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UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

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Abstract. Let $\mathcal{A}_n$ be the class of functions $f(z)$ which are analytic and n-fold symmetric in the open unit disk $\mathbb{U}$. The integral operator $G_\alpha(z)$ for $f(z) \in \mathcal{A}_n$ is considered. The object of the present paper is to derive univalence conditions of the integral operator $G_\alpha(z)$ for $f(z) \in \mathcal{A}_n$.

1. Introduction

Let $\mathcal{A}_n$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{nk+1} z^{nk+1} \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic and n-fold symmetric in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We denote by $\mathcal{S}_n$ the subclass of $\mathcal{A}_n$ consisting of functions $f(z)$ which are univalent in $\mathbb{U}$. Many authors studied the problem of integral operators for functions $f(z)$ in the class $\mathcal{S}_1$. In this sense, the following useful result is due to Pfaltzgraff [3].

Theorem 1.1. *If $f(z)$ is univalent in $\mathbb{U}$ and α is complex number with $|\alpha| \leqq \frac{1}{4}$, then the integral operator $G_\alpha(z)$ given by*

$$G_\alpha(z) = \int_0^z (f'(t))^\alpha dt \tag{1}$$

is also univalent in $\mathbb{U}$.

Further, Pascu and Pescar [2] gave

Theorem 1.2. *If $f(z) \in \mathcal{S}_1$ and α is a complex number with $|\alpha| \leqq \frac{1}{4n}$, then the integral operator $G_{\alpha,n}(z)$ given by*

$$G_{\alpha,n}(z) = \int_0^z (f'(t^n))^\alpha dt$$

is in the class $\mathcal{S}_n$ for all positive integer n.

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2. Properties of Integral Operators

To discuss our problems for integral operators, we need to recall here the following lemma due to Becker [1].

Lemma 2.1. *If $f(z) \in \mathcal{A}_1$ satisfies*

$$(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1 \quad (z \in \mathbb{U}), \tag{2}$$

then $f(z) \in \mathcal{S}_1$.

Applying the above lemma, we derive

Theorem 2.1. *If $f(z) \in \mathcal{A}_1$ satisfies the inequality* (2) *for all $z \in \mathbb{U}$, then the integral operator $G_\alpha(z)$ defined by* (1) *belongs to the class $\mathcal{S}_1$ for all α $(|\alpha| \leqq 1)$.*

Proof. Note that $G_\alpha(z) \in \mathcal{A}_1$ for $f(z) \in \mathcal{A}_1$ and that

$$\frac{zf''(z)}{f'(z)} = \frac{1}{a}\frac{zG''_\alpha(z)}{G'_\alpha(z)}.$$

It follows that

$$(1-|z|^2)\left|\frac{zG''_\alpha(z)}{G'_\alpha(z)}\right| = |\alpha|(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leqq |\alpha| \leqq 1$$

for $z \in \mathbb{U}$. Thus, using Lemma 2.1, we have $G_\alpha(z) \in \mathcal{S}_1$.

Next, we prove

Corollary 2.1. *If $f(z) \in \mathcal{A}_1$ satisfies*

$$\left|\frac{f''(z)}{f'(z)}\right| \leqq 1 \quad (z \in \mathbb{U}),$$

then the integral operator $G_\alpha(z)$ defined by (1) *is in the class $\mathcal{S}_1$ with $|\alpha| \leqq \frac{3\sqrt{3}}{2}$.*

Proof. In view of the proof of Theorem 2.1, we see that

$$(1-|z|^2)\left|\frac{zG''_\alpha(z)}{G'_\alpha(z)}\right| \leqq |\alpha|(1-|z|^2)|z| \leqq 1,$$

because $|\alpha| \leqq \frac{3\sqrt{3}}{2}$ and

$$\max_{|z|\leq 1}(1-|z|^2)|z| = \frac{2}{3\sqrt{3}}.$$

Thus, by Lemma 2.1, we prove that $G_\alpha(z) \in \mathcal{S}_1$.

Finally, we show

Theorem 2.2. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\left|\frac{f''(z)}{f'(z)}\right| \leqq |z|^{n-1} \quad (z \in \mathbb{U}),$$

then the integral operator $G_\alpha(z)$ defined by (1) *belongs to the class $\mathcal{S}_n$ with*

$$|\alpha| \leqq \frac{(n+2)^{\frac{n+2}{2}}}{2n^{\frac{n}{2}}}.$$

Proof. Since

$$\frac{zf''(z)}{f'(z)} = \frac{1}{\alpha}\frac{zG''_\alpha(z)}{G'_\alpha(z)} = n(n+1)a_{n+1}z^n + \cdots,$$

we have that

$$(1-|z|^2)\left|\frac{zG''_\alpha(z)}{G'_\alpha(z)}\right| = |\alpha|(1-|z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leqq |\alpha|(1-|z|^2)|z|^n \quad (z \in \mathbb{U}).$$

Note that

$$|\alpha| \leqq \frac{(n+2)^{\frac{n+2}{2}}}{2n^{\frac{n}{2}}}$$

and

$$(1-|z|^2)|z|^n \leqq \frac{2n^{\frac{n}{2}}}{(n+2)^{\frac{n+2}{2}}} \quad (z \in \mathbb{U}).$$

This gives us that

$$(1-|z|^2)\left|\frac{zG''_\alpha(z)}{G'_\alpha(z)}\right| \leqq 1 \quad (z \in \mathbb{U}).$$

Further, it is easy to see that $G_\alpha(z) \in \mathcal{A}_n$. This completes the proof of the theorem.

Remark. For $n = 1$, Theorem 2.2 becomes Theorem 2.1.

References

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