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COMMON FIXED POINT THEOREMS BY ALTERING DISTANCES

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Abstract. In this paper we obtain common fixed point theorems for weakly commuting pairs of self mappings by altering distances between the points under a ϕ-contractive condition.

1. Introduction

The study of common fixed points for self mappings on a metric space by altering distances between the points with the use of control functions has emerged as an area of wide interest. Khan *et al.* [2] established fixed point theorem for a single self map. Sastry and Babu [7] proved fixed point theorem for a pair of self maps. Sastry *et al.* [8] proved a unique common fixed point theorem for four mappings by using a control function in order to alter distances between the points. Pant *et al.* [5, 6] obtained an answer to the open problem of Sastry *et al.* [8] by establishing a connection between continuity and reciprocal continuity in the setting of control function.

The presence of control function creates certain difficulties in proving the existence of fixed point under contractive conditions. In view of these difficulties, known fixed-point theorems either employ a stronger contractive condition like the Banach contractive condition e.g. in Sastry *et al.* [8] or assume the existence of a convergent sequence of iterates e.g. in [2], [7]. The study of fixed points in the presence of control function under more general contractive conditions like Mier-Keeler type (ε, δ)-contractive condition or a ϕ-contractive condition is still an open area. In the present paper, we prove a common fixed point theorem assuming a ϕ-contractive condition. We employ a control function that unifies the choice of control function in [7], [8]. Also, in the settings of our theorem, we consider the open problem of [8] and provide an answer to the problem in the setting of a more general contractive condition than in Sastry *et al.* [8].

We have used the following notions.

Definition 1.1. A control function ψ is defined as $\psi : \Re_+ \to \Re_+$ which is continuous, monotonically increasing, $\psi(2t) \le 2\psi(t)$ and $\psi(t) = 0$ if and only if $t = 0$.

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Definition 1.2. Two self mappings A and S of a metric space (X, d) are called *weakly commuting* if $d(ASx, SAx) \leq d(Ax, Sx)$ for each x in X. This condition implies that $ASx = SAx$ whenever $Ax = Sx$.

Clearly, commuting and weakly commuting mappings are compatible, but the converses are not necessarily true [1].

Definition 1.3. ([8]) Two self mappings A and S of a metric space (X, d) are called *ψ-compatible* if $\lim_n \psi(d(ASx_n, SAx_n)) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X.

Definition 1.4. ([3]) Two self mappings A and S of a metric space (X, d) are said to be *reciprocally continuous* in X, if $\lim_n ASx_n = At$ and $\lim_n SAx_n = St$ whenever $\{x_n\}$ is a sequence such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X.

Notation 1.5. If A, B, S and T are four self mappings of (X, d) and ψ is a control function on $\Re_+$, we write

$$M_\psi(x, y) = \max\{\psi(d(Sx, Ty)), \psi(d(Ax, Sx)), \psi(d(By, Ty)), \\ [\psi(d(Ax, Ty)) + \psi(d(Sx, By))]/2\}.$$

2. Main Theorem

Theorem 2.1. *Let (A, S) and (B, T) be weakly commuting pairs of self mappings of a complete metric space (X, d) and ψ be as in Definition* 1.1 *satisfying*
(i) $AX \subset TX$, $BX \subset SX$ *and*
(ii) $\psi(d(Ax, By)) \leq \phi(M_\psi(x, y))$, *for all x, y in X whenever $M_\psi(x, y) > 0$ and*
$\phi : \Re_+ \to \Re_+$ *be an upper semi continuous function such that $\phi(t) < t$ for each $t > 0$.*

Suppose that (A, S) and (B, T) are ψ-compatible pairs of reciprocally continuous mappings. Then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1}; \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}. \tag{2.1.1}$$

We claim that $\{y_n\}$ is a Cauchy sequence. We write $\alpha_n = \psi(d(y_n, y_{n+1}))$. Then, using condition (ii), it follows that

$$\begin{aligned}\alpha_{2n} &= \psi(d(y_{2n}, y_{2n+1})) = \psi(d(Ax_{2n}, Bx_{2n+1})) \\ &\leq \phi(M_\psi(x_{2n}, x_{2n+1})) \\ &= \phi(\max\{\psi(d(Sx_{2n}, Tx_{2n+1})), \psi(d(Ax_{2n}, Sx_{2n})), \psi(d(Bx_{2n+1}, Tx_{2n+1})), \\ &\qquad [\psi(d(Sx_{2n}, Bx_{2n+1}))]/2\}). \\ &= \phi(\max\{\psi(d(y_{2n}, y_{2n+1})), \psi(d(y_{2n-1}, y_{2n})), \psi(d(y_{2n+1}, y_{2n})),\end{aligned}$$

$$
\begin{aligned}
&[\psi(\max\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n-1})\})]/2\}) \\
&= \phi(\psi(d(y_{2n-1}, y_{2n}))) \\
&\leq \phi(\psi(d(y_{2n-1}, y_{2n}))) = \phi(\alpha_{2n-1}).
\end{aligned}
$$

That is,

$$\alpha_{2n} \leq \phi(\alpha_{2n-1}) < \alpha_{2n-1}. \tag{2.1.2}$$

Similarly, $\alpha_{2n-1} < \alpha_{2n-2}$; $\alpha_{2n-2} < \alpha_{2n-3}$ and so on. Thus $\{\alpha_n\} = \{\psi(d(y_n, y_{n+1}))\}$ is a strictly decreasing sequence of positive numbers and hence converges, say, to $\alpha \geq 0$. Suppose $\alpha > 0$. Then the inequality (2.1.2) on making $n \to \infty$ and in view of upper semi continuity of ϕ yields $\alpha \leq \phi(\alpha) < \alpha$, a contradiction. Hence $\alpha = \lim_{n\to\infty} \psi(d(y_n, y_{n+1})) = 0$. This, by the monotonically increasing property of ψ, implies

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0, \tag{2.1.3}$$

and also $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive numbers. We now show that $\{y_n\}$ is a Cauchy sequence.

Suppose it is not. Then there exists an $\varepsilon > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_i+1}) > 2\varepsilon$. But since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, there exists an integer m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) \geq \varepsilon$. If not, then

$$d(y_{n_i}, y_{n_{i+1}}) \leq d(y_{n_i}, y_{n_{i+1}-1}) + d(y_{n_{i+1}}, y_{n_{i+1}-1}) < \varepsilon + d(y_{n_{i+1}}, y_{n_{i+1}-1}) < 2\varepsilon,$$

a contradiction. If m_i is the smallest integer such taht $d(y_{n_i}, y_{m_i}) \geq \varepsilon$, then

$$
\begin{aligned}
\varepsilon \leq d(y_{n_i}, y_{m_i}) &\leq d(y_{n_i}, y_{m_i-2}) + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i}) \\
&< \varepsilon + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i}).
\end{aligned}
$$

That is, there corresponds an integer m_i satisfying $n_i < m_i < n_{i+1}$ such that

$$d(y_{n_i}, y_{m_i}) \geq \varepsilon \quad \text{and} \quad \lim_{n_i\to\infty} d(y_{n_i}, y_{m_i}) = \varepsilon. \tag{2.1.4}$$

From the triangle inequality, we get $|d(y_{n_i}, y_{m_i+1}) - d(y_{n_i}, y_{m_i})| \leq d(y_{m_i+1}, y_{m_i})$. Thus, as $n_i \to \infty$, we obtain $d(y_{m_i+1}, y_{n_i}) \to \varepsilon$. Similarly, we can get $d(y_{n_i+1}, y_{m_i}) \to \varepsilon$. Applying the continuity of ψ, we get either

$$\psi(d(y_{m_i+1}, y_{n_i})) \to \psi(\varepsilon) \quad \text{or} \quad \psi(d(y_{n_i+1}, y_{n_i})) \to \psi(\varepsilon). \tag{2.1.5}$$

Moreover, m_i can be chosen in such a manner that m_i is even, when n_i is odd and m_i is odd, when n_i is even. Suppose that n_i is odd and m_i is even. Then by virtue of (ii), we get

$$\psi(d(y_{n_i+1}, y_{m_i+1})) = \psi(d(Ax_{n_i+1}, Bx_{m_i+1})) \leq \phi(M_\psi(x_{n_i+1}, x_{m_i+1})).$$

On letting $n_i \to \infty$ and in view of result (2.1.5) and applying the upper semi continuity of ϕ, the above inequality yields $\psi(\varepsilon) \leq \phi(\psi(\varepsilon)) < \psi(\varepsilon)$, a contradiction. Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there is a point z in X such that $y_n \to z$ as $n \to \infty$. Hence from (2.1.1), we have

$$y_{2n} = Ax_{2n+1} = Tx_{2n+1} \to z \quad \text{and} \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z. \tag{2.1.6}$$

Now suppose that (A, S) is a ψ-compatible pair of reciprocally continuous mappings. Since A and S are reciprocally continuous, by (2.1.6), we get

$$ASx_{2n} \to Az \quad \text{and} \quad SAx_{2n} \to Sz. \tag{2.1.7}$$

Also, ψ-compatibility of A and S implies that $\lim_n \psi(d(ASx_{2n}, SAx_{2n})) = 0$. We now show that $Az = Sz$.

Suppose $Az \neq Sz$. Let $\varepsilon = (d(Az, Sz))/2$. Then there exists N in Z^+ such that $\psi(d(ASx_{2n}, SAx_{2n})) < \psi(\varepsilon)$ for all $n \geq N$. This implies that $d(ASx_{2n}, SAx_{2n}) < \varepsilon$ for all $n \geq N$. Hence by (2.1.7), $d(Az, Sz) < \varepsilon = (d(Az, Sz))/2$, a contradiction.
Hence

$$Az = Sz. \tag{2.1.8}$$

Since $AX \subset TX$, there is a point w in X such that $Tw = Az$. By (2.1.8),

$$Tw = Az = Sz. \tag{2.1.9}$$

Now, we show that $Az = Bw$. Suppose $Az \neq Bw$. Then, by (ii), we have

$$\psi(d(Az, Bw)) \leq \phi(M_\psi(z, w)) \leq \phi(\psi(d(Bw, Tw))) < \psi(d(Bw, Az)),$$

a contradiction. Hence $Az = Bw$. Therefore, by (2.1.9),

$$Bw = Az = Sz = Tw. \tag{2.1.10}$$

Since A and S are weakly commuting, we have by (2.1.10),

$$ASz = SAz \quad \text{and} \quad AAz = ASz = SAz = SSz. \tag{2.1.11}$$

Since B and T are weakly commuting, we have

$$BBw = BTw = TBw = TTw. \tag{2.1.12}$$

We now show that $AAz = Az$. Suppose $AAz \neq Az$. Then by (ii), we get

$$\begin{aligned}\psi(d(Az, AAz)) &= \psi(d(Bw, AAz))\\ &\leq \phi(M\psi(Az, w))\\ &= \phi(\psi(d(Az, AAz))), \qquad \text{(by (2.1.10) \& (2.1.12))}\end{aligned}$$

a contradiction. Hence $AAz = Az$.

Also, we have $AAz = SAz$. Therefore, Az is a common fixed point for A and S. Also, suppose $BBw \neq Bw$. By (ii), we have

$$\begin{aligned} \psi(d(Bw, BBw)) &= \psi(d(Az, BBw)) \qquad \text{(by (2.1.10))} \\ &\leq \phi(M\psi(z, Bw)) \\ &= \phi(\psi(d(Bw, BBw))), \qquad \text{(by (2.1.10) \& (2.1.12))} \\ &< \psi(d(Bw, BBw)), \end{aligned}$$

a contradiction. Hence $BBw = Bw$ and since $TBw = BBw$, we have Bw as a common fixed point for B and T. Since $Az = Bw$, we have Az as a common fixed point for A, B, S and T. Uniqueness of a common fixed point follows by (ii). The proof is similar when the pair (B, T) is assumed ψ-compatible and reciprocally continuous. This completes the proof of the theorem.

In the above theorem, we replace reciprocal continuity of B and T by continuity of A and obtain result similar to Theorem 2.1,

Theorem 2.2. *Let (A, S) and (B, T) be weakly commuting pairs of self mappings of a complete metric space (X, d) and ψ be as in definition (1.1) satisfying*

(i) *$AX \subset TX$, $BX \subset SX$ and*

(ii) *$\psi(d(Ax, By)) \leq \phi(M_\psi(x, y))$, for all x, y in X whenever $M_\psi(x, y) > 0$ and $\phi : \Re_+ \to \Re_+$ be an upper semi continuous function such that $\phi(t) < t$ for each $t > 0$.*

Suppose that A and S are ψ-compatible and A is continuous mapping. Then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any fixed point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}. \tag{2.2.1}$$

This can be done by virtue of (i). Then applying the same proof as that in the Theorem 2.1, we can show that $\{y_n\}$ is a Cauchy seuqence. Since X is a complete metric space, there is a point z in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z \quad \text{and} \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z. \tag{2.2.2}$$

Now, suppose that (A, S) is ψ-compatible then we have

$$Ax_{2n} \to z \quad \text{and} \quad Sx_{2n} \to z \quad \text{implies that } \lim_n \psi(d(ASx_{2n}, SAx_{2n})) = 0. \tag{2.2.3}$$

Also, since A is continuous, so by (2.2.2), we get

$$AAx_{2n} \to Az \quad \text{and} \quad ASx_{2n} \to Az \quad \text{as} \quad n \to \infty. \tag{2.2.4}$$

We claim that $\lim_n SAx_{2n} = Az$. Using (2.2.3), we get

$$\psi(d(SAx_{2n}, Az)) \leq \psi(d(SAx_{2n}, ASx_{2n}) + d(ASx_{2n}, Az)) \to 0 \quad \text{as } n \to \infty.$$

Thus, we get $d(SAx_{2n}, Az) \to 0$ as $n \to \infty$, and so $\lim_n SAx_{2n} = Az$. Also, since $AX \subset TX$, for each n, there exists w_{2n} in X such that $AAx_{2n} = Tw_{2n}$ and $AAx_{2n} = Tw_{2n} \to Az$. Thus, $AAx_{2n} \to Az$, $SAx_{2n} \to Az$, $ASx_{2n} \to Az$ and $Tw_{2n} \to Az$ as $n \to \infty$. Again, we claim that $\lim_n Bw_{2n} \to Az$. If not, then there exist $\varepsilon > 0$ and a subsequence $\{n_k\}$ such that $d(AAx_{2n_k}, Bw_{2n_k}) > \varepsilon$ and $\psi(d(SAx_{2n_k}, ASx_{2n_k})) < \varepsilon$ for all n_k. Therefore,

$$\begin{aligned}\psi(\varepsilon) &\le \psi(d(AAx_{2n_k}, Bw_{2n_k}))\\ &\le \phi(M_\psi(Ax_{2n_k}, w_{2n_k}))\\ &= \phi(\max\{\psi(d(SAx_{2n_k}, Tw_{2n_k})), \psi(d(AAx_{2n_k}, SAx_{2n_k})), \psi(d(Bw_{2n_k}, Tw_{2n_k})),\\ &\qquad [\psi(d(AAx_{2n_k}, Tw_{2n_k})) + \psi(d(Bw_{2n_k}, SAx_{2n_k}))]/2\})\\ &= \phi(\max\{\psi(d(Bw_{2n_k}, Tw_{2n_k})), [\psi(d(Bw_{2n_k}, SAx_{2n_k}))]/2\}),\\ &= \phi(\psi(d(Bw_{2n_k}, AAx_{2n_k}))),\\ &< \psi(d(Bw_{2n_k}, AAx_{2n_k})), \quad \text{a contradiction.}\end{aligned}$$

Hence $\lim_n Bw_{2n} = Az$.

We claim that $Az = Sz$. For this, using (ii), we get

$$\begin{aligned}\psi(d(Sz, Bw_{2n})) &\le \phi(M_\psi(z, w_{2n}))\\ &= \phi(\max\{\psi(d(Sz, Tw_{2n})), \psi(d(Az, Sz)), \psi(d(Bw_{2n}, Tw_{2n})),\\ &\qquad [\psi(d(Az, Tw_{2n})) + \psi(d(Bz, Sw_{2n}))]/2\}),\\ &= \phi(\max\{\psi(d(Sz, Tw_{2n})), \psi(d(Az, Sz)), [\psi(d(Sz, Bw_{2n}))]/2\}).\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}\psi(d(Sz, Az)) &\le \phi(\max\{\psi(d(Sz, Az))), [\psi(d(Sz, Az))]/2\})\\ &= \phi(\psi(d(Sz, Az))), \quad \text{a contradiction.}\end{aligned}$$

Thus we have

$$Az = Sz. \tag{2.2.5}$$

Since $AX \subset TX$, there exists some w in X such that $Az = Tw$. Therefore, we have

$$Az = Sz = Tw. \tag{2.2.6}$$

Moreover, we show that $Az = Bw$. Suppose on the contrary that $Az \neq Bw$. Then, using (ii), we get

$$\begin{aligned}\psi(d(Az, Bw)) &\le \phi(M_\psi(z, w))\\ &= \phi(\max\{\psi(d(Sz, Tw)), \psi(d(Az, Sz)), \psi(d(Bw, Tw)),\\ &\qquad [\psi(d(Az, Tw)) + \psi(d(Bz, Sw))]/2\}),\\ &= \phi(\max\{\psi(d(Bw, Az)), [\psi(d(Bw, Az))]/2\}),\\ &= \phi(\psi(d(Bw, Az))), \quad \text{a contradiction.}\end{aligned}$$

Therefore, $Az = Bw$. Hence

$$Az = Sz = Tw = Bw. \tag{2.2.7}$$

Since A and S are weakly commuting, we have by (2.2.7), $ASz = SAz$ and hence

$$AAz = ASz = SAz = SSz \tag{2.2.8}$$

and by the weakly commuting property of B and T, we get

$$BBw = BTw = TBw = TTw. \tag{2.2.9}$$

We now show that $AAz = Az$. Suppose that $AAz \neq Az$ then by (ii), we get

$$\psi(d(Az, AAz)) = \psi(d(Bw, AAz)) \leq \phi(M_\psi(Az, w)) = \phi(\psi(d(Az, AAz))),$$

(using (2.2.7) & (2.2.8)), a contradiction. Hence $AAz = Az$. Also, we have $AAz = SAz$. Therefore, Az is a common fixed point of A and S. Again, suppose that $BBw \neq Bw$. Then using (ii), we get

$$\begin{aligned}\psi(d(Bw, BBw)) &= \psi(d(Az, BBw)) \quad \text{(by (2.2.6))}\\ &\leq \phi(M_\psi(z, Bw))\\ &= \phi(\psi(d(Bw, BBw))), \quad \text{(by using (2.2.7) \& (2.2.9))}\\ &< \psi(d(Bw, BBw)), \quad \text{a contradiction.}\end{aligned}$$

Hence $BBw = Bw$ and since $TBw = BBw$, we have Bw being a common fixed point for B and T. Finally, since $Az = Bw$, we have Az as a common fixed point for A, B, S and T. Moreover, the uniqueness of a common fixed point follows from (ii). This completes the proof of the theorem.

Remark. The proof is similar when the pair (A, S) is assumed ψ-compatible and S is continuous. Moreover, we can get the same result when the (B, T) is assumed ψ-compatible and either T or B is assumed to be continuous.

The following example shows that if A and S are not continuous in Theorem 2.2 then the result of Theorem 2.2 is not true. That is, all the mappings A, B, S and T do not have common fixed point.

Example 2.3. Let $X = [0, 1]$ with the Euclidean metric d. Define $A = B$ and $S = T : X \to X$ by the rule $A0 = 1/2$, $Ax = x/4$ for $0 < x \leq 1$ and $S0 = 1$, $Sx = x/2$ for $0 < x \leq 1$. Then A and S are weakly commuting mappings and hence they are ψ-compatible, with ψ being an identity mapping. Also, A and S satisfy both the conditions (i) and (ii) of Theorem 2.2 with $\phi(t) = t/2$. But A and S are not continuous and they do not have common fixed point.

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