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GROWTH OF COMPOSITE ENTIRE FUNCTIONS

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Abstract. The growth of maximum term of a composite entire function is compared with that of the maximum term of its left and right factors.

1. Introduction.

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be an entire function. Then as usual $\mu(r,f)=\max_{n\geq 0}|a_n|r^n$ is called the maximum term of $f(z)$ on $|z|=r$ and $M(r,f)=\max_{|z|=r}|f(z)|$ is called the maximum modulus of $f(z)$ on $|z|=r$.

The numbers $\rho_f(p,q)$ and $\lambda_f(p,q)$ are, respectively, called the (p,q)-order and lower (p,q)-order of $f(z)$ having index-pair (p,q) and are defined as [1]:

$$\lim_{r\to\infty}\frac{\sup \log^{[p]} M(r,f)}{\inf \log^{[q]} r}=\begin{matrix}\rho_f\equiv\rho_f(p,q)\\ \lambda_f\equiv\lambda_f(p,q)\end{matrix}, \tag{1.1}$$

where p and q are integers such that $p\geq q\geq 1$, $\log^{[0]}x=x$, and $\log^{[n]}x=\log(\log^{[n-1]}x)$ for $0<\log^{[n-1]}x<\infty$, $n=1,2,3,\ldots$.

Some theorems that will be of use to us are:

Theorem A. (Singh [2]). *For $0\leq r<R$, we have*

$$\mu(r,f)\leq M(r,f)\leq\frac{R}{R-r}\mu(R,f). \tag{1.2}$$

Theorem B. (Juneja, Kapoor and Bajpai [1]). *If $f(z)$ is an entire function then*

$$\lim_{r\to\infty}\frac{\sup \log^{[p]} \mu(r,f)}{\inf \log^{[q]} r}=\begin{matrix}\rho_f\equiv\rho_f(p,q)\\ \lambda_f\equiv\lambda_f(p,q)\end{matrix}. \tag{1.3}$$

Definition 1. Let $g(z)$ be an entire function of finite lower (p,q)-order λ_g. A function $\lambda_g(r)$ is called a lower proximate (p,q)-order of $g(z)$ relative to $\mu(r,g)$ if (i) $\lambda_g(r)$ is real,

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continuous and piecewise differentiable for sufficiently large values of $r \geq r_0$,

$$\text{(ii)} \ \lim_{r\to\infty} \lambda_g(r) = \lambda_g,$$
$$\text{(iii)} \ \lim_{r\to\infty} \wedge_{[q]}(r)\lambda_g'(r) = 0 \quad \text{and} \tag{1.4}$$
$$\text{(iv)} \ \liminf_{r\to\infty} \frac{\log^{[p-1]}\mu(r,g)}{(\log^{[q-1]} r)^{\lambda_g(r)}} = 1,$$

where $\wedge_{[q]}(r) = \prod_{i=0}^{q} \log^{[i]} r$.

The purpose of this paper is to compare the maximum term of a composite entire function with that of its left and right factors. Throughout this paper $f(z)$, $g(z)$ and $h(z)$ will stand for entire functions.

2. Main Results

Firstly, in some theorems we will compare the growth of the maximum term of a composite entire function with that of its left factor.

Theorem 1. *If ρ_f, ρ_g are finite and $\lambda_f > 0$ then for $x > \frac{\rho_g}{\lambda_f} - 1$ and $p > q$,*

$$\lim_{r\to\infty} \frac{\log^{[p]}\mu(r,fog)}{(\log^{[q]}\mu(r,f))^{1+x}} = 0.$$

Proof. Let $x > \frac{\rho_g}{\lambda_f} - 1$ and $0 < \varepsilon < \min\{\lambda_f, \frac{(1+x)\lambda_f - \rho_g}{x+2}\}$. Then in view of (1.3) it follows that for all sufficiently large values of r,

$$\mu(r,f) < \exp^{[p-1]}((\log^{[q-1]} r)^{\rho_f+\varepsilon}) \tag{2.1}$$

and

$$\mu(r,f) > \exp^{[p-1]}((\log^{[q-1]} r)^{\rho_f-\varepsilon}). \tag{2.2}$$

Now, from Lemma 1 [2] for all sufficiently large values of r,

$$\log\mu(r,fog) \leq 2\log\mu(4\mu(2r,g),f),$$

or,
$$\log^{[p-1]}\mu(r,fog) \leq 2\log^{[p-1]}\mu(4\mu(2r,g),f),$$

or,
$$\begin{aligned}\log^{[p]}\mu(r,fog) &< \log 2 + (\rho_f+\varepsilon)\log^{[q]}(4\mu(2r,g)) \\ &= \log 2 + (\rho_f+\varepsilon)\log^{[q]}\mu(2r,g) + o(1).\end{aligned} \tag{2.3}$$

Using (2.1), we have

$$\log^{[p]}\mu(r,fog) < \log 2 + (\rho_f+\varepsilon)\exp^{[p-q-1]}((\log^{[q-1]}(2r))^{\rho_g+\varepsilon}) + o(1). \tag{2.4}$$

Also from (2.2), we have

$$(\log^{[p]}\mu(r))^{1+x} > \{\exp^{[p-q-1]}(\log^{[q-1]} r)^{\lambda_f-\varepsilon}\}^{1+x}. \tag{2.5}$$

So for all sufficiently large values of r,

$$\frac{\log^{[p]}\mu(r,fog)}{(\log^{[q]}\mu(r))^{1+x}} < \frac{\log 2+(\rho_f+\varepsilon)\exp^{[p-q-1]}((\log^{[q-1]}2r)^{\rho_g+\varepsilon})+o(1)}{(\exp^{[p-q-1]}((\log^{[q-1]}r)^{\lambda_f-\varepsilon}))^{1+x}}$$

which implies that

$$\lim_{r\to\infty}\frac{\log^{[p]}\mu(r,fog)}{(\log^{[q]}\mu(r))^{1+x}}=0.$$

Theorem 2. *If ρ_f, ρ_g, λ_f, λ_g are finite and $\lambda_f>0$, then*

$$\limsup_{r\to\infty}\frac{\log^{[p]}\mu(r,fog)}{(\log^{[q]}\mu(r,f))^{1+x}}=\infty, \quad \text{where } x<\max\left\{\frac{\lambda_g}{\lambda_f}-1,\frac{\rho_g}{\rho_f}-1\right\} \quad \text{and} \quad p>q.$$

Proof. Let $x<\frac{\lambda_g}{\lambda_f}-1$ and $\varepsilon>0$ be such that $\varepsilon<\lambda_f$, if $2+x\le 0$ and $\varepsilon<\min\{\lambda_f,(\lambda_g-(1+x)\lambda_f)/(2+x)\}$ if $2+x>0$.

For all sufficiently large values of r, we get from Lemma 2 [2],

$$\begin{aligned}\log\mu(r,fog) &\ge \log\frac{1}{2}+\log\mu\left[\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|,f\right]\\ &\ge \frac{1}{2}\log\mu\left[\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|,f\right],\end{aligned}$$

or,

$$\log^{[p]}\mu(r,fog)\ge\frac{1}{2}\log^{[p]}\mu\left[\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|,f\right].$$

Using (1.3), we have,

$$\begin{aligned}\log^{[p]}\mu(r,fog) &> \frac{1}{2}(\lambda_f-\varepsilon)\log^{[q]}\left[\frac{1}{8}\mu\left(\frac{r}{4},g\right)\right]\\ &= \frac{1}{2}(\lambda_f-\varepsilon)\log^{[q]}\mu\left(\frac{r}{4},g\right)+o(1)\\ &> \frac{1}{2}(\lambda_f-\varepsilon)\exp^{[p-q-1]}\left(\left(\log^{[q-1]}\left(\frac{r}{4}\right)\right)^{\lambda_g-\varepsilon}\right)+o(1).\end{aligned} \tag{2.6}$$

Also, for a sequence of values of r tending infinity, we have

$$\log^{[q]}\mu(r,f)<\exp^{[p-q-1]}\left(\left(\log^{[q-1]}r\right)^{\lambda_f+\varepsilon}\right). \tag{2.7}$$

From (2.6) and (2.7), we get

$$\frac{\log^{[p]}\mu(r,fog)}{(\log^{[q]}\mu(r,f))^{1+x}} > \frac{\frac{1}{2}(\lambda_f-\varepsilon)\exp^{[p-q-1]}((\log^{[q-1]}(r/4))^{\lambda_g-\varepsilon})+o(1)}{(\exp^{[p-q-1]}((\log^{[q-1]}r)^{\lambda_f+\varepsilon}))^{1+x}} \tag{2.8}$$

for a sequence of values of r tending to infinity. This gives

$$\limsup_{r\to\infty} \frac{\log^{[p]} \mu(r, fog)}{(\log^{[q]} \mu(r, f))^{1+x}} = \infty.$$

We omit the proof for $x < \frac{\rho_g}{\rho_f} - 1$ because it runs parallel to that of the case for $x < \frac{\lambda_g}{\lambda_f} - 1$. This completes the proof of the theorem.

Theorem 3. *If ρ_f and ρ_g are finite, $p > q$, $\lambda_f > 0$ and either $\lambda_f = \rho_f$, or $\lambda_g = \rho_g$, or both, then*

$$T(x) = \limsup_{r\to\infty} \frac{\log^{[p]} \mu(r, fog)}{(\log^{[q]} \mu(r, f))^{1+x}},$$

has a jumped discontinuity with an infinite jump from zero to infinity at $x = \frac{\rho_g}{\lambda_f} - 1$.

Proof. Since under the conditions of the theorem $\frac{\rho_g}{\lambda_f} - 1 = \max\{\frac{\rho_g}{\rho_f} - 1, \frac{\lambda_g}{\lambda_f} - 1\}$, the theorem follows from Theorem 1 and Theorem 2.

Theorem 4. *If ρ_f, ρ_g are finite, $\lambda_f > 0$ and $\lambda_g \rho_f < \lambda_f \rho_g$, then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \mu(r, fog)}{(\log^{[q]} \mu(r, f))^{1+x}} = 0 \tag{2.9}$$

and

$$\limsup_{r\to\infty} \frac{\log^{[p]} \mu(r, fog)}{(\log^{[q]} \mu(r, f))^{1+x}} = \infty, \tag{2.10}$$

for any x, with $\frac{\lambda_g}{\lambda_f} - 1 < x < \frac{\rho_g}{\rho_f} - 1$ and so the corresponding limit does not exist.

Proof. Let $x > \frac{\lambda_g}{\lambda_f} - 1$ and $0 < \varepsilon < \min\{\lambda_f, \frac{(1+x)\lambda_f - \lambda_g}{x+2}\}$. From (2.3) and (1.3) we get for all sufficiently large values of r,

$$\log^{[p]} \mu(r, fog) < \log 2 + (\rho_f + \varepsilon) + (\exp^{[p-q-1]}((\log^{[q-1]}(2r))^{\lambda_g - \varepsilon})) + o(1). \tag{2.11}$$

Dividing (2.11) by (2.5) and taking limit infimum, we get

$$\liminf_{r\to\infty} \frac{\log^{[p]} \mu(r, fog)}{(\log^{[q]} \mu(r, f))^{1+x}} = 0.$$

Since under the given conditions $\frac{\rho_g}{\rho_f} - 1 = \max[\frac{\lambda_g}{\lambda_f} - 1, \frac{\rho_g}{\rho_f} - 1] > \frac{\lambda_g}{\lambda_f} - 1$, it follows from Theorem 2 that

$$\limsup_{r\to\infty} \frac{\log^{[p]} \mu(r, fog)}{(\log^{[q]} \mu(r, f))^{1+x}} = \infty.$$

Hence the corresponding limit does not exist.

Corollary 1. *If* $\lambda_g < \lambda_f \leq \rho_f < \rho_g < \infty$, *then*

$$\liminf_{r\to\infty} \frac{\log^{[p]} \mu(r, fog)}{(\log^{[q]} \mu(r, f))} = 0 \quad \textit{and} \quad \limsup_{r\to\infty} \frac{\log^{[p]} \mu(r, fog)}{(\log^{[q]} \mu(r, f))} = \infty$$

and so the corresponding limit does not exist.

3. In this section we shall compare the growth of the maximum term of a composite entire function with that of its right factor. In first three theorems of this section we use the following definition:

Definition 2. For the entire functions $f(z)$ and $g(z)$, we define

$$A(x) = \limsup_{r\to\infty} \frac{\log^{[p]} \mu(r, fog)}{\log^{[q]} \mu((1 + x)r, g)}, \quad \text{for } x \geq 0 \text{ and } p > q.$$

Obviously $A(x)$ is a non-increasing function of x.

Theorem 5. *If* $\rho_g < \infty$, *then* $A(0) \leq \rho_f$.

Proof. Since for $\rho_f = \infty$ the result is trivially true, we suppose that $\rho_f < \infty$. By the maximum modulus principle, we have

$$M(r, fog) \leq M(M(r, g), f). \tag{3.1}$$

(1.2) and (3.1) give,

$$\log^{[p]} \mu(r, fog) \leq \log^{[p]} M(M(r, g), f).$$

Thus, for given $\varepsilon > 0$, we get for all sufficiently large values of r,

$$\log^{[p]} \mu(r, fog) \leq (\rho_f + \varepsilon) \log^{[q]} M(r, g). \tag{3.2}$$

Since $\rho_g < \infty$, $\lim_{r\to\infty} \frac{\log^{[q]} M(r,g)}{\log^{[q]} \mu(r,g)} = 1$ by [3], so that for all sufficiently large values of r,

$$\log^{[q]} M(r, g) < (1 + \varepsilon) \log^{[q]} \mu(r, g). \tag{3.3}$$

Therefore, from (3.2) and (3.3), we get for all sufficiently large values of r,

$$\frac{\log^{[p]} \mu(r, fog)}{\log^{[q]} \mu(r, g)} < (1 + \varepsilon)(\rho_f + \varepsilon).$$

From which the theorem follows because $\varepsilon > 0$ is arbitrary.

Theorem 6. $\lim_{x\to 0^+} A(x) \leq \rho_f$.

Proof. Since for $\rho_f = \infty$ the result is trivially true, we suppose that $\rho_f < \infty$.

Putting $R = (1+x)r,\ x > 0$, in (1.2), we get

$$\mu(r,g) \le \left(1 + \frac{1}{x}\right)\mu((1+x)r,g),$$

or,
$$\log \mu(r,g) \le \log\left(1+\frac{1}{x}\right) + \log \mu((1+x)r,g),$$

or,
$$\log^{[q]} \mu(r,g) \le \log^{[q]} \mu((1+x)r,g) + o(1). \tag{3.4}$$

From (3.2), (3.3) and (3.4), we get

$$\log^{[p]} \mu(r,fog) < (1+\varepsilon)(\rho_f+\varepsilon)(\log^{[q]} \mu((1+x)r,g) + o(1))$$

for all sufficiently large values of r.

Since $g(z)$ is non-constant and $\varepsilon > 0$ is arbitrary, it follows from above that $A(x) \le \rho_f$ for every $x > 0$. Also since $A(x)$ is a non-increasing function of x, $\lim_{x\to 0^+} A(x)$ exists and $\lim_{x\to 0^+} A(x) \le \rho_f$.

Theorem 7. *If* $\sup_{r>0} \frac{\log^{[p]} \mu(r,fog)}{\log^{[q]} \mu((1+x)r,g)}$ *is not attained for any* $x \ge 0$ *and* $p > q$*, then* $A(0) \le \rho_f$.

Proof. Let $B(x) = \sup_{r>0} \frac{\log^{[p]} \mu(r,fog)}{\log^{[q]} \mu((1+x)r,g)}$ for $x \ge 0$. Since $B(x)$ is not attained, for each $x \ge 0$ there exists a sequence $\{r_n\}$, $n = 1,2,3,\ldots$ tending to infinity such that

$$B(x) - \frac{1}{n} < \frac{\log^{[p]} \mu(r_n,fog)}{\log^{[q]} \mu((1+x)r_n,g)},$$

which implies that $B(x) \le A(x)$ and so $B(x) = A(x)$ for all $x \ge 0$ because $B(x) \ge A(x)$ follows easily from the definitions.

Now, for given $\varepsilon > 0$ there exists a $\xi > 0$ such that

$$A(0) - \varepsilon = B(0) - \varepsilon < \frac{\log^{[p]} \mu(\xi,fog)}{\log^{[q]} \mu(\xi,g)}. \tag{3.5}$$

Also,
$$\lim_{x\to 0^+} \frac{\log^{[p]} \mu(\xi,fog)}{\log^{[q]} \mu((1+x)\xi,g)} = \frac{\log^{[p]} \mu(\xi,fog)}{\log^{[q]} \mu(\xi,g)},$$

so there exists $x_1 > 0$ such that

$$\frac{\log^{[p]} \mu(\xi,fog)}{\log^{[q]} \mu(\xi,g)} < \frac{\log^{[p]} \mu(\xi,fog)}{\log^{[q]} \mu((1+x_1)\xi,g)} + \varepsilon.$$

Therefore, from (3.5) we get

$$A(0) - \varepsilon < \frac{\log^{[p]} \mu(\xi,fog)}{\log^{[q]} \mu((1+x_1)\xi,g)} + \varepsilon \le B(x_1) + \varepsilon = A(x_1) + \varepsilon \le \lim_{x\to 0^+} A(x) + \varepsilon.$$

Since ε is arbitrary, the theorem follows from Theorem 6.

Theorem 8. *If ρ_f and λ_g are finite, then*

$$\liminf_{r\to\infty} \frac{\log^{[p-q]}(\log^{[p-1]}\mu(r,fog))^{1/\rho_f}}{\log^{[p-1]}\mu(r,g)} \leq \begin{matrix} 2^{\lambda_g}, & \text{if } (p,q)=(2,1) \\ 1, & \text{if } (p,q)\neq(2.1) \end{matrix}.$$

Proof. From (1.2) and (3.1), we have

$$\log^{[p]}\mu(r,fog) \leq \log^{[p]} M(M(r,g),f). \tag{3.6}$$

Also, from (1.1) for all sufficiently large values of r and for any given $\varepsilon > 0$,

$$\log^{[p]} M(r,f) < (\rho_f+\varepsilon)\log^{[q]} r. \tag{3.7}$$

(3.6) and (3.7) give

$$\log^{[p-q]}(\log^{[p-1]}\mu(r,fog))^{1/(\rho_f+\varepsilon)} < \log^{[p-1]} M(r,g)$$

for all sufficiently large values of r. This implies that

$$\liminf_{r\to\infty} \frac{\log^{[p-q]}(\log^{[p-1]}\mu(r,fog))^{1/(\rho_f+\varepsilon)}}{\log^{[p-1]}\mu(r,g)} \leq \liminf_{r\to\infty} \frac{\log^{[p-1]} M(r,g)}{\log^{[p-1]}\mu(r,g)}. \tag{3.8}$$

Now, for a sequence of value of r tending to infinity, (1.2) and (1.4) give,

$$\begin{aligned} \log^{[p-1]} M(r,g) &\leq \log^{[p-1]}\mu(2r,g) + o(1) \\ &\leq (1+\varepsilon)(\log^{[q-1]}(2r))^{\lambda_g(r)} + o(1) \\ &= (1+\varepsilon)\frac{(\log^{[q-1]}(2r))^{\lambda_g+\delta}}{(\log^{[q-1]}(2r))^{\lambda_g+\delta-\lambda_g(r)}} + o(1), \end{aligned} \tag{3.9}$$

where $\delta > 0$ is arbitrary.

Since

$$\frac{d}{dr}\{(\log^{[q-1]} r)^{\lambda_g+\delta-\lambda_g(r)}\} = \{\lambda_g+\delta-\lambda_g(r) - \lambda_g'(r)\wedge_{[q]}(r)\}\frac{(\log^{[q-1]} r)^{\lambda_g+\delta-\lambda_g(r)}}{\wedge_{[q-1]}(r)} > 0$$

for all sufficiently large values of r and $\delta > 0$. This implies that $(\log^{[q-1]} r)^{\lambda_g+\delta-\lambda_g(r)}$ is an increasing function of r. Therefore, for a sequence of values of r tending to infinity, (3.9) gives

$$\begin{aligned} \log^{[p-1]} M(r,g) &< (1+\varepsilon)\frac{(\log^{[q-1]}(2r))^{\lambda_g+\delta}}{(\log^{[q-1]} r)^{\lambda_g+\delta-\lambda_g(r)}} \\ &\leq (1+\varepsilon)\frac{(\log^{[q-1]} r)^{\lambda_g+\delta}(1+L_{q-1}(r))^{\lambda_g+\delta}}{(\log^{[q-1]} r)^{\lambda_g+\delta-\lambda_g(r)}} + o(1) \\ &= (1+\varepsilon)(\log^{[q-1]} r)^{\lambda_g(r)}(1+L_{q-1}(r))^{\lambda_g+\delta} + o(1), \end{aligned}$$

where $L_0(r) = 1$, $L_1(r) = \frac{\log 2}{\log r}$ and $L_{q-1}(r) = \{\log(1 + L_{q-2}(r))\}/(\log^{[q-1]} r)$, $q = 3, 4, 5, \ldots$.

Again, for all sufficiently large values of r, (1.4) gives

$$\log^{[p-1]} \mu(r, g) > (1 - \varepsilon)(\log^{[q-1]} r)^{\lambda_g(r)}.$$

Therefore, for a sequence of values of r tending to infinity, we find

$$\log^{[p-1]} M(r, g) < \frac{1 + \varepsilon}{1 - \varepsilon}(1 + L_{q-1}(r))^{\lambda_g+\delta} \log^{[p-1]} \mu(r, g) + o(1). \tag{3.10}$$

Since ε and δ are arbitrary, the theorem follows from (3.8) and (3.10).

Now, we study the growth of the maximum term of two composite entire functions.

Theorem 9. *If ρ_h, ρ_g and λ_f are finite, then*

$$\lim_{r\to\infty} \frac{(\log^{[p-2]} \mu(r, hog))^{(\log^{[q-1]} M(r,g))^x}}{\log^{[p-2]} \mu(r, fog)} = 0, \quad \text{for } x < \lambda_f - \rho_h.$$

Proof. Let $x < \lambda_f - \rho_h$ and $0 < \varepsilon < (\lambda_f - \rho_h - x)/2$. From (1.2), we have, for all sufficiently large values of r,

$$\begin{aligned} \mu(r, hog) &\le M(r, hog) \\ &\le M(M(r, g), h) \\ &< \exp^{[p-1]}((\log^{[q-1]} M(r, g))^{\rho_h+\varepsilon}). \end{aligned}$$

or,

$$\log^{[p-2]} \mu(r, hog) < \exp((\log^{[q-1]} M(r, g))^{\rho_h+\varepsilon}). \tag{3.11}$$

Also, we can easily prove that for all sufficiently large values of r,

$$\log^{[p-2]} \mu(r, fog) > \exp((\log^{[q-1]} M(r, g))^{\lambda_f-\varepsilon}. \tag{3.12}$$

From (3.11) and (3.12) for all sufficiently large values of r, we get

$$\begin{aligned} \frac{(\log^{[p-2]} \mu(r, hog))^{(\log^{[q-1]} M(r,g))^x}}{\log^{[p-2]} \mu(r, fog)} &< \frac{\exp((\log^{[q-1]} M(r, g))^{\rho_h+\varepsilon+x})}{\exp((\log^{[q-1]} M(r, g))^{\lambda_f-\varepsilon})} \\ &= \frac{1}{\exp((\log^{[q-1]} M(r, g))^{\lambda_f-\rho_h-x-2\varepsilon})}. \end{aligned}$$

Since $\lambda_f - \rho_h - x - 2\varepsilon > 0$ and $g(z)$ is non-constant,

$$\lim_{r\to\infty} \frac{(\log^{[p-2]} \mu(r, hog))^{(\log^{[q-1]} M(r,g))^x}}{\log^{[p-2]} \mu(r, fog)} = 0.$$

Corollary 2. *Using* (1.2) *we get under the assumptions of Theorem* 9 *that*

$$\lim_{r\to\infty} \frac{(\log^{[p-2]} \mu(r, hog))^{(\log^{[q-1]} \mu(r,g))^x}}{\log^{[p-2]} \mu(r, fog)} = 0, \quad \text{for } x < \lambda_f - \rho_h.$$

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