| \leq 2e^{-\frac{1}{2}} \approx 1.21 \cdots,$$

$$|a_0 + a_1 + a_2| \leq e^{-t_0} \left[1 + 2t_0 + \frac{t_0}{2(2-t_0)}\right] \approx 1.33 \cdots,$$

and $t_0 = 0.66 \cdots$ is the root of equation: $-4t^3 + 19t^2 - 26t + 10 = 0$.

Based on above lemma, we give further improvement in Section 2.

2. Main Results

We claim the following theorem for the Landau problem of B^s.

Theroem. *If $f(z) \in B^s$, then*

$$|a_0 + a_1| \leq 1.202 \cdots, \tag{1}$$

$$|a_0 + a_1 + a_2| \leq e^{-t_0}(1 + 2t_0^2) \leq 1.238 \cdots, \tag{2}$$

with $t_0 = 1.707 \cdots$ is the root of equation: $2t^2 - 4t + 1 = 0$.

Proof. By the representation formula for function $f \in B^s$

$$f(z) = \exp\left(-t\frac{1 + w(z)}{1 - w(z)}\right) = a_0 + a_1 z + a_2 z^2 + \cdots, \quad w \in \Omega,$$

we can get the relations:

$$a_0 = e^{-t}, \quad a_1 = -2tc_1 e^{-t}, \quad a_2 = -2te^{-t}[c_2 + (1-t)c_1^2]; \tag{3}$$

Applying the inequality $|a_0| \leq 1$, $|a_1| \leq \frac{4|a_0|(1-|a_0|)}{1+|a_0|}$ and composition of univalent and let $x = |a_0|$, we have

$$|a_0 + a_1| \leq |a_0| + \frac{4|a_0|(1-|a_0|)}{1+|a_0|} = x + \frac{4x(1-x)}{1+x} := g_1(x);$$

After elementary calculations of one extreme we arrive at following inequalities:

$$|a_0 + a_1| \leq g_1(x_0) = 11 - 4\sqrt{6} \approx 1.202 \cdots,$$

where $x_0 = \sqrt{\frac{8}{3}} - 1$. It is easy to see that upper bound in (1) is sharp.

Using the representation (3) we have

$$a_0 + a_1 + a_2 = e^{-t} - 2te^{-t}[c_1 + c_2 + (1-t)c_1^2] = e^{-t} - 2te^{-t}\Phi(c_1, c_2, t).$$

Applying the inequality and composition of univalent: $|c_1| \le 1$, $|c_2| \le 2|c_1|(1 - |c_1|)$, we have

$$\begin{aligned} &-2|c_1|(1-|c_1|) \le c_2 \le 2|c_1|(1-|c_1|),\\ &\max \Phi(c_1, c_2, t) = |c_1| + 2|c_1|(1-|c_1|) + (1-t)c_1^2 = H_1(c_1, t),\\ &\min \Phi(c_1, c_2, t) = -|c_1| - 2|c_1|(1-|c_1|) + (1-t)c_1^2 = H_2(c_1, t). \end{aligned}$$

There are two cases of the problem.

a). If $t \le \frac{3}{2}$, $x = |c_1|$, then

$$\begin{aligned} H_2(c_1, t) = H_2(x, t) = (3-t)x^2 - 3x,\\ H_1(c_1, t) = H_1(x, t) = -(1+t)x^2 + 3x. \end{aligned}$$

After elementary calculations of one extreme we can obtain in the following expressions

$$\begin{aligned} \min \Phi(c_1, c_2, t) = H_2\left(\frac{3}{2(3-t)}, t\right) = -\frac{9}{4(3-t)};\\ \max \Phi(c_1, c_2, t) = H_1\left(\frac{3}{2(1+t)}, t\right) = \frac{9}{4(1+t)};\\ e^{-t}\left(1 - \frac{9t}{2(1+t)}\right) \le a_0 + a_1 + a_2 \le e^{-t}\left(1 + \frac{9t}{2(3-t)}\right);\\ -0.461\cdots \le a_0 + a_1 + a_2 \le 1.227\cdots. \end{aligned}$$

Hence

$$|a_0 + a_1 + a_2| \le e^{-t}\left(1 + \frac{9t_0}{2(3-t_0)}\right) \approx 1.227\cdots. \tag{4}$$

b). From the inequality $|c_2| \le 2|c_1|(1-|c_1|)$ and if $t > \frac{3}{2}$, $x = |c_1|$, then we have

$$|a_0 + a_1 + a_2| \le e^{-t}\{2t(t-3)x^2 + 6tx + 1\} := e^{-t}g_2(x).$$

We arrive at following after elementary calculations of one extreme

$$g_2(x) \le g_2(1) = e^{-t}(1 + 2t^2) \le 1.238\cdots. \tag{5}$$

According to **a)** and **b)** we obtain

$$|a_0 + a_1 + a_2| \le e^{-t}\begin{cases} 1 + \dfrac{9t}{2(3-t)}, & t \le \frac{3}{2},\\ 1 + 2t^2, & t > \frac{3}{2}. \end{cases} \tag{6}$$

Hence

$$|a_0 + a_1 + a_2| \le e^{-t_0}(1 + 2t_0^2) \le 1.238\cdots, \quad \text{where } t_0 = 1.707\cdots.$$

Remark. We conjecture that for any $f \in B^s$ and $n \in N$ there exists an absolute constant $L > 1$ such that

$$\sup_{f\in B^s} |a_0 + a_1 + \cdots + a_n| \le L < k \cong 1.616\cdots.$$

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