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A NEW GENERAL IDEA FOR STARLIKE AND CONVEX FUNCTIONS

SHIGEYOSHI OWA, H. M. SRIVASTAVA, TOSHIO HAYAMI AND KAZUO KUROKI

Abstract. Let $\mathscr{A}$ be the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U}$ with $f(0)=0$ and $f'(0)=1$. For the class $\mathscr{A}$, a new general class $\mathscr{A}_k$ is defined. With this general class $\mathscr{A}_k$, two interesting classes $\mathscr{S}_k^*(\alpha)$ and $\mathscr{K}_k(\alpha)$ concerning classes of starlike of order α in $\mathbb{U}$ and convex of order α in $\mathbb{U}$ are considered.

1. Introduction

Let $\mathscr{A}$ be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ and $f(0)=f'(0)-1=0$. If $f(z)\in\mathscr{A}$ satisfies $f(z_1)\neq f(z_2)$ for any $z_1\in\mathbb{U}$ and $z_2\in\mathbb{U}$ with $z_1\neq z_2$, then $f(z)$ is said to be univalent in $\mathbb{U}$ and denoted by $f(z)\in\mathscr{S}$. If $f(z)\in\mathscr{A}$ satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha \qquad (z\in\mathbb{U}) \tag{1.2}$$

for some real α $(0\leqq\alpha<1)$, then we say that $f(z)$ is starlike of order α in $\mathbb{U}$ and denoted by $f(z)\in\mathscr{S}^*(\alpha)$. Further, if $f(z)\in\mathscr{A}$ satisfies the following inequality:

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (z\in\mathbb{U}) \tag{1.3}$$

for some real α $(0\leqq\alpha<1)$, then $f(z)$ is said to be convex of order α in $\mathbb{U}$. We also write $f(z)\in\mathscr{K}(\alpha)$ for convex functions $f(z)$ of order α in $\mathbb{U}$ (see, for details, [1], [2], [5], [6] and [7]). In the literature on Geometric Function Theory in Complex Analysis, there are many interesting results for univalent functions, starlike functions and convex functions (see, for example, [3], [4] and [8]).

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Corresponding author: Shigeyoshi Owa.

In view of the definitions for the function classes $\mathcal{S}$, $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, it is known that

$$\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S} \qquad \text{and} \qquad f(z) \in \mathcal{K}(\alpha) \iff zf'(z) \in \mathcal{S}^*(\alpha)$$

and

$$f(z) \in \mathcal{S}^*(\alpha) \iff \int_0^z \frac{f(t)}{t}\,dt \in \mathcal{K}(\alpha).$$

It is well known that the Koebe function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} nz^n \tag{1.4}$$

is the extremal function for the class $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and that a function $f(z)$ given by

$$f(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \tag{1.5}$$

is the extremal function for the class $\mathcal{K}(0) \equiv \mathcal{K}$.

Taking the principal value for $\sqrt{z}$, we consider a function $f(z)$ given by

$$f(z) = \frac{z}{\left(1-\sqrt{z}\right)^2} = z + \sum_{n=1}^{\infty} (n+1) z^{1+\frac{n}{2}} \qquad (z \in \mathbb{U}). \tag{1.6}$$

Then we find that

$$\Re\left(\frac{zf'(z)}{f(z)}\right) = \Re\left(\frac{1}{1-\sqrt{z}}\right) > \frac{1}{2} \qquad (z \in \mathbb{U}), \tag{1.7}$$

that is, that $f(z)$ is starlike of order $\frac{1}{2}$ in $\mathbb{U}$. Also, if we consider a function given by

$$f(z) = \frac{z\left(2-\sqrt{z}\right)}{2\left(1-\sqrt{z}\right)^2} = z + \sum_{n=1}^{\infty} \left(1+\frac{n}{2}\right) z^{1+\frac{n}{2}} \qquad (z \in \mathbb{U}), \tag{1.8}$$

then $f(z)$ satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) = \Re\left(\frac{4-3\sqrt{z}+z}{2\left(2-\sqrt{z}\right)\left(1-\sqrt{z}\right)}\right) > 0 \qquad (z \in \mathbb{U}), \tag{1.9}$$

which imples that $f(z)$ is starlike in $\mathbb{U}$.

Furthermore, if we take a function given by

$$f(z) = \frac{z}{1-\sqrt{z}} = z + \sum_{n=1}^{\infty} z^{1+\frac{n}{2}} \qquad (z \in \mathbb{U}), \tag{1.10}$$

then $f(z)$ satisfies the following inequalities:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) = \Re\left(\frac{2-\sqrt{z}}{2\left(1-\sqrt{z}\right)}\right) > \frac{3}{4} \qquad (z \in \mathbb{U}), \tag{1.11}$$

so that $f(z)$ is starlike of order $\dfrac{3}{4}$ in $\mathbb{U}$, and

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)=\Re\left(\frac{4-3\sqrt{z}+z}{2\left(2-\sqrt{z}\right)\left(1-\sqrt{z}\right)}\right)>0 \qquad (z\in\mathbb{U}), \tag{1.12}$$

which imples that $f(z)$ is convex in $\mathbb{U}$.

In view of the above observations, we introduce the general function classes $\mathscr{A}_k$ $(k=1,2,3,\dots)$ as follows.

Let $\mathscr{A}_k$ be the class of functions $f(z)$ given by

$$f(z)=z+\sum_{n=1}^{\infty}a_{1+\frac{n}{k}}z^{1+\frac{n}{k}} \qquad (k=1,2,3,\dots), \tag{1.13}$$

which are analytic in the *punctured* open unit disk

$$\mathbb{U}_0=\mathbb{U}\setminus\{0\}=\{z: z\in\mathbb{C} \quad \text{and} \quad 0<|z|<1\},$$

where we consider the principal value for $z^{\frac{1}{k}}$.

If $f(z)\in\mathscr{A}_k$ satisfies the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha \qquad (z\in\mathbb{U}) \tag{1.14}$$

for some real α $(0\leqq\alpha<1)$, then we say that $f(z)\in\mathscr{S}_k^*(\alpha)$. Further, if $f(z)\in\mathscr{A}_k$ satisfies the following inequality:

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (z\in\mathbb{U}) \tag{1.15}$$

for some real α $(0\leqq\alpha<1)$, then we write $f(z)\in\mathscr{K}_k(\alpha)$. With the above definitions, we see that

$$f(z)\in\mathscr{K}_k(\alpha)\Longleftrightarrow zf'(z)\in\mathscr{S}_k^*(\alpha)$$

and

$$f(z)\in\mathscr{S}_k^*(\alpha)\Longleftrightarrow\int_0^z\frac{f(t)}{t}\,dt\in\mathscr{K}_k(\alpha).$$

2. Coefficient inequalities

First of all, for the above-defined new general function classes, we consider the coefficient inequalities for functions in $\mathscr{S}_k^*(\alpha)$ and $\mathscr{K}_k(\alpha)$.

Theorem 1. *If $f(z) \in \mathscr{A}_k$ satisfies the following inequality:*

$$\sum_{n=1}^{\infty}\left(\frac{n}{k}+1-\alpha\right)\left|a_{1+\frac{n}{k}}\right| \leqq 1-\alpha \tag{2.1}$$

for some real α $(0 \leqq \alpha < 1)$, then $f(z) \in \mathscr{S}_k^*(\alpha)$. The equality in (2.1) is attained for

$$f(z) = z + \sum_{n=1}^{\infty}\frac{(1-\alpha)k\varepsilon}{n(n+1)(n+(1-\alpha)k)}z^{1+\frac{n}{k}} \qquad (|\varepsilon| = 1). \tag{2.2}$$

Proof. It follows that the function $f(z) \in \mathscr{S}_k^*(\alpha)$ when $f(z) \in \mathscr{A}_k$ satisfies the following inequality:

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\alpha \qquad (z \in \mathbb{U}). \tag{2.3}$$

Indeed, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{\sum_{n=1}^{\infty}\frac{n}{k}a_{1+\frac{n}{k}}z^{\frac{n}{k}}}{1+\sum_{n=1}^{\infty}a_{1+\frac{n}{k}}z^{\frac{n}{k}}}\right| < \frac{\sum_{n=1}^{\infty}\frac{n}{k}\left|a_{1+\frac{n}{k}}\right|}{1-\sum_{n=1}^{\infty}\left|a_{1+\frac{n}{k}}\right|} \leqq 1-\alpha \tag{2.4}$$

if $f(z)$ satisfies the following condition:

$$\sum_{n=1}^{\infty}\frac{n}{k}\left|a_{1+\frac{n}{k}}\right| \leqq (1-\alpha)\left(1-\sum_{n=1}^{\infty}\left|a_{1+\frac{n}{k}}\right|\right), \tag{2.5}$$

which is equivalent to the inequality (2.1). Further, we consider the function $f(z) \in \mathscr{A}_k$ which satisfies the following condition:

$$\sum_{n=1}^{\infty}\left(\frac{n}{k}+1-\alpha\right)\left|a_{1+\frac{n}{k}}\right| = (1-\alpha)\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\alpha. \tag{2.6}$$

This yields

$$\left(\frac{n}{k}+1-\alpha\right)\left|a_{1+\frac{n}{k}}\right| = \frac{1-\alpha}{n(n+1)} \tag{2.7}$$

for all $n \geqq 1$. Therefore, we have

$$a_{1+\frac{n}{k}} = \frac{(1-\alpha)k\varepsilon}{n(n+1)(n+(1-\alpha)k)} \qquad (|\varepsilon| = 1), \tag{2.8}$$

which shows us that the function $f(z)$ given by (2.2) satisfies the equality in (2.1). ☐

Taking $k = 1$ in Theorem 1, we have the following corollary.

Corollary 1. *If $f(z) \in \mathscr{A}$ satisfies the following inequality:*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leqq 1-\alpha \tag{2.9}$$

for some real α $(0 \leqq \alpha < 1)$, *then* $f(z) \in \mathcal{S}^*(\alpha)$. *The inequality in* (2.9) *is attained for the function* $f(z)$ *given by*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\alpha)\varepsilon}{n(n-1)(n-\alpha)} z^n \qquad (|\varepsilon| = 1). \tag{2.10}$$

Next, we derive Theorem 2 below.

Theorem 2. *If* $f(z) \in \mathcal{A}_k$ *satisfies the following inequality:*

$$\sum_{n=1}^{\infty} \left(\frac{n}{k}+1\right)\left(\frac{n}{k}+1-\alpha\right)\left|a_{1+\frac{n}{k}}\right| \leqq 1-\alpha \tag{2.11}$$

for some real α $(0 \leqq \alpha < 1)$, *then* $f(z) \in \mathcal{K}_k(\alpha)$. *The equality in* (2.11) *holds true for* $f(z)$ *given by*

$$f(z) = z + \sum_{n=1}^{\infty} \frac{(1-\alpha)k^2\varepsilon}{n(n+1)(n+k)(n+(1-\alpha)k)} z^{1+\frac{n}{k}} \qquad (|\varepsilon| = 1). \tag{2.12}$$

Proof. Noting that

$$f(z) \in \mathcal{K}_k(\alpha) \iff zf'(z) \in \mathcal{S}_k^*(\alpha),$$

we immediately see that $zf'(z) \in \mathcal{S}_k^*(\alpha)$, that is, that $f(z) \in \mathcal{K}_k(\alpha)$ if $f(z) \in \mathcal{A}_k$ satisfies the following inequality:

$$\sum_{n=1}^{\infty} \left(\frac{n}{k}+1\right)\left(\frac{n}{k}+1-\alpha\right)\left|a_{1+\frac{n}{k}}\right| \leqq 1-\alpha \tag{2.13}$$

for some real α $(0 \leqq \alpha < 1)$. Also, the equality in (2.11) is attained for $f(z)$ given by (2.12). □

Upon setting $k = 1$ in Theorem 2, we deduce the following corollary.

Corollary 2. *If* $f(z) \in \mathcal{A}$ *satisfies the following inequality:*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leqq 1-\alpha \tag{2.14}$$

for some real α $(0 \leqq \alpha < 1)$, *then* $f(z) \in \mathcal{K}(\alpha)$. *The equality in* (2.14) *is attained for the function* $f(z)$ *given by*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\alpha)\varepsilon}{n^2(n-1)(n-\alpha)} z^n \qquad (|\varepsilon| = 1). \tag{2.15}$$

Corollary 3. *If* $f(z) \in \mathcal{A}_k$ *satisfies the coefficient inequality* (2.11) *for some real* α $(0 \leqq \alpha < 1)$, *then* $f(z) \in \mathcal{S}_k^*(\beta)$ *with*

$$\beta = \frac{1+k}{1+(2-\alpha)k} < 1. \tag{2.16}$$

Proof. If $f(z) \in \mathscr{A}_k$ constrained by (2.11) satisfies the following inequality:

$$\sum_{n=1}^{\infty} \left(\frac{n}{k}+1-\beta\right) \left|a_{1+\frac{n}{k}}\right| \leqq 1-\beta \tag{2.17}$$

for some real β $(0 \leqq \beta < 1)$, then we say that $f(z) \in \mathscr{S}_k^*(\beta)$. Therefore, we consider some real β such that

$$\frac{\frac{n}{k}+1-\beta}{1-\beta} \leqq \frac{\left(\frac{n}{k}+1\right)\left(\frac{n}{k}+1-\alpha\right)}{1-\alpha} \tag{2.18}$$

for all $n = 1,2,3,\ldots$. This yields

$$\beta \leqq \frac{n+k}{n+(2-\alpha)k} \qquad (n = 1,2,3,\ldots).$$

Therefore, we see that

$$\beta \leqq \min_{n \geqq 1} \left\{ \frac{n+k}{n+(2-\alpha)k} \right\} = \frac{1+k}{1+(2-\alpha)k}.$$

□

3. A general class of functions

Noting that the Koebe function $f(z)$ given by (1.4) is the extremal function for the class $\mathscr{S}^*$, we consider the function $f(z)$ given by

$$f(z) = \frac{z}{\left(1-z^{\frac{1}{k}}\right)^2} = z + \sum_{n=1}^{\infty} (n+1) z^{1+\frac{n}{k}} \tag{3.1}$$

for $k = 1,2,3,\ldots$. If $k = 1$ in (3.1), then $f(z) \in \mathscr{S}_1^*$. Moreover, if $k = 2$ in (3.1), then $f(z) \in \mathscr{S}_2^*\left(\frac{1}{2}\right)$.

For such $f(z)$ given by (3.1), we have the following result.

Theorem 3. *If $f(z)$ is given by (3.1), then $f(z) \in \mathscr{S}_k^*\left(\frac{k-1}{k}\right)$.*

Proof. It follows that

$$\Re\left(\frac{zf'(z)}{f(z)}\right) = \Re\left(1 + \frac{2z^{\frac{1}{k}}}{k\left(1-z^{\frac{1}{k}}\right)}\right) = \Re\left(\frac{k-2}{k} + \frac{2}{k\left(1-z^{\frac{1}{k}}\right)}\right) > \frac{k-1}{k} \tag{3.2}$$

for $z \in \mathbb{U}$. □

Remark 1. From $f(z)$ in (3.1), we see that

$$\sum_{n=1}^{\infty} \left(\frac{n}{k}+1-\alpha\right) \left|a_{1+\frac{n}{k}}\right| = \sum_{n=1}^{\infty} \left(\frac{n}{k}+1-\alpha\right)(n+1)$$

$$= (1-\alpha)\sum_{n=1}^{\infty}(n+1) + \sum_{n=1}^{\infty}\frac{n}{k}(n+1)$$

$$> 1-\alpha.$$

Therefore, $f(z)$ does not satisfy the inequality (2.1) for any α and k.

We now derive the following result.

Theorem 4. *If $f(z)$ is given by (3.1) with $k \geqq 4$, then $f(z) \in \mathscr{K}_k\left(\frac{k-4}{2k}\right)$.*

Proof. We note that the function $f(z)$ given by (3.1) satisfies the following condition:

$$\begin{aligned} 1+\frac{zf''(z)}{f'(z)} &= 1+\frac{3z^{\frac{1}{k}}}{k\left(1-z^{\frac{1}{k}}\right)} - \frac{(k-2)z^{\frac{1}{k}}}{k\left(k-(k-2)z^{\frac{1}{k}}\right)} \\ &= 1+\frac{3}{k\left(z^{-\frac{1}{k}}-1\right)} - \frac{k-2}{k\left(kz^{-\frac{1}{k}}-(k-2)\right)} \qquad (z \in \mathbb{U}_0). \end{aligned} \tag{3.3}$$

If we take $z=0$, then the left-hand side of (3.3) becomes 1. Therefore, we consider

$$z^{\frac{1}{k}} = e^{i\frac{\theta}{k}} = e^{i\varphi} \qquad \left(\varphi = \frac{\theta}{k}\right).$$

We then have

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) = 1-\frac{3}{k}+\frac{(k-2)((k-2)-k\cos\varphi)}{k((k-2)^2+k^2-2k(k-2)\cos\varphi)}. \tag{3.4}$$

It is easy to see that the right-hand side of (3.4) is decreasing for $\cos\varphi$ with $k \geqq 4$. This obviously yields

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{k-4}{2k} \qquad (z \in \mathbb{U}). \tag{3.5}$$

□

Next, we define a function $f(z)$ by

$$f(z) = \frac{z}{1-z^{\frac{1}{k}}} = z + \sum_{n=1}^{\infty} z^{1+\frac{n}{k}} \tag{3.6}$$

for $k = 1,2,3,\ldots$. If $k=1$ in (3.6), then $f(z) \in \mathscr{K}$.

Theorem 5. *If $f(z)$ is given by (3.6), then $f(z) \in \mathscr{S}_k^*\left(\frac{2k-1}{2k}\right)$ and $f(z) \in \mathscr{K}_k(0)$.*

Proof. Noting that

$$\frac{zf'(z)}{f(z)} = 1+\frac{z^{\frac{1}{k}}}{k\left(1-z^{\frac{1}{k}}\right)}, \tag{3.7}$$

we have

$$\Re\left(\frac{zf'(z)}{f(z)}\right) = \Re\left(\frac{k-1}{k} + \frac{1}{k\left(1-z^{\frac{1}{k}}\right)}\right) > \frac{2k-1}{2k} \tag{3.8}$$

for $z \in \mathbb{U}$. Further, we readily find that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{k-(k-2)z^{\frac{1}{k}}}{k\left(1-z^{\frac{1}{k}}\right)} - \frac{(k-1)z^{\frac{1}{k}}}{k\left(k-(k-1)z^{\frac{1}{k}}\right)}, \tag{3.9}$$

which shows that

$$\begin{aligned}\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) &= \Re\left(\frac{k-1}{k} + \frac{2}{k\left(1-z^{\frac{1}{k}}\right)} - \frac{1}{k-(k-1)z^{\frac{1}{k}}}\right)\\ &> \frac{k-1}{k} + \frac{1}{k} - \frac{1}{k-(k-1)}\\ &= 0\end{aligned}$$

for $z \in \mathbb{U}$. □

Remark 2. From $f(z)$ in (3.6), we see that

$$\sum_{n=1}^{\infty}\left(\frac{n}{k}+1-\alpha\right)\left|a_{1+\frac{n}{k}}\right| = \sum_{n=1}^{\infty}\left(\frac{n}{k}+1-\alpha\right) > 1-\alpha, \tag{3.10}$$

which shows that $f(z)$ does not satisfy (2.1) for any α and k.

In view of the function $f(z)$ given by (1.8), we introduce a function $f(z)$ as follows:

$$f(z) = \frac{z\left(k-(k-1)z^{\frac{1}{k}}\right)}{k\left(1-z^{\frac{1}{k}}\right)^2} = z + \sum_{n=1}^{\infty}\left(1+\frac{n}{k}\right)z^{1+\frac{n}{k}} \tag{3.11}$$

with $k = 1, 2, 3, \ldots$. If $k = 1$, then $f(z)$ becomes the Koebe function given by (1.4).

We derive the following result.

Theorem 6. *If $f(z)$ is given by (3.11), then $f(z) \in \mathscr{S}_k^*(0)$.*

Proof. Note that, for $f(z)$ given by (3.11), we get

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= 1 + \frac{z^{\frac{1}{k}}\left((k+1)-(k-1)z^{\frac{1}{k}}\right)}{k\left(1-z^{\frac{1}{k}}\right)\left(k-(k-1)z^{\frac{1}{k}}\right)}\\ &= 1 + \frac{1}{k\left(z^{-\frac{1}{k}}-1\right)} + \frac{1}{k\left(z^{-\frac{1}{k}}-1\right)\left(k-(k-1)z^{\frac{1}{k}}\right)} \qquad (z \in \mathbb{U}_0).\end{aligned}$$

Since

$$\frac{zf'(z)}{f(z)} = 1 \quad \text{for} \quad z = 0,$$

we consider

$$z^{\frac{1}{k}} = e^{i\frac{\theta}{k}} = e^{i\varphi} \qquad \left(\varphi = \frac{\theta}{k}\right).$$

We then find that

$$\Re\left(\frac{zf'(z)}{f(z)}\right) = \frac{2k-1}{2k}\left(1 - \frac{1-\cos\varphi}{\left(2k^2-2k+1\right) - (2k-1)^2\cos\varphi + 2k(k-1)\cos^2\varphi}\right). \tag{3.12}$$

Letting

$$x = \cos\varphi \qquad (-1 \leqq x \leqq 1),$$

we consider a function $g(x)$ given by

$$g(x) = \frac{1-x}{\left(2k^2-2k+1\right) - (2k-1)^2 x + 2k(k-1)x^2}. \tag{3.13}$$

Then, since $g'(x) \geqq 0$, we obtain

$$g(x) \leqq \lim_{x\to 1} g(x) = 1.$$

Consequently, we obtain

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}), \tag{3.14}$$

which that $f(z) \in \mathscr{S}_k^*(0)$. □

Remark 3. From $f(z)$ in (3.11), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{n}{k}+1-\alpha\right)\left|a_{1+\frac{n}{k}}\right| &= \sum_{n=1}^{\infty}\left(\frac{n}{k}+1-\alpha\right)\left(\frac{n}{k}+1\right)\\ &= (1-\alpha)\sum_{n=1}^{\infty}\left(\frac{n}{k}+1\right) + \sum_{n=1}^{\infty}\frac{n}{k}\left(\frac{n}{k}+1\right)\\ &> 1-\alpha. \end{aligned}\tag{3.15}$$

Thus, clearly, the function $f(z)$ does not satisfy the inequality (2.1) for any α and k.

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Department of Mathematics, Faculty of Education, Yamato University, Katayama 2-5-1, Suita, Osaka 564-0082, Japan.

E-mail: owa.shigeyoshi@yamato-u.ac.jp

Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada.

China Medical University, Taichung 40402, Taiwan, Republic of China.

E-mail: harimsri@math.uvic.ca

Department of Mathematics and Physics, Setsunan University, Neyagawa, Osaka 572-8508, Japan.

E-mail: ha_ya_to112@hotmail.com

Study Supporting Room, Osaka University of Health and Sport Sciences, Kumatori, Sennan, Osaka 590-0496, Japan.

E-mail: freedom@sakai.zaq.ne.jp