TAMKANG JOURNAL OF MATHEMATICS
Volume 35, Number 2, Summer 2004

FOUR DIMENSIONAL CHARACTERIZATION OF BOUNDED DOUBLE SEQUENCES

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Abstract. In 1945 Brudno presented the following important theorem: If A and B are regular summability matrix methods such that every bounded sequence summed by A is also summed by B, then it is summed by B to the same value. R. G. Cooke suggested that a simpler proof would be desirable. Petersen presented such a proof. The goal of the paper is to present an accessible multidimensional analog of Brudno theorem for double sequences using four dimensional matrix transformations.

1. Introduction

In [4] Brudno presented an elegant, but inaccessible proof of the following theorem: If A and B are regular summability matrix methods such that every bounded sequence summed by A is also summed by B, then it is summed by B to the same value. Petersen presented an accessible proof of Brudno's theorem in [5]. The goal of this paper is to present an accessible multidimensional analog of Brudno theorem for double sequences using four dimensional matrices transformations.

2. Definitions, Notations and Preliminary Results

Definition 2.1. [Pringsheim, 1900] A double sequence $x = [x_{k,l}]$ has *Pringsheim limit* L (denoted by $P\text{-}\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k,\ l > N$. We shall describe such an x more briefly as "P-convergent".

Definition 2.2. A double sequence x is called *definite divergent*, if for every (arbitrarily large) $G > 0$ there exist two natural numbers n_1 and n_2 such that $|x_{n,k}| > G$ for $n \geq n_1$, $k \geq n_2$.

Definition 2.3. Let A denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the mn-th term to

Received September 24, 2002.
2000 *Mathematics Subject Classification.* Primary 40B05.
Key words and phrases. Double sequences, RH-regular.

Ax is as follow:

$$(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}.$$

Definition 2.4. A double sequence x is *divergent in the Pringsheim sense* (P-divergent) provided that x does not converge in the Pringsheim sense (P-convergent).

Definition 2.5. A double sequence x is bounded if and only if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l.

Definition 2.6. The four dimensional matrix A is said to be *RH-regular* if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

Theorem 2.1. (Hamilton 1936, Robison 1926) *The four dimensional matrix A is RH-regular if and only if*

RH_1: $P\text{-}\lim_{m,n} a_{m,n,k,l} = 0$ for each k and l;

RH_2: $P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;

RH_3: $P\text{-}\lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0$ for each l;

RH_4: $P\text{-}\lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0$ for each k;

RH_5: $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ is P-convergent; and

RH_6: there exist finite positive integers A and B such that $\sum_{k,l>B} |a_{m,n,k,l}| < A$.

3. Main Result

Theorem 3.1. *If two RH-regular summability matrices $A = [a_{m,n,k,l}]$ and $B = [b_{m,n,k,l}]$ sum a bounded double sequence $[s_{k,l}]$ to different sums, then there exists a bounded double sequnce which is summed by A but not summed by B.*

Proof. Without loss of generality we can choose $[s_{k,l}]$ such that

$$P\text{-}\lim_{m,n}(As)_{m,n} = 0 \quad \text{and} \quad P\text{-}\lim_{m,n}(Bs)_{m,n} = 1.$$

Let

$$\Phi_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} s_{k,l} \quad \text{and} \quad \prod_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} b_{m,n,k,l} s_{k,l}.$$

Since A and B are RH-regular matrices and are summed to 0 and 1 respectively, then there exists $\delta_{m,n} > 0$ with Pringsheim limit zero such that

$$|\Phi_{m,n}| \leq \delta_{m,n} \quad \text{and} \quad |\prod_{m,n} -1| \leq \delta_{m,n}.$$

Let

$$H := \max_{k,l} |s_{k,l}| \quad \text{and} \quad M := \max_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|\,|s_{k,l}|.$$

In addition, let us consider the following double sequence

$$h_{i,j} = \begin{cases} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \cdot \end{cases}$$

Let $[F_{k,l}]$ denote the following double partial sum $\sum_{i,j=1,1}^{k,l} h_{i,j}$. This definition implies that $[F_{k,l}]$ has the following properties: $|F_{k,k}| \leq 1$, $|F_{k,k} - F_{k,k+1}| = 0$, $|F_{k,k} - F_{k+1,k}| = 0$, and $|F_{k,k} - F_{k+1,k+1}| = \mu_{k,k}$ with $P\text{-}\lim_{k,k} \mu_{k,k} = 0$. By the RH-regularity conditions of A and B we can choose seven index sequences $[m_x]$, $[n_y]$, $[\alpha_m]$, $[\bar{\alpha}_m]$, $[\beta_n]$, $[\bar{\beta}_n]$ and $[\epsilon_{m,n}]$ with $[\alpha_m]$, $[\bar{\alpha}_m]$, $[\beta_n]$, and $[\bar{\beta}_n]$ being strictly increasing and $P\text{-}\lim_{m,n} \epsilon_{m,n} = 0$, such that first RH_1 grant us the following:

$$\sum_{k=1,l=1}^{\alpha_m-1,\beta_n-1} |a_{m,n,k,l}| < \epsilon_{m,n},$$

and

$$\sum_{k=1,l=1}^{\alpha_m-1,\beta_n-1} |b_{m,n,k,l}| < \epsilon_{m,n},$$

next RH_3 and RH_4 also grant us the following:

$$\sum_{k=1,l=\beta_n}^{\alpha_m-1,\infty} |a_{m,n,k,l}| < \epsilon_{m,n},$$

$$\sum_{k=\alpha_m,l=1}^{\infty,\beta_n-1} |a_{m,n,k,l}| < \epsilon_{m,n},$$

$$\sum_{k=\bar{\alpha}_m,l=\beta_n}^{\infty,\bar{\beta}_n-1} |a_{m,n,k,l}| < \epsilon_{m,n},$$

$$\sum_{k=\alpha_m,l=\bar{\beta}_n-1}^{\bar{\alpha}_m-1,\infty} |a_{m,n,k,l}| < \epsilon_{m,n},$$

$$\sum_{k=1,l=\beta_n}^{\alpha_m-1,\infty} |b_{m,n,k,l}| < \epsilon_{m,n},$$

$$\sum_{k=\alpha_m,l=1}^{\infty,\beta_n-1} |b_{m,n,k,l}| < \epsilon_{m,n},$$

$$\sum_{k=\bar{\alpha}_m,l=\beta_n}^{\infty,\bar{\beta}_n-1} |b_{m,n,k,l}| < \epsilon_{m,n},$$

and

$$\sum_{k=\alpha_m,l=\bar{\beta}_n-1}^{\bar{\alpha}_m-1,\infty} |b_{m,n,k,l}| < \epsilon_{m,n},$$

and finally RH_5 grant us the following:

$$\sum_{k=\bar{\alpha}_m-1,l=\beta_n-1}^{\infty,\infty} |a_{m,n,k,l}| < \epsilon_{m,n},$$

and

$$\sum_{k=\bar{\alpha}_m-1,l=\beta_n-1}^{\infty,\infty} |b_{m,n,k,l}| < \epsilon_{m,n}.$$

In addition, the index sequences $[\alpha_{m_x}]$, $[\bar{\alpha}_{m_x}]$, $[\beta_{n_y}]$, and $[\bar{\beta}_{n_y}]$ are chosen such that $\alpha_{m_x} = \bar{\alpha}_{m_{x-1}}$ and $\beta_{n_y} = \bar{\beta}_{n_{y-1}}$ for x, $y = 1, 2, 3, \ldots$. Let us consider the following double sequence

$$s'_{k,l} = \begin{cases} s_{k,l} & \text{for } k < \bar{\alpha}_{m_1} \ \& \ l < \bar{\beta}_{n_1} \\ F_{x,y} s_{k,l} & \text{for } \alpha_{m_x} \leq k \leq \bar{\alpha}_{m_x} \ \& \ \beta_{n_y} \leq l \leq \bar{\beta}_{n_y}. \end{cases}$$

For $m_x \leq m < m_{x+1}$ and $n_y \leq n < n_{y+1}$ the above definition of $[s'_{k,l}]$ grant us the following:

$$\begin{aligned}
\Phi'_{m,n} &= \sum_{m,n=1,1}^{\infty,\infty} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} s'_{k,l} \\
&= \sum_{k,l=1,1}^{\alpha_m-1,\beta_n-1} a_{m,n,k,l} s'_{k,l} + \sum_{\alpha_m \leq k < \infty \& 1 \leq l \leq \beta_n - 1} a_{m,n,k,l} s'_{k,l} \\
&\quad + \sum_{1 \leq k < \alpha_m - 1 \& \beta_n \leq 1 < \infty} a_{m,n,k,l} s'_{k,l} + F_{x,y} \sum_{\alpha_m \leq k < \bar{\alpha}_m \& \beta_n \leq l \leq \bar{\beta}_n} a_{m,n,k,l} s_{k,l} \\
&\quad + (F_{x+1,y} - F_{x,y}) \sum_{\alpha_{m_x} \leq k < \bar{\alpha}_m \& \beta_n \leq l \leq \bar{\beta}_{n_y}} a_{m,n,k,l} s_{k,l} \\
&\quad + (F_{x,y+1} - F_{x,y}) \sum_{\alpha_m \leq k < \bar{\alpha}_{m_x} \& \bar{\beta}_{n_y} \leq l \leq \bar{\beta}_{n_y}} a_{m,n,k,l} s_{k,l}
\end{aligned}$$

$$+(F_{x+1,y+1}-F_{x,y})\sum_{\bar{\alpha}_{m_x}\leq k<\bar{\alpha}_m\&\bar{\beta}_{n_y}\leq l\leq\bar{\beta}_n} a_{m,n,k,l}s_{k,l}$$
$$+\sum_{\bar{\alpha}_m\leq k<\infty\&\beta_n\leq l\leq\bar{\beta}_n} a_{m,n,k,l}s'_{k,l}+\sum_{\alpha_m\leq k<\bar{\alpha}_m\&\bar{\beta}_n\leq l\leq\infty} a_{m,n,k,l}s'_{k,l}$$
$$+\sum_{\bar{\alpha}_m<k<\infty\&\bar{\beta}_n\leq l\leq\infty} a_{m,n,k,l}s'_{k,l}.$$

The properties of absolute value grant us the following:

$$|\Phi'_{m,n}|\leq\sum_{k,l=1,1}^{\alpha_m-1,\beta_n-1}|a_{m,n,k,l}|\,|s'_{k,l}|+\sum_{\alpha_m\leq k<\infty\&1\leq l\leq\beta_n-1}|a_{m,n,k,l}|\,|s'_{k,l}|$$
$$+\sum_{1\leq k<\alpha_m-1\&\beta_n\leq 1<\infty}|a_{m,n,k,l}|\,|s'_{k,l}|+|F_{x,y}|\sum_{\alpha_m\leq k<\bar{\alpha}_m\&\beta_n\leq l\leq\bar{\beta}_n}|a_{m,n,k,l}|\,|s_{k,l}|$$
$$+|F_{x+1,y}-F_{x,y}|\sum_{\alpha_{m_x}\leq k<\bar{\alpha}_m\&\beta_n\leq l\leq\bar{\beta}_{n_y}}|a_{m,n,k,l}|\,|s_{k,l}|$$
$$+|F_{x,y+1}-F_{x,y}|\sum_{\alpha_m\leq k<\bar{\alpha}_{m_x}\&\bar{\beta}_{n_y}\leq l\leq\bar{\beta}_{n_y}}|a_{m,n,k,l}|\,|s_{k,l}|$$
$$+|F_{x+1,y+1}-F_{x,y}|\sum_{\bar{\alpha}_{m_x}\leq k<\bar{\alpha}_m\&\bar{\beta}_{n_y}\leq l\leq\bar{\beta}_n}|a_{m,n,k,l}|\,|s_{k,l}|$$
$$+\sum_{\bar{\alpha}_m\leq k<\infty\&\beta_n\leq l\leq\bar{\beta}_n}|a_{m,n,k,l}|\,|s'_{k,l}|+\sum_{\alpha_m\leq k<\bar{\alpha}_m\&\bar{\beta}_n\leq l\leq\infty}|a_{m,n,k,l}|\,|s'_{k,l}|$$
$$+\sum_{\bar{\alpha}_m<k<\infty\&\bar{\beta}_n\leq l\leq\infty}|a_{m,n,k,l}|\,|s'_{k,l}|.$$

The inequalities conditions above the properties of $[s'_{k,l}]$ grant us the following:

$$|\Phi'_{m,n}|\leq 6H\epsilon_{m,n}$$
$$+|F_{x,y}|\sum_{\alpha_m\leq k<\bar{\alpha}_m\&\beta_n\leq l\leq\bar{\beta}_n}|a_{m,n,k,l}|\,|s_{k,l}|$$
$$+|F_{x+1,y}-F_{x,y}|\sum_{\alpha_{m_x}\leq k<\bar{\alpha}_m\&\beta_n\leq l\leq\bar{\beta}_{n_y}}|a_{m,n,k,l}|\,|s_{k,l}|$$
$$+|F_{x,y+1}-F_{x,y}|\sum_{\alpha_m\leq k<\bar{\alpha}_{m_x}\&\bar{\beta}_{n_y}\leq l\leq\bar{\beta}_{n_y}}|a_{m,n,k,l}|\,|s_{k,l}|$$
$$+|F_{x+1,y+1}-F_{x,y}|\sum_{\bar{\alpha}_{m_x}\leq k<\bar{\alpha}_m\&\bar{\beta}_{n_y}\leq l\leq\bar{\beta}_n}|a_{m,n,k,l}|\,|s_{k,l}|.$$

Thus

$$|\Phi'_{m,n}| \leq 6H\epsilon_{m,n} + \delta_{m,n} + 3M\mu_{x,y}.$$

Hence $P\text{-}\lim_{m,n}(As')_{m,n} = 0$. Also, note that the above sequence $[h]$ has the following properties: if $x = y = p^2$ then $F_{x,y} = 1$ and if $x = y = p(p+1)$ then $F_{x,y} = 0$. In the expression $\prod$ above, let us perform the following substitution: let $l = k = p^2$, $m = m_x$, $n = n_y$, $a = b$, $\Phi'_{m,n} = \Phi'_{m,n} - 1$ if

$$F_{x,y} \sum_{\alpha_m \leq k < \bar{\alpha}_m \& \beta_n \leq l \leq \bar{\beta}_n} a_{m,n,k,l} s_{k,l} = F_{p^2,p^2} \sum_{\alpha_m \leq k < \bar{\alpha}_m \& \beta_n \leq l \leq \bar{\beta}_n} b_{m,n,k,l} s_{k,l} - 1.$$

Then $\lim_p \Phi'_{p^2,p^2} = 1$. On the other hand if

$$F_{x,y} \sum_{\alpha_m \leq k < \bar{\alpha}_m \& \beta_n \leq l \leq \bar{\beta}_n} a_{m,n,k,l} s_{k,l} = F_{p(p+1),p(p+1)} \sum_{\alpha_m \leq k < \bar{\alpha}_m \& \beta_n \leq l \leq \bar{\beta}_n} b_{m,n,k,l} s_{k,l}$$

the $\lim_p \Phi'_{p(p+1),p(p+1)} = 0$. Thus s' is not B summable. This completes the proof of the theorem.

The above theorem grants us the following multidimensional analog of Brudno in [4].

Theorem 3.2. *Let every bounded double sequence summable by a RH-regular summability matrix A also be summable by a RH-regular summability matrix B. Then it is summable to the same value by B as by A.*

Proof. This theorem follow directly from the above theorem.

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