TAMKANG JOURNAL OF MATHEMATICS
Volume 47, Number 4, 455-475, December 2016
doi:10.5556/j.tkjm.47.2016.2163

This paper is available online at http://journals.math.tku.edu.tw/index.php/TKJM/pages/view/onlinefirst

# EXISTENCE THEOREMS FOR GENERALIZED VECTOR EQUILIBRIA WITH VARIABLE ORDERING RELATION

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**Abstract**. In this paper we study the solvability of the generalized vector equilibrium problem (for short, GVEP) with a variable ordering relation in reflexive Banach spaces. The existence results of strong solutions of GVEPs for monotone multifunctions are established with the use of the KKM-Fan theorem. We also investigate the GVEPs without monotonicity assumptions and obtain the corresponding results of weak solutions by applying the Brouwer fixed point theorem. These results are also the extension and improvement of some recent results in the literature.

## 1. Introduction

A partially order set $(X,\preceq)$ is a set $X$ equipped with a partial order $\preceq$, that is, $\preceq$ is a transitive, reflexive, antisymmetric relation. An ordered vector space $X$ is a real vector space with a partial order $\preceq$ such that if $x, y \in X$ and $x \preceq y$, then

(i) $x+z \preceq y+z$ for each $z \in X$; and

(ii) $\alpha x \preceq \alpha y$ for each $\alpha \geq 0$.

A nonempty subset $P$ of a vector space $X$ is a convex cone if $\alpha P \subset P$ for all $\alpha \geq 0$ and $P+P \subset P$. A convex cone $P$ is pointed if $P \cap (-P) = \{0\}$. A cone $P$ is proper if it is properly contained in $X$. Note that $P$ is a proper cone if and only if $0 \not\in \mathrm{int}P$, where $\mathrm{int}E$ denotes the interior of a set $E$. A pointed convex cone $P$ induces a partial order $\leq_P$ on $X$ defined by $x \leq_P y$ whenever $y-x \in P$. In this case, $(X,\leq_P)$ is an ordered vector space $(X,\leq_P)$ with an order relation $\leq_P$. The weak order $\not\leq_{\mathrm{int}P}$ on an ordered vector space $(X,\leq_P)$ with $\mathrm{int}P \neq \emptyset$ is defined by $x \not\leq_{\mathrm{int}P} y$ whenever $y-x \not\in \mathrm{int}P$.

Let $X$ and $Y$ be two Banach spaces. The space of all continuous linear operators from a Banach space $X$ into a Banach space $Y$ is denoted by $\mathscr{L}(X,Y)$. For $S \in \mathscr{L}(X,Y)$ and $x \in X$,

Received March 9, 2016, accepted April 26, 2016.
2010 *Mathematics Subject Classification.* 49J30, 47H10, 47H17, 49M05.
*Key words and phrases.* Generalized vector equilibrium problem, variable ordering relation, cone mapping, kkm-fan theorem, brouwer fixed point theorem, monotonicity, complete continuity.
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$\langle S, x\rangle$ denotes the value of $S$ at $x$. Let $K$ be a nonempty closed convex subset of $X$ and let $C: K \to 2^Y$ be a cone mapping, i.e., $C(x)$ is a proper closed pointed convex cone and $\mathrm{int}C(x) \neq \emptyset$ for each $x \in K$. Suppose that $A: K \times \mathscr{L}(X,Y) \to \mathscr{L}(X,Y)$ and $f: K \to Y$ are single-valued mappings and $T: K \to 2^{\mathscr{L}(X,Y)}$ is a set-valued mapping. In 2010, Ceng and Huang [16] introduced and considered the generalized vector variational inequality (for short, GVVI), which is to find $x_0 \in K$ with the following property: there exists $u_0 \in T(x_0)$ such that

$$\langle A(x_0, u_0), y - x_0\rangle + f(y) - f(x_0) \nleq_{\mathrm{int}C(x_0)} 0, \quad \forall y \in K. \tag{1.1}$$

Such an $x_0$ is also called a strong solution of GVVI (1.1). If $T$ is single-valued, then the GVVI reduces to the vector variational inequality (VVI). In recent years there has been an increasing interest in VVI; mainly this study in finite-dimensional Euclidean spaces was first introduced by Giannessi in [6]. It has shown to be an effective and powerful tool in the mathematical investigation of a wide class of problems arising in pure and applied sciences. Various classes of VVIs have been intensively analyzed both in finite- and infinite-dimensional spaces; see [2]-[4], [7]-[11], [14]-[18], [20]-[26] and the references therein. In [19], Zheng posed the concept of semimonotonicity and applied Fan-Glicksberg fixed point theorem to generalize the existence results for VVI obtained by Chen [4] which is to find a point $x_0 \in K$ such that

$$\langle \eta(x_0, x_0), y - x_0\rangle + f(y) - f(x_0) \nleq_{\mathrm{int}C(x_0)} 0, \quad \forall y \in K, \tag{1.2}$$

where $\eta: K \times K \to \mathscr{L}(X,Y)$. Most of the latest existence results for VVI problems are based on KKM-Fan Theorem [5], which requires the feasible set to be closed and bounded in the strong topology and the mapping to possess certain monotonicity type properties; see [2, 9, 10, 18]. It is noteworthy that Huang and Fang [9] studied the following VVI in reflexive Banach spaces not only with but also without monotonicity assumptions: find $x_0 \in K$ such that

$$\langle T x_0, y - x_0\rangle + f(y) - f(x_0) \nleq \mathrm{int}C0, \quad \forall y \in K, \tag{1.3}$$

where $T: K \to \mathscr{L}(X,Y)$ and $C$ is a proper closed pointed convex cone with $\mathrm{int}C \neq \emptyset$. Furthermore, Zeng and Yao [18] defined the concepts of the complete and strong semicontinuities and extended the results of Huang and Fang [9] to the GVVI, i.e., find $x_0 \in K$ and $u_0 \in T(x_0)$ such that

$$\langle A u_0, y - x_0\rangle + f(y) - f(x_0) \nleq_{\mathrm{int}C} 0, \quad \forall y \in K, \tag{1.4}$$

where $A: \mathscr{L}(X,Y) \to \mathscr{L}(X,Y)$ and $T: K \to 2^{\mathscr{L}(X,Y)}$.

Motivated and inspired by generalized vector equilibrium problems considered in Zeng and Yao [27, 28], this work is to further extend the results of Ceng and Huang [16] to the setting of the following generalized vector equilibrium problem (GVEP):

**GVEP.** Let $K$ be a nonempty closed convex subset of $X$ and let $C : K \to 2^Y$ be a cone mapping. Let $\Phi : \mathscr{L}(X,Y) \times K \times K \to Y$ is a equilibrium-like function, i.e., $\Phi(w,x,y) + \Phi(w,y,x) = 0$ for each $(w,x,y) \in \mathscr{L}(X,Y) \times K \times K$. Suppose that $A : K \times \mathscr{L}(X,Y) \to \mathscr{L}(X,Y)$ and $f : K \to Y$ are single-valued mappings and $T : K \to 2^{\mathscr{L}(X,Y)}$ is a set-valued mapping. Then the objective is to find $x_0 \in K$ with the following property: there exists $w_0 \in A(x_0, T(x_0))$ such that

$$\Phi(w_0, x_0, y) + f(y) - f(x_0) \not\leq_{\operatorname{int} C(x_0)} 0, \quad \forall y \in K. \tag{1.5}$$

Such an $x_0$ is called a strong solution of GVEP (1.5).

In particular, if $\Phi(w,x,y) = \langle w, y-x \rangle$ for each $(w,x,y) \in \mathscr{L}(X,Y) \times K \times K$, then GVEP (1.5) reduces to GVVI (1.1). We first establish the existence results of the GVEP (1.5) for monotone multifunction $T : K \to 2^{\mathscr{L}(X,Y)}$ with the use of the KKM-Fan theorem. To this end, we need to provide a parallel version of the existence of strong solutions to GVEP (1.5). It is somewhat difficult to derive a corresponding result of strong solutions to our GVEP (1.5) without assuming monotonicity. Instead, we investigate the following problem: find a point $x_0 \in K$, called a weak solution, such that for each $y \in K$ there exists $w \in A(x_0, T(y))$ satisfying

$$\Phi(w, x_0, y) + f(y) - f(x_0) \not\leq_{\operatorname{int} C(x_0)} 0. \tag{1.6}$$

Each strong solution is of course a weak solution of GVEP (1.5), but the converse is false. This problem for the case where $\Phi(w,x,y) = \langle w, y-x \rangle$, $A(x,u) = u$ and $f \equiv 0$ was introduced by Lin, Yang and Yao [12]. Being based upon the characterization of upper semicontinuity together with the Brouwer fixed point theorem, we present several new results which are the extension of those in [2, 3, 9, 10, 16, 17, 18].

The paper is organized as follows. In Section 2 we set notation and give some background. In Section 3 we prove the existence results of GVEP (1.5) for vector monotone multifunctions in reflexive Banach spaces. Finally, in Section 4 we study the GVEP (1.5) without monotonicity assumptions. In the remainder of this paper, for simplicity, we denote GVEP (1.5) by GVEP.

## 2. Notation, definitions and basic properties

Let $X$ and $Y$ be topological spaces. A multifunction $\varphi : X \to 2^Y$ is upper semicontinuous at $x$ if for every open set $V$ containing $\varphi(x)$, there is a neighborhood $U$ of $x$ such that $z \in U$ implies $\varphi(z) \subset V$. We say that $\varphi$ is upper semicontinuous on $X$ if it is upper semicontinuous at every point of $X$. The mapping $\varphi$ is closed, or has closed graph if its graph given by

$$\mathscr{G}(\varphi) = \{(x,y) \in X \times Y : y \in \varphi(x)\}$$

is a closed subset of $X \times Y$. We recall the following well-known facts.

**Theorem 2.1.**

(a) *An upper semicontinuous multifunction* $\varphi : X \to 2^Y$ *is closed if either*

  (i) $\varphi$ *is closed-valued and* $Y$ *is regular, or*

  (ii) $\varphi$ *is compact-valued and* $Y$ *is Hausdorff.*

(b) *A compact-valued multifunction* $\varphi : X \to 2^Y$ *is upper semicontinuous if and only if for every net* $\{(x_\alpha, y_\alpha)\}$ *in* $\mathcal{G}(\varphi)$ *that satisfies* $x_\alpha \to x$ *for some* $x \in X$ *the net* $\{y_\alpha\}$ *has a subnet converging to a point in* $\varphi(x)$.

Let $(X, \|\cdot\|)$ be a normed vector space so that its norm induces a metric $d$. For any pair of nonempty subsets $A$ and $B$ of $X$, define

$$d_H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\}.$$

This extended real number $d_H(A,B)$ is the Hausdorff distance between $A$ and $B$ induced by $d$. The distance function $d_H$ turns the collection of all nonempty closed and bounded subsets of $X$, denoted $CB(X)$, into a metric space. Note that [13] if $A$ and $B$ are nonempty subsets of $X$ with $B$ compact, then for each $a \in A$ there exists $b \in B$ such that

$$\|a-b\| \le d_H(A,B).$$

**Definition 2.1** ([18])**.** Let $X$ and $Y$ be two real Banach spaces and $K$ be a nonempty closed convex subset of $X$. A compact-valued multifunction $T : K \to 2^{\mathcal{L}(X,Y)}$ is $H$-hemicontinuous if the mapping $\alpha \mapsto T(x+\alpha y)$ is continuous at $0^+$, where $CB(\mathcal{L}(X,Y))$ is equipped with the metric topology induced by $d_H$.

The concept of $H$-hemicontinuity is interesting and useful in connection to nonlinear mappings of monotone type.

**Definition 2.2.** Let $X$ and $Y$ be real Banach spaces. A function $f : X \to Y$ is completely continuous if $\{f(x_n)\}$ converges to $f(x)$ in $Y$ whenever $\{x_n\}$ converges weakly to some $x \in X$, i.e., $f$ is weak-to-norm sequentially continuous.

A completely continuous linear operator $T$ from a Banach space $X$ into a Banach space $Y$ is also known as a Dunford-Pettis operator and is continuous. Hence the collection of all completely continuous linear operators from $X$ into $Y$, denoted $\mathcal{L}_{cc}(X,Y)$, is a subspace of $\mathcal{L}(X,Y)$. Sequential continuity does not in general imply continuity. In fact not all completely continuous operators are weak-to-norm continuous. It does of course follow from the definition of complete continuity that every weak-to-norm continuous linear operator from $X$ into $Y$ is completely continuous.

**Definition 2.3.** Let $X$ and $Y$ be real Banach spaces, $K$ be a nonempty subset of $X$ and $C$ be a convex cone.

(i) A single-valued mapping $T : K \to \mathscr{L}(X,Y)$ is $C$-monotone if

$$\langle T(x) - T(y), x - y\rangle \geq_C 0, \quad \text{for all } x, y \in K.$$

(ii) A set-valued mapping $T : K \to 2^{\mathscr{L}(X,Y)}$ is $C$-monotone if

$$\langle x^* - y^*, x - y\rangle \geq_C 0, \quad \text{for all } x, y \in K,\ x^* \in Tx,\ y^* \in Ty.$$

(iii) A set-valued mapping $T : K \to 2^{\mathscr{L}(X,Y)}$ is $C$-monotone with respect to a mapping $A : \mathscr{L}(X,Y) \to \mathscr{L}(X,Y)$ (see [18]) if

$$\langle Ax^* - Ay^*, x - y\rangle \geq_C 0, \quad \text{for all } x, y \in K,\ x^* \in Tx,\ y^* \in Ty.$$

(iv) A mapping $f : K \to Y$ is $C$-convex if $K$ is convex and

$$f(tx + (1-t)y) \leq_C t f(x) + (1-t) f(y), \quad \text{for all } x, y \in K,\ t \in [0,1].$$

**Definition 2.4.** Let $X$ and $Y$ be real Banach spaces, $K$ be a nonempty subset of $X$ and $C$ be a convex cone. Let $T : K \to 2^{\mathscr{L}(X,Y)}$ be a set-valued mapping and $A : K \times \mathscr{L}(X,Y) \to \mathscr{L}(X,Y)$ be a single-valued mapping. A equilibrium-like function $\Phi : \mathscr{L}(X,Y) \times K \times K \to Y$ is called $C$-monotone with respect to $T$ and $A$ if for each $y \in K$

$$\Phi(w_1, x_2, x_1) + \Phi(w_2, x_1, x_2) \geq_C 0, \quad \text{for all } x_1, x_2 \in K,\ w_1 \in A(y, Tx_1),\ w_2 \in A(y, Tx_2).$$

## 3. Strong solutions of GVEP with monotonicity

We turn attention to the question of the solvability to GVEPs for vector monotone multifunctions in reflexive Banach spaces by applying the KKM-Fan theorem.

Let $E$ be a nonempty subset of a topological vector space $X$. A multifunction $\varphi : E \to 2^X$ is a KKM mapping if for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of $E$,

$$\operatorname{co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} \varphi(x_i),$$

where $\operatorname{co}\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \ldots, x_n\}$. In a topological vector space, the convex hull of a finite union of compact convex sets is compact.

**Lemma 3.1** (KKM-Fan Theorem [5])**.** *Let $E$ be a nonempty convex subset of a Hausdorff topological vector space $X$ and let $\varphi : E \to 2^X$ be a KKM mapping with closed values. If there is a point $x_0 \in E$ such that $\varphi(x_0)$ is compact, then $\bigcap_{x \in E} \varphi(x) \neq \emptyset$.*

**Lemma 3.2** ([3])**.** *Let $C$ be a closed pointed convex cone with $\operatorname{int}C \neq \emptyset$ and let $(X, \leq_C)$ be a real ordered Banach space. For any $a, b, c \in X$, we have*

(i) *$c \not\leq_{\operatorname{int}C} a$ and $a \geq_C b$ imply that $c \not\leq_{\operatorname{int}C} b$;*

(ii) *$c \geq_{\operatorname{int}C} a$ and $a \leq_C b$ imply that $c \not\geq_{\operatorname{int}C} b$.*

A key to our problem is shown as follows. It also generalizes [9, Lemma 2.5], [18, Lemma 2.3] and [16, Lemma 3.3].

**Lemma 3.3.** *Let $X$ and $Y$ be real Banach spaces, $K$ be a nonempty closed convex subset of $X$, $\Phi : \mathscr{L}(X,Y) \times K \times K \to Y$ be a equilibrium-like function, $C : K \to 2^Y$ and $T : K \to 2^{\mathscr{L}(X,Y)}$ be two multifunctions, and $f : K \to Y$ and $A : K \times \mathscr{L}(X,Y) \to \mathscr{L}(X,Y)$ be two single-valued functions. Suppose that:*

(i) *$C$ is a cone mapping such that $\operatorname{int}C_- \neq \emptyset$, where $C_- = \bigcap_{x\in K} C(x)$;*

(ii) *$T$ is $H$-hemicontinuous with nonempty compact values;*

(iii) *$f$ is $C_-$-convex;*

(iv) *$A$ is continuous in the second variable;*

(v) *$\Phi$ is continuous in the first variable, and $C_-$-convex in the third variable;*

(vi) *$\Phi$ is $C_-$-monotone with respect to $T$ and $A$.*

*Then a point $x_0 \in K$ is a strong solution of GVEP, i.e., there exists $w_0 \in A(x_0, T(x_0))$ such that*

$$\Phi(w_0, x_0, y) + f(y) - f(x_0) \not\leq_{\operatorname{int}C(x_0)} 0, \quad \text{for all } y \in K,$$

*if and only if*

$$\Phi(w, x_0, y) + f(y) - f(x_0) \not\leq_{\operatorname{int}C(x_0)} 0, \quad \text{for all } y \in K \text{ and } w \in A(x_0, T(y)).$$

**Proof.** Suppose that there exist $x_0 \in K$ and $w_0 \in A(x_0, T(x_0))$ such that

$$\Phi(w_0, x_0, y) + f(y) - f(x_0) \not\leq_{\operatorname{int}C(x_0)} 0, \quad \text{for all } y \in K.$$

Let $y \in K$ and $w \in A(x_0, T(y))$. Since $\Phi$ is $C_-$-monotone with respect to $T$ and $A$, it follows that

$$\Phi(w, x_0, y) + \Phi(w_0, y, x_0) \geq_{C_-} 0.$$

Also, since $\Phi$ is a equilibrium-like function, we have

$$\Phi(w, x_0, y) \geq_{C_-} \Phi(w_0, x_0, y),$$

and so

$$\Phi(w, x_0, y) + f(y) - f(x_0) \geq_{C_-} \Phi(w_0, x_0, y) + f(y) - f(x_0),$$

which immediately leads to

$$\Phi(w, x_0, y) + f(y) - f(x_0) \geq_{C(x_0)} \Phi(w_0, x_0, y) + f(y) - f(x_0).$$

Therefore by Lemma 3.2,

$$\Phi(w, x_0, y) + f(y) - f(x_0) \not\leq_{\operatorname{int} C(x_0)} 0, \quad \text{for all } y \in K,\ w \in A(x_0, T(y)).$$

For the converse, suppose that there exists $x_0 \in K$ such that

$$\Phi(w, x_0, y) + f(y) - f(x_0) \not\leq_{\operatorname{int} C(x_0)} 0, \quad \text{for all } y \in K,\ w \in A(x_0, T(y)),$$

that is,

$$\Phi(A(x_0, v), x_0, y) + f(y) - f(x_0) \not\leq_{\operatorname{int} C(x_0)} 0, \quad \text{for all } y \in K,\ v \in T(y). \tag{3.1}$$

For any $y \in K,\ y_t = (1-t)x_0 + ty \in K$, for all $t \in (0,1)$, because $K$ is convex. Let $v_t \in T(y_t)$. Using $y_t$ and $v_t$ in place of $y$ and $v$ in Eq. (3.1) respectively yields

$$\Phi(A(x_0, v_t), x_0, y_t) + f(y_t) - f(x_0) \not\leq_{\operatorname{int} C(x_0)} 0. \tag{3.2}$$

On the other hand, since $f$ is $C_-$-convex and $\Phi$ is $C_-$-convex in the third variable, we have

$$\begin{aligned} &\Phi(A(x_0, v_t), x_0, y_t) + f(y_t) - f(x_0) \\ &\quad \leq_{C_-} (1-t)\Phi(A(x_0, v_t), x_0, x_0) + t\Phi(A(x_0, v_t), x_0, y) + (1-t)f(x_0) + tf(y) - f(x_0) \\ &\quad = t[\Phi(A(x_0, v_t), x_0, y) + f(y) - f(x_0)]. \end{aligned}$$

In particular,

$$\Phi(A(x_0, v_t), x_0, y_t) + f(y_t) - f(x_0) \leq_{C(x_0)} t[\Phi(A(x_0, v_t), x_0, y) + f(y) - f(x_0)]. \tag{3.3}$$

By Eqs. (3.2) and (3.3) and Lemma 3.2, we obtain

$$\Phi(A(x_0, v_t), x_0, y) + f(y) - f(x_0) \not\leq_{\operatorname{int} C(x_0)} 0, \quad \text{for all } v_t \in T(y_t),\ t \in (0,1). \tag{3.4}$$

Since $T$ is compact-valued, for each $v_t \in T(y_t)$ there exists $u_t \in T(x_0)$ such that

$$\|v_t - u_t\| \leq d_H(T(y_t), T(x_0)).$$

We may assume without loss of generality that $\{u_t\}$ converges strongly to some $u_0 \in T(x_0)$ as $t \to 0^+$. Since

$$\|v_t - u_0\| \leq \|v_t - u_t\| + \|u_t - u_0\| \leq d_H(T(y_t), T(x_0)) + \|u_t - u_0\|,$$

by the $H$-hemicontinuity of $T$ we deduce that $v_t \to u_0$ as $t \to 0^+$. Also, since $A$ is continuous in the second variable, we know that $A(x_0, v_t) \to A(x_0, u_0)$ as $t \to 0^+$, which together with the continuity of $\Phi$ in the first variable, implies that

$$\lim_{t\to 0^+} \|\Phi(A(x_0, v_t), x_0, y) \to \Phi(A(x_0, u_0), x_0, y)\| = 0. \tag{3.5}$$

Since $Y \setminus (-\mathrm{int}C(x_0))$ is closed, we have by Eq. (3.4) that

$$\Phi(A(x_0, u_0), x_0, y) + f(y) - f(x_0) \in Y \setminus (-\mathrm{int}C(x_0)).$$

Hence

$$\Phi(A(x_0, u_0), x_0, y) + f(y) - f(x_0) \not\leq_{\mathrm{int}C(x_0)} 0, \quad \text{for all } y \in K.$$

That is, there exists $w_0 (= A(x_0, u_0)) \in A(x_0, T(x_0))$ such that

$$\Phi(w_0, x_0, y) + f(y) - f(x_0) \not\leq_{\mathrm{int}C(x_0)} 0, \quad \text{for all } y \in K.$$

This completes the proof. □

We are now in a position to discuss the solvability of GVEPs for monotone mappings.

**Theorem 3.1.** *Let $X$ be a real reflexive Banach space, $Y$ be a real Banach space, $K$ be a nonempty bounded closed convex subset of $X$, $\Phi : \mathscr{L}(X,Y) \times K \times K \to Y$ be a equilibrium-like function, $C : K \to 2^Y$, $D : K \to 2^Y$ and $T : K \to 2^{\mathscr{L}(X,Y)}$ be three multifunctions, where $D$ is defined by $D(x) = Y \setminus (-\mathrm{int}C(x))$, and $f : K \to Y$ and $A : K \times \mathscr{L}(X,Y) \to \mathscr{L}(X,Y)$ be two single-valued functions. Suppose that:*

- (i) *$C$ is a cone mapping such that $\mathrm{int}C_- \neq \emptyset$, where $C_- = \bigcap_{x\in K} C(x)$;*
- (ii) *$D$ has weakly closed graph;*
- (iii) *$T$ is $H$-hemicontinuous with nonempty compact values;*
- (iv) *$f$ is weakly sequentially continuous and $C_-$-convex;*
- (v) *$A$ is completely continuous in the first variable and continuous in the second variable;*
- (vi) *$\Phi$ is Lipschitz continuous in the first variable, weakly sequentially continuous and $C_-$-concave in the second variable, and $C_-$-convex in the third variable;*
- (vii) *$\Phi$ is $C_-$-monotone with respect to $T$ and $A$.*

*Then there exist $x_0 \in K$ and $w_0 \in A(x_0, T(x_0))$ such that*

$$\Phi(w_0, x_0, y) + f(y) - f(x_0) \not\leq_{\mathrm{int}C(x_0)} 0, \quad \text{for all } y \in K.$$

**Proof.** Let $E, F : K \to 2^K$ be two multifunctions defined by, for $y \in K$,

$$E(y) = \{x \in K : \Phi(\hat{w}, x, y) + f(y) - f(x) \not\leq_{\text{int}C(x)} 0, \text{ for some } \hat{w} \in A(x, T(x))\}$$

and

$$F(y) = \{x \in K : \Phi(w, x, y) + f(y) - f(x) \not\leq_{\text{int}C(x)} 0, \text{ for all } w \in A(x, T(y))\}.$$

Then $E(y)$ and $F(y)$ are nonempty due to $y \in E(y) \cap F(y)$. We claim that $E$ is a KKM mapping. Assume on the contrary that there exist a finite subset $\{x_1, \dots, x_n\}$ of $K$ and nonnegative numbers $t_1, \dots, t_n$ with $\sum_{i=1}^n t_i = 1$ such that

$$x = \sum_{i=1}^n t_i x_i \notin \bigcup_{i=1}^n E(x_i).$$

Then for any $\hat{w} \in A(x, T(x))$,

$$\Phi(\hat{w}, x, x_i) + f(x_i) - f(x) \leq_{\text{int}C(x)} 0, \quad i = 1, 2, \dots, n;$$

hence by the $C_-$-convexity of $f$ and $C_-$-concavity of $\Phi$ in the second variable,

$$\begin{aligned} 0 &= \Phi(\hat{w}, x, x) + f(x) - f(x) \\ &\geq_{C(x)} \sum_{i=1}^n t_i \Phi(\hat{w}, x_i, x) + f(x) - \sum_{i=1}^n t_i f(x_i) \\ &= \sum_{i=1}^n t_i [\Phi(\hat{w}, x_i, x) + f(x) - f(x_i)] \\ &\geq_{\text{int}C(x)} 0, \end{aligned}$$

which leads to a contradiction that $C(x) = Y$. So $E$ is a KKM mapping. Further $E(y) \subset F(y)$ for every $y \in K$. Furthermore, if $x \in E(y)$, then there exists $\hat{w} \in A(x, T(x))$ such that

$$\Phi(\hat{w}, x, y) + f(y) - f(x) \not\leq_{\text{int}C(x)} 0.$$

Since $\Phi$ is $C_-$-monotone with respect to $T$ and $A$, it follows that

$$\Phi(w, x, y) + \Phi(\hat{w}, y, x) \geq_{C_-} 0, \quad \text{for all } y \in K, \ w \in A(x, T(y)).$$

Also, since $\Phi$ is a equilibrium-like function, we have

$$\Phi(w, x, y) \geq_{C_-} \Phi(\hat{w}, x, y), \quad \text{for all } y \in K, \ w \in A(x, T(y)),$$

and so

$$\Phi(w, x, y) + f(y) - f(x) \geq_{C_-} \Phi(\hat{w}, x, y) + f(y) - f(x), \quad \text{for all } y \in K, \ w \in A(x, T(y)),$$

which immediately leads to

$$\Phi(w, x, y) + f(y) - f(x) \geq_{C(x)} \Phi(\hat{w}, x, y) + f(y) - f(x), \quad \text{for all } y \in K, \ w \in A(x, T(y)).$$

Hence Lemma 3.2 asserts that

$$\Phi(w,x,y)+f(y)-f(x)\not\leq_{\mathrm{int}C(x)} 0,\quad \text{for all } y\in K,\ w\in A(x,T(y)).$$

This shows that $E(y)\subset F(y)$ for all $y\in K$, and so $F$ is also a KKM mapping.

We next prove that for each $y\in K$, the set $F(y)$ is closed in the weak topology of $X$. Note that the weak closure $\overline{F(y)}^w$ of $F(y)$ is weakly compact because $K$ is weakly compact. Thus for any $x\in\overline{F(y)}^w$, there is a sequence $\{x_n\}$ in $F(y)$ which converges weakly to $x$. The definition of $F(y)$ assures that for all $n\in\mathbf{N},\ v\in T(y)$,

$$\Phi(A(x_n,v),x_n,y)+f(y)-f(x_n)\in D(x_n)=Y\setminus(-\mathrm{int}C(x_n)). \tag{3.6}$$

For any fixed $v\in T(y)$, we observe that

$$\begin{aligned}&\Phi(A(x_n,v),x_n,y)-\Phi(A(x,v),x,y)\\&=\Phi(A(x_n,v),x_n,y)-\Phi(A(x,v),x_n,y)+\Phi(A(x,v),x_n,y)-\Phi(A(x,v),x,y).\end{aligned}$$

Since $\Phi$ is Lipschitz continuous in the first variable and $A$ is completely continuous in the first variable, there exists a constant $L>0$ such that

$$\|\Phi(A(x_n,v),x_n,y)-\Phi(A(x,v),x_n,y)\|\le L\|A(x_n,v)-A(x,v)\|\to 0,\quad \text{as } n\to\infty.$$

Also, $\Phi(A(x,v),x_n,y)\to\Phi(A(x,v),x,y)$ weakly as $n\to\infty$ because $\Phi$ is weakly sequentially continuous in the second variable. Now the weak-to-weak sequential continuity of $f$ implies that the sequence $\{\Phi(A(x_n,v),x_n,y)+f(y)-f(x_n)\}$ converges weakly to $\Phi(A(x,v),x,y)+f(y)-f(x)$. Since the graph of $D$ is weakly closed, it follows from Eq. (3.6) that

$$\Phi(A(x,v),x,y)+f(y)-f(x)\in D(x),\quad \text{for all } v\in T(y).$$

That is,

$$\Phi(w,x,y)+f(y)-f(x)\in D(x),\quad \text{for all } w\in A(x,T(y)).$$

We conclude that $x\in F(y)$. Therefore for each $y\in K,\ F(y)$ is weakly closed and so is weakly compact. According to the KKM-Fan Theorem (Lemma 3.1),

$$\bigcap_{y\in K}F(y)\neq\emptyset;$$

hence there exists $x_0\in K$ such that

$$\Phi(w,x_0,y)+f(y)-f(x_0)\not\leq_{\mathrm{int}C(x_0)} 0,\quad \text{for all } y\in K,\ w\in A(x_0,T(y)).$$

Equivalently, by Lemma 3.3 there exists $w_0\in A(x_0,T(x_0))$ such that

$$\Phi(w_0,x_0,y)+f(y)-f(x_0)\not\leq_{\mathrm{int}C(x_0)} 0,\quad \text{for all } y\in K.$$

This completes the proof. □

When the underlying space $X$ is a finite dimensional normed space, the norm and weak topologies of $X$ coincide, and the continuity and sequential continuity from $X$ into a topological space are also the same. In this case, each $F(y)$ is compact if we assume that $f$ is continuous. In addition, the same argument of Theorem 3.1 works provided that $D$ has the closed graph. This result is stated as follows.

**Corollary 3.1.** *Let $Y$ be a real Banach space, $K$ be a nonempty bounded closed convex subset of $\mathbf{R^n}$, $\Phi : \mathscr{L}(\mathbf{R^n}, Y) \times K \times K \to Y$ be a equilibrium-like function, $C : K \to 2^Y$, $D : K \to 2^Y$ and $T : K \to 2^{\mathscr{L}(\mathbf{R^n},Y)}$ be three multifunctions, where $D$ is defined by $D(x) = Y \setminus (-\mathrm{int}C(x))$, and $f : K \to Y$ and $A : K \times \mathscr{L}(\mathbf{R^n}, Y) \to \mathscr{L}(\mathbf{R^n}, Y)$ be two single-valued functions. Suppose that:*

(i) *$C$ is a cone mapping such that $\mathrm{int}C_- \neq \emptyset$, where $C_- = \bigcap_{x\in K} C(x)$;*

(ii) *$D$ has closed graph;*

(iii) *$T$ is $H$-hemicontinuous with nonempty compact values;*

(iv) *$f$ is continuous and $C_-$-convex;*

(v) *$A$ is continuous;*

(vi) *$\Phi$ is Lipschitz continuous in the first variable, continuous and $C_-$-concave in the second variable, and $C_-$-convex in the third variable;*

(vii) *$\Phi$ is $C_-$-monotone with respect to $T$ and $A$.*

*Then the GVEP has a strong solution.*

To guarantee the existence of strong solutions to the GVEP for a weak-to-norm upper semicontinuous mapping $D$, we require that $A$ is a function from $K \times \mathscr{L}(X, Y)$ into $\mathscr{L}_{cc}(X, Y)$, instead of $\mathscr{L}(X, Y)$.

**Theorem 3.2.** *Let $X$ be a real reflexive Banach space, $Y$ be a real Banach space, $K$ be a nonempty bounded closed convex subset of $X$, $\Phi : \mathscr{L}_{cc}(X, Y) \times K \times K \to Y$ be a equilibrium-like function, $C : K \to 2^Y$, $D : K \to 2^Y$ and $T : K \to 2^{\mathscr{L}(X,Y)}$ be three multifunctions, where $D$ is defined by $D(x) = Y \setminus (-\mathrm{int}C(x))$, and $f : K \to Y$ and $A : K \times \mathscr{L}(X, Y) \to \mathscr{L}_{cc}(X, Y)$ be two single-valued functions. Suppose that:*

(i) *$C$ is a cone mapping such that $\mathrm{int}C_- \neq \emptyset$, where $C_- = \bigcap_{x\in K} C(x)$;*

(ii) *$D$ is weak-to-norm upper semicontinuous;*

(iii) *$T$ is $H$-hemicontinuous with nonempty compact values;*

(iv) *$f$ is completely continuous and $C_-$-convex;*

(v) *$A$ is completely continuous in the first variable and continuous in the second variable;*

(vi) $\Phi$ *is Lipschitz continuous in the first variable, completely continuous and $C_-$-concave in the second variable, and $C_-$-convex in the third variable;*

(vii) $\Phi$ *is $C_-$-monotone with respect to $T$ and $A$.*

*Then the GVEP has a strong solution.*

**Proof.** This result can be proved from similar arguments to those employed in the proof of Theorem 3.1. Denote the space $X$ endowed with the weak topology by $X^w$. Since $D$ is a closed-valued weak-to-norm upper semicontinuous multifunction with the a regular range space, it follows that $\mathscr{G}(D)$ is a closed subset of $X^w \times Y$. By adapting the same notation as in Theorem 3.1, we see from Eq. (**??**) that for each $n \in \mathbf{N}$,

$$
\begin{aligned}
&\|\Phi(A(x_n, v), x_n, y) - \Phi(A(x, v), x, y)\| \\
&\le \|\Phi(A(x_n, v), x_n, y) - \Phi(A(x, v), x_n, y)\| + \|\Phi(A(x, v), x_n, y) - \Phi(A(x, v), x, y)\| \\
&\le L\|A(x_n, v) - A(x, v)\| + \|\Phi(A(x, v), x_n, y) - \Phi(A(x, v), x, y)\|.
\end{aligned}
$$

Since $A(\cdot, v)$, $\Phi(A(x, v), \cdot, y)$ and $f$ are completely continuous, the above inequality implies that the sequence $\{\Phi(A(x_n, v), x_n, y) + f(y) - f(x_n)\}$ converges strongly to $\Phi(A(x, v), x, y)$. This shows that $F(y)$ is weakly closed. The remaining claims in the theorem are proved by the same arguments of Theorem 3.1. □

We can extend the previous results to the case where the set $K$ is closed and convex but not necessarily bounded under a coercive condition.

**Theorem 3.3.** *Let $X$ be a real reflexive Banach space, $Y$ be a real Banach space, $K$ be a nonempty closed convex subset of $X$ such that $K \cap B_r \neq \emptyset$, for some $r > 0$, where $B_r = \{x \in X : \|x\| \le r\}$, $\Phi : \mathscr{L}(X, Y) \times K \times K \to Y$ be a equilibrium-like function, $C : K \to 2^Y$, $D : K \to 2^Y$ and $T : K \to 2^{\mathscr{L}(X,Y)}$ be three multifunctions, where $D$ is defined by $D(x) = Y \setminus (-\mathrm{int}C(x))$, and $f : K \to Y$ and $A : K \times \mathscr{L}(X, Y) \to \mathscr{L}(X, Y)$ be two single-valued functions. Suppose that:*

(i) *$C$ is a cone mapping such that $\mathrm{int}C_- \neq \emptyset$, where $C_- = \bigcap_{x\in K} C(x)$;*

(ii) *$D$ has weakly closed graph;*

(iii) *$T$ is $H$-hemicontinuous with nonempty compact values;*

(iv) *$f$ is weakly sequentially continuous and $C_-$-convex;*

(v) *$A$ is completely continuous in the first variable and continuous in the second variable;*

(vi) *for each $x \in K$ with $\|x\| = r$ and each $w \in A(x, T(x))$, there exists $y \in K \cap B_r$ such that*

$$\Phi(w, x, y) + f(y) - f(x) \le_{\mathrm{int}C(x)} 0;$$

(vii) $\Phi$ *is Lipschitz continuous in the first variable, weakly sequentially continuous and* $C_-$*-concave in the second variable, and* $C_-$*-convex in the third variable;*

(viii) $\Phi$ *is* $C_-$*-monotone with respect to* $T$ *and* $A$.

*Then there exist* $x_0 \in K$ *and* $w_0 \in A(x_0, T(x_0))$ *such that*

$$\Phi(w_0, x_0, y) + f(y) - f(x_0) \not\leq_{\operatorname{int}C(x_0)} 0, \quad \text{for all } y \in K.$$

**Proof.** By Theorem 3.1, there exist $x_r \in K \cap B_r$ and $w_r \in A(x_r, T(x_r))$ such that

$$\Phi(w_r, x_r, y) + f(y) - f(x_r) \not\leq_{\operatorname{int}C(x_r)} 0, \quad \text{for all } y \in K \cap B_r. \tag{3.7}$$

It follows from assumption (vi) that $\|x_r\| < r$. To prove that $x_r$ is a strong solution, let $z \in K$ and choose $t \in (0,1)$ small enough such that $(1-t)x_r + tz \in K \cap B_r$. In Eq. (3.7), using $(1-t)x_r + tz$ in place of $y$ yields

$$\Phi(w_r, x_r, (1-t)x_r + tz) + f((1-t)x_r + tz) - f(x_r) \not\leq_{\operatorname{int}C(x_r)} 0. \tag{3.8}$$

Since $\Phi$ is $C_-$-convex in the third variable and $f$ is $C_-$-convex, we have

$$\begin{aligned}
&\Phi(w_r, x_r, (1-t)x_r + tz) + f((1-t)x_r + tz) - f(x_r) \\
&\quad \leq_{C(x_r)} (1-t)\Phi(w_r, x_r, x_r) + t\Phi(w_r, x_r, z) + (1-t)f(x_r) + tf(z) - f(x_r) \\
&\quad = t[\Phi(w_r, x_r, z) + f(z) - f(x_r)].
\end{aligned} \tag{3.9}$$

Therefore, Eqs. (3.8), (3.9) and Lemma 3.2 imply that

$$\Phi(w_r, x_r, z) + f(z) - f(x_r) \not\leq_{\operatorname{int}C(x_r)} 0,$$

as required. □

**Corollary 3.2.** *Let* $Y$ *be a real Banach space,* $K$ *be a nonempty closed convex subset of* $\mathbf{R^n}$ *such that* $K \cap B_r \neq \emptyset$, *for some* $r > 0$, *where* $B_r = \{x \in \mathbf{R^n} : \|x\| \leq r\}$, $\Phi : \mathcal{L}(X,Y) \times K \times K \to Y$ *be a equilibrium-like function,* $C : K \to 2^Y$, $D : K \to 2^Y$ *and* $T : K \to 2^{\mathcal{L}(\mathbf{R^n},Y)}$ *be three multifunctions, where* $D$ *is defined by* $D(x) = Y \setminus (-\operatorname{int}C(x))$, *and* $f : K \to Y$ *and* $A : K \times \mathcal{L}(\mathbf{R^n}, Y) \to \mathcal{L}(\mathbf{R^n}, Y)$ *be two single-valued functions. Suppose that:*

(i) $C$ *is a cone mapping such that* $\operatorname{int}C_- \neq \emptyset$, *where* $C_- = \bigcap_{x \in K} C(x)$;

(ii) $D$ *has closed graph;*

(iii) $T$ *is* $H$*-hemicontinuous with nonempty compact values;*

(iv) $f$ *is continuous and* $C_-$*-convex;*

(v) $A$ *is continuous;*

(vi) *for each* $x \in K$ *with* $\|x\| = r$ *and each* $w \in A(x, T(x))$*, there exists* $y \in K \cap B_r$ *such that*

$$\Phi(w, x, y) + f(y) - f(x) \leq_{\mathrm{int}C(x)} 0;$$

(vii) $\Phi$ *is Lipschitz continuous in the first variable, continuous and* $C_-$*-concave in the second variable, and* $C_-$*-convex in the third variable;*

(viii) $\Phi$ *is* $C_-$*-monotone with respect to* $T$ *and* $A$*.*

*Then the GVEP has a strong solution.*

**Proof.** This follows immediately from Theorem 3.2. $\square$

We shall give an example in finite dimensional Euclidean spaces where the multifunction $T : K \to 2^{\mathscr{L}(X,Y)}$ and the single-valued function $A : K \times \mathscr{L}(X,Y) \to \mathscr{L}(X,Y)$ satisfy conditions (iii) and (v)-(vii) in Theorem 3.1.

**Example 3.1.** Let $X = Y = \mathbf{R^2}$, $K = [0,1] \times [0,1]$ and $C : K \to 2^{\mathbf{R^2}}$ be a multifunction defined by

$$C(x) = C(x_1, x_2) = \{(r\cos\theta, r\sin\theta) \in \mathbf{R^2} : r \geq 0, 0 \leq \theta \leq \frac{\pi}{8}(x_1 + x_2 + 4)\},$$

for $x = (x_1, x_2) \in K$. Then $C$ is a cone mapping and $C_- = \bigcap_{x \in K} C(x) = \{x = (x_1, x_2) \in \mathbf{R^2} : x_1 \geq 0, x_2 \geq 0\}$. Given any matrix $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{L}(\mathbf{R^2}, \mathbf{R^2})$, we define $\|v\| = |a| + |b| + |c| + |d|$ so that $\|\cdot\|$ induces a norm on $\mathscr{L}(\mathbf{R^2}, \mathbf{R^2})$. Let $A : K \times \mathscr{L}(\mathbf{R^2}, \mathbf{R^2}) \to \mathscr{L}(\mathbf{R^2}, \mathbf{R^2})$ be defined by

$$A(x, u) = \begin{pmatrix} u_{11} & x_1 \\ x_2 & u_{22} \end{pmatrix},$$

where $x = (x_1, x_2) \in K$ and $u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in \mathscr{L}(\mathbf{R^2}, \mathbf{R^2})$, and let $T : K \to 2^{\mathscr{L}(\mathbf{R^2}, \mathbf{R^2})}$ be defined by

$$Tx = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_1 & x_2 \end{pmatrix}, \begin{pmatrix} x_1 & x_1 \\ x_2 & x_2 \end{pmatrix} \right\},$$

where $x = (x_1, x_2) \in K$.

Let $\Phi : \mathscr{L}(\mathbf{R^2}, \mathbf{R^2}) \times K \times K \to \mathbf{R^2}$ be defined by

$$\begin{aligned} \Phi(u, x, y) &= \langle u, y - x \rangle = \langle u, (y_1, y_2) - (x_1, x_2) \rangle \\ &= ((y_1 - x_1)u_{11} + (y_2 - x_2)u_{21}, (y_1 - x_1)u_{12} + (y_2 - x_2)u_{22}), \end{aligned}$$

where $x = (x_1, x_2) \in K$, $y = (y_1, y_2) \in K$ and $u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in \mathscr{L}(\mathbf{R^2}, \mathbf{R^2})$. It is clear that $\Phi$ is a equilibrium-like function.

We first show that $\Phi$ is $C_-$-monotone with respect to $T$ and $A$. Indeed, take three points $x=(x_1,x_2),\ y=(y_1,y_2)$ and $z=(z_1,z_2)$ in $K$ arbitrarily. If $\bar{w}\in A(z,Tx)$ and $\hat{w}\in A(z,Ty)$, then there exist $u\in Tx$ and $v\in Ty$ such that $\bar{w}=A(z,u)$ and $\hat{w}=A(z,v)$. Hence we have

$$A(z,u)-A(z,v)=A((z_1,z_2),u)-A((z_1,z_2),v)=\begin{pmatrix} x_1-y_1 & 0\\ 0 & x_2-y_2\end{pmatrix},$$

and

$$\begin{aligned}\Phi(\bar{w},y,x)+\Phi(\hat{w},x,y)&=\langle\bar{w},x-y\rangle+\langle\hat{w},y-x\rangle\\ &=\langle\bar{w}-\hat{w},x-y\rangle=\langle A(z,u)-A(z,v),x-y\rangle\\ &=\langle A((z_1,z_2),u)-A((z_1,z_2),v),(x_1,x_2)-(y_1,y_2)\rangle\\ &=((x_1-y_1)^2,(x_2-y_2)^2)\\ &\geq_{C_-}(0,0).\end{aligned}$$

We claim that $T$ is $H$-hemicontinuous. Indeed, if $x=(x_1,x_2)\in K,\ y=(y_1,y_2)\in K$ and $\alpha>0$, then

$$d_H(T(x+\alpha y),Tx)=d_H(T((x_1,x_2)+\alpha(y_1,y_2)),T(x_1,x_2))\leq 2\alpha(y_1+y_2)$$

which implies that $d_H(T(x+\alpha y),Tx)\to 0$ as $\alpha\to 0^+$.

On the other hand, for any fixed $u\in\mathscr{L}(\mathbf{R^2},\mathbf{R^2})$, if a sequence $(x_n,y_n)$ in $K$ converges weakly (equivalently, strongly) to $(a,b)$, we deduce that as $n\to\infty$

$$A((x_n,y_n),u)-A((a,b),u)=\|\begin{pmatrix} 0 & x_n-a\\ y_n-b & 0\end{pmatrix}\|=|x_n-a|+|y_n-b|\to 0.$$

Hence $A$ is completely continuous in the first variable, and is of course continuous in the second variable.

Finally, we show that $\Phi$ is Lipschitz continuous in the first variable, weakly sequentially continuous and $C_-$-concave in the second variable, and $C_-$-convex in the third variable. Indeed, since $K$ is a bounded set in $\mathbf{R^2}$, from the definition of $\Phi$ it follows that $\Phi$ is Lipschitz continuous in the first variable. In addition, according to the definition of $\Phi$, it is easy to see that $\Phi$ is weakly sequentially continuous (equivalently, continuous) and $C_-$-concave in the second variable, and $C_-$-convex in the third variable.

## 4. Weak solutions of GVEP without monotonicity

We start with the Brouwer fixed point theorem which enables us to investigate the solvability of the GVEP without monotonicity assumptions.

**Lemma 4.1** (Brouwer's fixed point theorem [1])**.** *Let $K$ be a nonempty compact convex subset of $\mathbf{R^n}$ and let $f:K\to K$ be a continuous function. Then $f$ has a fixed point, i.e., there exists $x\in K$ such that $f(x)=x$.*

**Theorem 4.1.** *Let $X$ be a real reflexive Banach space, $Y$ be a real Banach space, $K$ be a nonempty bounded closed convex subset of $X$, $\Phi : \mathscr{L}(X,Y)\times K\times K \to Y$ be a equilibrium-like function, $C : K \to 2^Y$, $D : K \to 2^Y$ and $T : K \to 2^{\mathscr{L}(X,Y)}$ be three multifunctions, where $D$ is defined by $D(x) = Y \setminus (-\mathrm{int}C(x))$, and $f : K \to Y$ and $A : K \times \mathscr{L}(X,Y) \to \mathscr{L}(X,Y)$ be two single-valued functions. Suppose that:*

(i) *$C$ is a cone mapping such that $\mathrm{int}C_- \neq \emptyset$, where $C_- = \bigcap_{x\in K} C(x)$;*

(ii) *$D$ has weakly closed graph;*

(iii) *$T$ is weakly upper semicontinuous with nonempty weakly compact values;*

(iv) *$f$ is weakly sequentially continuous and $C_-$-convex;*

(v) *$A$ is completely continuous;*

(vi) *$\Phi$ is Lipschitz continuous in the first variable, and weakly sequentially continuous and $C_-$-concave in the second variable.*

*Then the GVEP has a weak solution $x_0 \in K$, that is, for each $y \in K$ there exists $w \in A(x_0, T(x_0))$ such that*

$$\Phi(w, x_0, y) + f(y) - f(x_0) \not\leq_{\mathrm{int}C(x_0)} 0.$$

**Proof.** Suppose on the contrary that this GVEP has no weak solutions. Let $N : K \to 2^K$ be a multifunction defined by, for $y \in K$,

$$N(y) = \{x \in K : \Phi(w, x, y) + f(y) - f(x) \leq_{\mathrm{int}C(x)} 0, \text{ for all } w \in A(x, T(x))\}.$$

To prove each $N(y)$ is weakly open, we consider the complement of $N(y)$ and simply write $M(y) = K \setminus N(y)$. Fix $y \in K$. For any $x$ in the weak closure $\overline{M(y)}^w$ of $M(y)$ which is weakly compact, there is a sequence $\{x_n\}$ in $M(y)$ converging weakly to $x$. Then, for each $n \in \mathbf{N}$ there exists $w_n \in A(x_n, T(x_n))$ satisfying

$$\Phi(w_n, x_n, y) + f(y) - f(x_n) \in D(x_n).$$

It follows that there exists $u_n \in T(x_n)$ such that $w_n = A(x_n, u_n)$ and

$$\Phi(A(x_n, u_n), x_n, y) + f(y) - f(x_n) \in D(x_n). \tag{4.1}$$

Since $T$ is weakly upper semicontinuous with weakly compact values, the sequence $\{u_n\}$ has a subsequence $\{u_{n_j}\}$ that converges weakly to some point $u$ in $T(x)$. For each $n_j$, we have

$$\begin{aligned}&\Phi(A(x_{n_j}, u_{n_j}), x_{n_j}, y) - \Phi(A(x, u), x, y)\\ &\quad = \Phi(A(x_{n_j}, u_{n_j}), x_{n_j}, y) - \Phi(A(x, u), x_{n_j}, y) + \Phi(A(x, u), x_{n_j}, y) - \Phi(A(x, u), x, y).\end{aligned}$$

Since $\Phi$ is Lipschitz continuous in the first variable, there exists a constant $L > 0$ such that

$$\|\Phi(A(x_{n_j}, u_{n_j}), x_{n_j}, y) - \Phi(A(x,u), x_{n_j}, y)\| \leq L\|A(x_{n_j}, u_{n_j}) - A(x,u)\|,$$

which together with the complete continuity of $A$, implies that

$$\lim_{j\to\infty} \|\Phi(A(x_{n_j}, u_{n_j}), x_{n_j}, y) - \Phi(A(x,u), x_{n_j}, y)\| = 0.$$

Also, since $\Phi$ is weakly sequentially continuous in the second variable and $f$ is weakly sequentially continuous, the sequence $\{\Phi(A(x,u), x_{n_j}, y) + f(y) - f(x_{n_j})\}$ converges weakly to $\Phi(A(x,u), x, y) + f(y) - f(x)$. Consequently, the sequence $\{\Phi(A(x_{n_j}, u_{n_j}), x_{n_j}, y) + f(y) - f(x_{n_j})\}$ converges weakly to $\Phi(A(x,u), x, y) + f(y) - f(x)$. Since the graph of $D$ is weakly closed, it follows from Eq. (4.1) that

$$\Phi(A(x,u), x, y) + f(y) - f(x) \in D(x)$$

which means $x \in M(y)$. This shows that $M(y)$ is weakly closed and so $N(y)$ is weakly open.

By our assumption for each $x \in K$ there exists some $y \in K$ such that $x \in N(y)$; hence $K = \bigcup_{y\in K} N(y)$ and $\{N(y) : y \in K\}$ is a weakly open cover of $K$. Since $K$ is weakly compact, there exists a finite subset $\{y_1, \dots, y_n\}$ of $K$ such that

$$K = \bigcup_{i=1}^{n} N(y_i).$$

Then there exists a family of functions $\{\beta_1, \dots, \beta_n\}$ with the following properties:

(a) for each $j$, $\beta_j : K \to [0,1]$ is continuous with respect to the weak topology $\tau$ of $X$;

(b) $\beta_j$ vanishes on $K \setminus N(y_j)$;

(c) $\sum_{j=1}^{n} \beta_j(x) = 1$, for all $x \in K$.

That is, $\{\beta_1, \dots, \beta_n\}$ is a $\tau$-continuous partition of unity subordinated to this finite cover $\{N(y_1), \dots, N(y_n)\}$. Define a function $\varphi : K \to X$ by

$$\varphi(x) = \sum_{j=1}^{n} \beta_j(x) y_j, \quad \forall x \in K,$$

so that $\varphi$ is $\tau$-continuous. Let $S = \mathrm{co}\{y_1, \dots, y_n\} \subset K$. Then $S$ is a compact convex subset of a finite dimensional space and $\varphi$ maps $S$ into $S$. By the Brouwer fixed point theorem (Lemma 4.1), there exists $x_0 \in S$ such that

$$x_0 = \varphi(x_0) = \sum_{j=1}^{n} \beta_j(x_0) y_j.$$

Let $x \in K$. Consider the nonempty set of natural numbers

$$k(x) = \{j \in \mathbf{N} : x \in N(y_j)\}.$$

Since $f$ is $C_-$-convex and $\Phi$ is $C_-$-concave in the second variable, for $w \in A(x_0, T(x_0))$ we have

$$\begin{aligned}
0 &= \Phi(w, x_0, x_0) + f(x_0) - f(x_0) \\
&= \Phi(w, \sum_{j=1}^{n} \beta_j(x_0) y_j, x_0) + f(x_0) - f(\sum_{j=1}^{n} \beta_j(x_0) y_j) \\
&\geq_{C(x_0)} \sum_{j=1}^{n} \beta_j(x_0)[\Phi(w, y_j, x_0) + f(x_0) - f(y_j)] \\
&= \sum_{j\in k(x_0)} \beta_j(x_0)[\Phi(w, y_j, x_0) + f(x_0) - f(y_j)] \\
&\geq_{\mathrm{int}C(x_0)} 0,
\end{aligned}$$

contrary to our hypothesis. Therefore the GVEP has a weak solution. □

The same proof also yields the following result. Just notice that the range space of the mapping $A$ is not $\mathscr{L}(X, Y)$, but $\mathscr{L}_{cc}(X, Y)$ instead. Let $X^w$ denote the space $X$ equipped with the weak topology. We also remark that if $D$ is weak-to-norm upper semicontinuous, then $\mathscr{G}(D)$ is a closed subset of $X^w \times Y$ because $Y$ is regular.

**Theorem 4.2.** *Let $X$ and $Y$ be real Banach spaces, $K$ be a nonempty compact convex subset of $X$, $\Phi : \mathscr{L}(X, Y) \times K \times K \to Y$ be a equilibrium-like function, $C : K \to 2^Y$, $D : K \to 2^Y$ and $T : K \to 2^{\mathscr{L}(X,Y)}$ be three multifunctions, where $D$ is defined by $D(x) = Y \setminus (-\mathrm{int}C(x))$, and $f : K \to Y$ and $A : K \times \mathscr{L}(X, Y) \to \mathscr{L}_{cc}(X, Y)$ be two single-valued functions. Suppose that:*

- (i) *$C$ is a cone mapping such that $\mathrm{int}C_- \neq \emptyset$, where $C_- = \bigcap_{x\in K} C(x)$;*
- (ii) *$\mathscr{G}(D)$ is closed in $X^w \times Y$;*
- (iii) *$T$ is weak-to-norm upper semicontinuous with nonempty compact values;*
- (iv) *$f$ is completely continuous and $C_-$-convex;*
- (v) *$A$ is completely continuous;*
- (vi) *$\Phi$ is Lipschitz continuous in the first variable, and completely continuous and $C_-$-concave in the second variable.*

*Then the GVEP has a weak solution.*

We obtain the following as an immediate consequence of Theorem 4.1.

**Corollary 4.1.** *Let $Y$ be a real Banach space, $K$ be a nonempty bounded closed convex subset of $\mathbf{R^n}$, $\Phi : \mathscr{L}(\mathbf{R^n}, Y) \times K \times K \to Y$ be a equilibrium-like function, $C : K \to 2^Y$, $D : K \to 2^Y$ and $T : K \to 2^{\mathscr{L}(\mathbf{R^n},Y)}$ be three multifunctions, where $D$ is defined by $D(x) = Y \setminus (-\mathrm{int}C(x))$, and $f : K \to Y$ and $A : K \times \mathscr{L}(\mathbf{R^n}, Y) \to \mathscr{L}(\mathbf{R^n}, Y)$ be two single-valued functions. Suppose that:*

- (i) *$C$ is a cone mapping such that $\mathrm{int}C_- \neq \emptyset$, where $C_- = \bigcap_{x\in K} C(x)$;*

(ii) *$D$ has closed graph;*

(iii) *$T$ is upper semicontinuous with nonempty compact values;*

(iv) *$f$ is continuous and $C_-$-convex;*

(v) *$A$ is continuous;*

(vi) *$\Phi$ is Lipschitz continuous in the first variable, and continuous and $C_-$-concave in the second variable.*

*Then the GVEP has a weak solution.*

Theorem 4.1 can be generalized to the case where the set $K$ is closed and convex but not necessarily bounded under a coercive condition.

**Theorem 4.3.** *Let $X$ be a real reflexive Banach space, $Y$ be a real Banach space, $K$ be a nonempty bounded closed convex subset of $X$, $\Phi : \mathcal{L}(X,Y) \times K \times K \to Y$ be a equilibrium-like function, $C : K \to 2^Y$, $D : K \to 2^Y$ and $T : K \to 2^{\mathcal{L}(X,Y)}$ be three multifunctions, where $D$ is defined by $D(x) = Y \setminus (-\mathrm{int}C(x))$, and $f : K \to Y$ and $A : K \times \mathcal{L}(X,Y) \to \mathcal{L}(X,Y)$ be two single-valued functions. Suppose that:*

(i) *$C$ is a cone mapping such that $\mathrm{int}C_- \neq \emptyset$, where $C_- = \bigcap_{x\in K} C(x)$;*

(ii) *$D$ has weakly closed graph;*

(iii) *$T$ is weakly upper semicontinuous with nonempty weakly compact values;*

(iv) *$f$ is weakly sequentially continuous and $C_-$-convex;*

(v) *$A$ is completely continuous;*

(vi) *for each $x \in K$ with $\|x\| = r$ and each $y \in K \cap B_r$, there exists $w \in A(x, T(x))$ such that*

$$\Phi(w,x,y) + f(y) - f(x) \leq_{\mathrm{int}C(x)} 0;$$

(vii) *$\Phi$ is Lipschitz continuous in the first variable, weakly sequentially continuous and $C_-$-concave in the second variable, and $C_-$-convex in the third variable.*

*Then the GVEP has a weak solution.*

**Proof.** By Theorem 4.1, there exists a point $x_r \in K \cap B_r$ with the property that for each $y \in K \cap B_r$, there exists $w \in A(x_r, T(x_r))$ such that

$$\Phi(w,x_r,y) + f(y) - f(x_r) \not\leq_{\mathrm{int}C(x_r)} 0. \tag{4.2}$$

It follows from assumption (vi) that $\|x_r\| < r$. To prove that $x_r$ is a weak solution of the GVEP on $K$, let $z \in K$ and choose $t \in (0,1)$ small enough such that $(1-t)x_r + tz \in K \cap B_r$. In Eq. (4.2), substituting $(1-t)x_r + tz$ for $y$ yields

$$\Phi(w,x_r,(1-t)x_r + tz) + f((1-t)x_r + tz) - f(x_r) \not\leq_{\mathrm{int}C(x_r)} 0, \tag{4.3}$$

for some point $w \in A(x_r, T(x_r))$. Since $f$ is $C_-$-convex and $\Phi$ is $C_-$-convex in the third variable, we have

$$\begin{aligned}
&\Phi(w, x_r, (1-t)x_r + tz) + f((1-t)x_r + tz) - f(x_r) \\
&\quad \leq_{C(x_r)} (1-t)\Phi(w, x_r, x_r) + t\Phi(w, x_r, z) + (1-t)f(x_r) + tf(z) - f(x_r) \\
&\quad = t[\Phi(w, x_r, z) + f(z) - f(x_r)].
\end{aligned} \tag{4.4}$$

Therefore Eqs. (4.3), (4.4) and Lemma 3.2 imply that

$$\Phi(w, x_r, z) + f(z) - f(x_r) \not\leq_{\mathrm{int}C(x_r)} 0.$$

This completes the proof. ☐

## Acknowledgements

L.-C. Ceng was partially supported by the Innovation Program of Shanghai Municipal Education Commission (15ZZ068), Ph.D. Program Foundation of Ministry of Education of China (20123127110002), and Program for Outstanding Academic Leaders in Shanghai City (15XD1503100). Y.-C. Liou was supported in part by MOST 103-2923-E-037-001-MY3 and MOST 104-2221-E-230-004. C.-F. Wen was supported in part by MOST 104-2115-M-037-001.

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