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NORMAL EDGE-TRANSITIVE AND $\frac{1}{2}-$ARC$-$TRANSITIVE CAYLEY GRAPHS ON NON-ABELIAN GROUPS OF ODD ORDER $3pq$, p AND q ARE PRIMES

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Abstract. Suppose p and q are odd prime numbers. In this paper, the connected Cayley graph of groups of order $3pq$, for primes p and q, are investigated and all connected normal $\frac{1}{2}-$arc-transitive Cayley graphs of group of these orders will be classified.

1. Introduction

Throughout this paper all groups are assumed to be finite. Terms and definitions not defined here are follow from Biggs [2]. Suppose $\Gamma = (V, E)$ is a simple graph with vertex set $V = V(\Gamma)$ and edge set $E = E(\Gamma)$. The automorphism group $Aut(\Gamma)$ is acting naturally on the set of all vertices, edges and arcs of Γ. If this action is transitive on vertices, edges or arcs of Γ, then the graph Γ is said to be vertex$-$, edge$-$ or arc$-$transitive, respectively. If Γ is vertex$-$ and edge$-$transitive but not arc$-$transitive, then Γ is called $1/2-$arc$-$transitive.

Suppose G is a finite group and $S \subseteq G$ is non-empty. We assume further that $S = S^{-1}$ and $S \subseteq G \setminus \{1\}$. The Cayley graph $\Gamma = Cay(G, S)$ is defined by $V(\Gamma) = G$ and $E(\Gamma) = \{\{g, sg\} | g \in G, s \in S\}$. It is easy to see that for each element $g \in G$, the mapping $\rho_g : G \to G$ given by $\rho_g(x) = xg$ is an automorphism of Γ. This implies that $R(G) = \{\rho_g \mid g \in G\}$ is a subgroup of $Aut(\Gamma)$ isomorphic to G. Moreover, $Aut(G, S) = \{\alpha \in Aut(G) \mid \alpha(S) = S\}$ is a subgroup of $Aut(\Gamma)$. Following Xu [13], the Cayley graph $\Gamma = Cay(G, S)$ is called normal, if $R(G) \trianglelefteq Aut(\Gamma)$. A Cayley graph Γ is called normal edge$-$transitive or normal arc$-$transitive if $N_{Aut(\Gamma)}(R(G))$ acts transitively on the set of edges or arcs of Γ, respectively. If $\Gamma = Cay(G, S)$ is normal edge$-$transitive, but not normal arc$-$transitive, then Γ is called normal $1/2-$arc$-$transitive. Wang et al. [12] in their seminal paper, constructed all disconnected normal Cayley graphs on a finite group and so for studying the problem of normality in Cayley graphs, it suffices to consider the connected Cayley graphs. For the sake of completeness, we mention here a collection of results which are crucial throughout this paper:

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Theorem 1.1. *Let $\Gamma = Cay(G,S)$ and $A = Aut(\Gamma)$, then the following hold:*

1. ([6]) $N_A(R(G)) = R(G) \rtimes Aut(G,S)$. *The group $R(G)$ is normal in A if and only if $A = R(G) \rtimes Aut(G,S)$.*
2. ([6]) Γ *is normal if and only if $A_1 = Aut(G,S)$;*
3. ([11]) *Let $\Gamma = Cay(G,S)$ be a connected Cayley graph on S. Then Γ is normal edge-transitive if and only if $Aut(G,S)$ is either transitive on S, or has two orbits in S in the form of T and T^{-1}, where T is a non-empty subset of S such that $S = T \bigcup T^{-1}$;*
4. ([3, Corollary 2.3]) *Let $\Gamma = Cay(G,S)$ and H be the subset of all involutions of the group G. If $< H > \neq G$ and Γ is connected normal edge–transitive, then its valency is even;*
5. ([6]) *If $\Gamma = Cay(G,S)$ is a connected Cayley graph on S then Γ is normal arc–transitive if and only if $Aut(G,S)$ acts transitively on S;*
6. ([3, Corollary 2.5]) *If G is a Cayley graph of an abelian group, then G is not a normal $\frac{1}{2}$–arc-transitive Cayley graph.*

It is well-known that there are two non-abelian groups of order 27 presented as follows:

$$G_1 = \langle a,b \mid a^9 = b^3 = 1, b^{-1}ab = a^4\rangle,$$
$$G_2 = \langle a,b,c \mid a^3 = b^3 = c^3 = 1, [a,b] = c, [a,c] = [b,c] = 1\rangle.$$

Suppose U_n denotes the group of units of the ring Z_n. Then,

Theorem 1.2 (See [7])**.** *Up to isomorphism, there are three non-abelian groups of order $9p$, for a prime $p > 3$. These are presented as follows:*

$$G_3 = \langle a,b \mid a^p = b^9 = 1, b^{-1}ab = a^r\rangle,\ where\ r \in U_p\ and\ o(r) = 3;$$
$$G_4 = \langle a,b \mid a^p = b^9 = 1, b^{-1}ab = a^s\rangle,\ where\ s \in U_p\ and\ o(s) = 9;$$
$$G_5 = \langle a,b,c \mid a^p = b^3 = c^3 = [b,c] = [a,b] = 1, c^{-1}ac = a^t\rangle,\ where\ t \in U_p\ and\ o(t) = 3.$$

The automorphism groups of these five groups can be computed as follows:

$$Aut(G_1) = \{\sigma_{i,j,k} \mid \sigma_{i,j,k}(a) = a^i b^j, \sigma_{i,j,k}(b) = a^{3k}b, (i,9) = 1, 0 \le j \le 2, 0 \le k \le 2\},$$
$$Aut(G_2) = \langle \alpha_1, \alpha_2 \mid \alpha_1(a) = b,\ \alpha_1(b) = a^{-1},\ \alpha_1(c) = c, \alpha_2(a) = b,\ \alpha_2(b) = a,\ \alpha_2(c) = c\rangle$$
$$\cong D_8,$$
$$Aut(G_3) = \{\sigma_{i,j,k} \mid \sigma_{i,j,k}(a) = a^i, \sigma_{i,j,k}(b) = a^j b^{3k+1}, 1 \le i \le p-1, 0 \le j \le p-1, 0 \le k \le 2\},$$
$$Aut(G_4) = \{\sigma_{i,j} \mid \sigma_{i,j}(a) = a^i, \sigma_{i,j}(b) = a^j b, 1 \le i \le p-1, 0 \le j \le p-1\},$$

$$Aut(G_5) = \{\sigma_{i,j,k,l} \mid \sigma_{i,j,k,l}(a) = a^i, \sigma_{i,j,k,l}(b) = b^j, \sigma_{i,j,k,l}(c) = a^k b^l c,\ 1 \le i \le p-1,$$
$$1 \le j \le 2, 0 \le k \le p-1, 0 \le l \le 2\}.$$

If p is a prime and $q|p-1$, then we define $F_{p,q}$ to be a group of order pq presented by $F_{p,q} = \langle a, b \mid a^p = b^q = 1, b^{-1}ab = a^u\rangle$, where u is an element of order q in U_p, see [9] for details. We denote this group by $T_{p,q}$, when q is also prime. Ghorbani and Nowroozi Laraki [5] calculated all groups of order $3pq$ and their automorphism groups, p and q are distinct primes. They proved that:

Theorem 1.3. *A group of order $3pq$, $p > q$ are primes, is isomorphic to one of the following groups:*

$$H_1 = Z_{3pq},$$
$$H_2 = Z_3 \times T_{p,q}\ (q \mid p-1),$$
$$H_3 = Z_q \times T_{p,3}\ (3 \mid p-1),$$
$$H_4 = F_{p,3q}\ (3q \mid p-1),$$
$$H_5 = Z_p \times T_{q,3}\ (3 \mid q-1),$$
$$H_6 = \langle a, b, c \mid a^p = b^q = c^3 = 1, [a,b] = 1, c^{-1}bc = b^w, c^{-1}ac = a^s\rangle,$$
$$H_7 = \langle a, b, c \mid a^p = b^q = c^3 = 1, [a,b] = 1, c^{-1}bc = b^{w^2}, c^{-1}ac = a^s\rangle,$$

where s and w are elements of order 3 in U_p and U_q, respectively.

By [5, Theorem 2.7] and some easy calculations, one can see that:

Theorem 1.4. *The automorphism groups of H_2, H_3, H_4, H_5, H_6 and H_7 are computed as follows:*

$$Aut(H_2) = \{\sigma_{i,j,k} \mid \sigma_{i,j,k}(a) = a^i, \sigma_{i,j,k}(b) = ba^j, \sigma_{i,j,k}(c) = c^k, 1 \le i \le p-1, 0 \le j \le p-1,$$
$$1 \le k \le 2\} \cong Z_2 \times F_{p,p-1},$$
$$Aut(H_3) = \{\sigma_{i,j,k} \mid \sigma_{i,j,k}(a) = a^i, \sigma_{i,j,k}(b) = b^j, \sigma_{i,j,k}(c) = ca^k, 1 \le i \le p-1, 1 \le j \le q-1,$$
$$0 \le k \le p-1\} \cong Z_{q-1} \times F_{p,p-1},$$
$$Aut(H_4) = \{\sigma_{i,j} \mid \sigma_{i,j}(a) = a^i, \sigma_{i,j}(b) = ba^j, 1 \le i \le p-1, 0 \le j \le p-1, 0 \le k \le p-1\}$$
$$\cong F_{p,p-1},$$
$$Aut(H_5) = \{\sigma_{i,j,k} \mid \sigma_{i,j,k}(a) = a^i, \sigma_{i,j,k}(b) = b^j, \sigma_{i,j,k}(c) = cb^k, 1 \le i \le p-1, 1 \le j \le q-1,$$
$$0 \le k \le q-1\} \cong Z_{p-1} \times F_{q,q-1},$$
$$Aut(H_6) = \{\sigma_{i,j,k,l} \mid \sigma_{i,j,k,l}(a) = a^i, \sigma_{i,j,k}(b) = b^j, \sigma_{i,j,k}(c) = ca^k b^l, 1 \le i \le p-1, 1 \le j \le q-1,$$
$$0 \le k \le p-1, 0 \le l \le p-1\} \cong F_{p,p-1} \times F_{q,q-1}.$$

Note that $Aut(H_6) \cong Aut(H_7) \cong F_{p,p-1} \times F_{q,q-1}$.

We encourage the interested readers to consult [4, 7, 8, 10] for more information on this topic. Our work is a continuation of recent papers [3, 1]. We will classify all normal edge-transitive and $\frac{1}{2}-$arc$-$transitive Cayley graphs on non-abelian groups of orders $9p$ and $3pq$, when p and q are distinct odd primes.

2. Cayley graphs on groups of odd order $9p$, p is prime

It is clear that a Cayley graph $\Gamma = Cay(G,S)$ is connected if and only if G is generated by S. In this section, the connected Cayley graphs of groups of odd order $9p$, p is prime, are investigated. All Cayley graphs considered here are assumed to be undirected.

Theorem 2.1. *The Cayley graph $\Gamma_1 = Cay(G_1,S)$ is normal $\frac{1}{2}-$arc transitive if and only if the following conditions are satisfied:*

1. *$|S| > 2$ is even, $G_1 = \langle S\rangle$ and $S = S^{-1}$,*
2. *$S = T \cup T^{-1}$, where T is an orbit of $Aut(G_1,S)$ and $T \subseteq \{a^ib \mid (i,9) = 1\}$ or $T \subseteq \{a^ib \mid i = 3k, k = 1,2\}$.*

Moreover, if $\Gamma_1 = Cay(G_1,S)$ is normal $\frac{1}{2}-$arc transitive and $|S| = 2d$ then $d|54$.

Proof. Since G_1 does not have elements of order two, by Theorem 1.1(4), $|S|$ is an even integer. If $|S| = 2$ then G_1 is cyclic which is not possible. So, we can assume that $|S| > 2$. By Table 1, there is no automorphism that maps a^ib^j to $a^{3k}b^{j'}$, where $(i,9) = 1$. On the other hand, there is no automorphism that maps a^lb to $(a^lb)^{-1}$. To prove, we first note that $(a^lb)^{-1} = a^{-4l}b^2 = b^2a^{-4^3l}$ and $\sigma_{i,j,k}(a^lb) = b^{lj+1}a^{4(i(4^{lj}+\cdots+4+1))+3k}$. Suppose $b^{lj+1}a^{4(i(4^{lj}+\cdots+4+1))+3k} = b^2a^{-4^3l}$. Then $b^{2-lj-1} = a^{4(i(4^{lj}+\cdots+4+1))+3k+4^3l} = a^{4(i(4^{lj}+\cdots+4+1)+4^2l)+3k}$ which implies by Table 1 that $2-lj-1 \equiv 0 \bmod 3)$ and $4(i(4^{lj}+\cdots+4+1)+4^2l)+3k \equiv 0 \bmod 9)$. Thus, $lj \equiv 1 \bmod 3)$ and the following cases can be occurred:

$$\begin{cases} j=1, l=1 \\ j=1, l=4 \\ j=1, l=7 \end{cases} \text{and} \begin{cases} j=2, l=2 \\ j=2, l=5 \\ j=2, l=8 \end{cases} ;\ 0 \le j \le 2\ \&\ 1 \le l \le 8$$

We now consider the following cases:

1. $j = l = 1$. Since $4(i(4^{lj}+\cdots+4+1)+4^2l)+3k \equiv\ 0 \bmod 9)$, $2i+3k+1 \equiv 0 \bmod 9)$. So, we have the following three cases: $k = 0$, $i = 4$; $k = 1, i = 7$ or $k = 2, i = 1$. If $k = 0$, $i = 4$ then $\sigma_{4,1,0}(a^{-4}b^2) = \sigma_{4,1,0}(a^5b^2) = b^7a^{4^3(4^5+\cdots+4+1)} = ab = ba^4$ and so $a^2 = 1$, a contradiction. We now assume that $k = 1$ and $i = 7$. Then $\sigma_{7,1,1}(a^{-4}b^2) = \sigma_{7,1,1}(a^5b^2) =$

Table 1: The Orders of Elements in G_i, $1 \le i \le 5$.

Order	G_1	G_2	G_3	G_4	G_5
a^i	$\begin{cases} 9 & (i,9)=1 \\ 3 & 3\vert i \end{cases}$	3	p	p	p
b^j	3	3	9	$\begin{cases} 9 & (j,9)=1 \\ 3 & 3\vert j \end{cases}$	3
a^ib^j	$\begin{cases} 9 & (i,9)=1 \\ 3 & 3\vert i \end{cases}$	3	$\begin{cases} 9 & (j,9)=1 \\ 3 & 3\vert j \end{cases}$	9	$3p$
a^ic^k	−	3	−	−	3
b^jc^k	−	3	−	−	3
$a^ib^jc^k$	−	3	−	−	3
c^k	−	3	−	−	3

$b^7a^{7\cdot 4^2(4^5+\cdots+4+1)+6} = ab = ba^4$ leaded us to $a = 1$ which is impossible. Finally, we assume that $k = 2$ and $i = 1$. Then $\sigma_{1,1,2}(a^{-4}b^2) = \sigma_{1,1,2}(a^5b^2) = b^7a^{4^2(4^5+\cdots+4+1)+120} = ab = ba^4$ that implies our final contradiction $a^5 = 1$.

2. $j = 1, l = 4$. In this case, $5i + 3k + 4 \equiv 0 \bmod 9)$ that leaded us to the following subcases: $k = 0, i = 1$; $k = 1, i = 4$ and $k = 2, i = 7$. If $k = 0, i = 1$ then $\sigma_{1,1,0}(a^{-16}b^2) = \sigma_{1,1,0}(a^2b^2) = b^4a^{4^3\cdot 5} = ba^7$ and so $a^2 = 1$, a contradiction. If $k = 2, i = 7$ then $\sigma_{4,1,1}(a^2b^2) = b^4a^8 = ba^7$ which leads to the contradiction $a = 1$. Finally, we assume that $k = 2, i = 7$. Hence $\sigma_{7,1,2}(a^2b^2) = b^4a^2 = ba^7$ and so $a^5 = 1$ which is our final contradiction.

3. By a similar argument as above, the cases $j = 1, l = 7$; $j = l = 2$; $j = 2, l = 5$ and $j = 2, l = 8$ cannot be occurred.

This proves that there is no automorphism that maps a^lb to $(a^lb)^{-1}$. Since $Aut(G_1,S) \le Aut(G_1)$, each orbit of $Aut(G_1,S)$ under its natural action on S is a subset of an orbit of $Aut(G_1)$ under its action on G_1. So, $S = T \cup T^{-1}$, where $T \subseteq \{a^ib \mid (i,9) = 1\}$ or $T \subseteq \{a^ib \mid i = 3k, k = 1,2\}$. If $|S| = 2d$ then since $Aut(G_1,S)$ has a transitive action on T,
$|T| \mid |Aut(G_1,S)| \mid |Aut(G_1)| = 2.3^3$. On the other hand, the equation $|S| = |T| + |T^{-1}| = 2|T|$ implies that $|T| = d$ and so $d \mid 2.3^3$. □

In the following example, we apply previous theorem to prove that $Cay(G_1,S)$, $|S| = 4$, is normal $\frac{1}{2}$−arc transitive.

Example 2.2. Suppose $S = \{a^ib, a^{-i}b, (a^ib)^{-1}, (a^{-i}b)^{-1}\}$. Since $(a^ib)(a^{-i}b)^{-1} = a^iba^{4i}b^2 = a^{2i} \in \langle S\rangle$, $a^{-4i} \in \langle S\rangle$. This shows that $a^{-4i}a^{4i}b^2 \in \langle S\rangle$ and so $b^2 \in \langle S\rangle$. Thus, $b^{-1} \in \langle S\rangle$. On the other hand, $a^ibb^{-1} = a^i \in \langle S\rangle$ and so $a \in \langle S\rangle$. Hence $G = \langle S\rangle$ which proves that $Cay(G_1,S)$ is connected. Consider the automorphism $\sigma_{-1,0,0}$. Since $\sigma_{-1,0,0}(a^ib) = a^{-i}b$, $\sigma_{-1,0,0}(a^{4i}b^2) =$

$a^{-4i}b^2$, $\sigma_{-1,0,0}(a^{-i}b) = a^ib$ and $\sigma_{-1,0,0}(a^{-4i}b^2) = a^{4i}b^2$, $T = \{a^ib, a^{-i}b\}$ and $T^{-1} = \{a^{4i}b^2, a^{-4i}b^2\}$ are orbits of $Aut(G_1, S)$ on S. So, by previous theorem, $Cay(G_1, S)$ is normal $\frac{1}{2}-$arc transitive.

Theorem 2.3. *The Cayley graph $\Gamma_2 = Cay(G_2, S)$ is normal $\frac{1}{2}-$arc transitive if and only if the following conditions are satisfied:*

1. $|S| > 2$ *is even,* $G_2 = \langle S \rangle$ *and* $S = S^{-1}$,
2. $S = T \cup T^{-1}$, *where T is an orbit of $Aut(G_2, S)$ and* $T \subseteq \{ca^ib^j \mid i \neq j\ ;\ 1 \leq i, j \leq 2\}$ *or* $T \subseteq \{c^2a^ib^i \mid 1 \leq i \leq 2\}$ *or* $T = \{ca, ca^2, cb, cb^2\}$.

Moreover, if $\Gamma_2 = Cay(G_2, S)$ is normal $\frac{1}{2}-$arc transitive and $|S| = 2d$ then $d = 2, 4$.

Proof. The proof of Part (1) is similar to Theorem 2.1 and so it is omitted. To prove (2), we note that the orbits of $Aut(G_2)$ on G_2 are as follows:

$$\{1\}, \{c\}, \{c^2\}, \{cab, ca^2b, cab^2, ca^2b^2\}, \{c^2ab, c^2a^2b, c^2ab^2, c^2a^2b^2\},$$
$$\{ca, ca^2, cb, cb^2\}, \{c^2a, c^2a^2, c^2b, c^2b^2\}, \{ab, a^2b, ab^2, a^2b^2\}, \{a, a^2, b, b^2\}.$$

Since there is no orbit L of size four such that $G = \langle L \rangle$ and $L = L^{-1}$, by Theorem 1.1 (3) S has the form $T \cup T^{-1}$ for an orbit T. Finally, we assume that $\Gamma_2 = Cay(G_2, S)$ is normal $\frac{1}{2}-$arc transitive and $|S| = 2d$. Since $|S| = |T| + |T^{-1}|$ and T is an orbit of $Aut(G_2, S)$ on S, $|T| \mid |Aut(G_2, S)| \mid |Aut(G_2)| = 8$. Therefore, $|T| = 2, 4, 8$. But T is a subset of an orbit of $Aut(G_2)$ on G_2, so $|T| = 2$ or 4. Hence the result. □

To explain the previous theorem, we investigates the case of $|S| = 4$.

Example 2.4. Suppose $S = \{ca^2b, cab^2, a^2b, ab^2\}$. Since $\alpha_1^2(ca^2b) = cab^2$, $\alpha_1^2(a^2b) = ab^2$, $\alpha_1^2(cab^2) = ca^2b$ and $\alpha_1^2(ab^2) = a^2b$, $\alpha_1^2 \in Aut(G_2, S)$. Thus, if $T = \{ca^2b, cab^2\}$ then $S = T \cup T^{-1}$ and so $Cay(G_2, S)$ is normal $\frac{1}{2}-$arc transitive.

Theorem 2.5. *The Cayley graph $\Gamma_3 = Cay(G_3, S)$ is normal $\frac{1}{2}-$arc transitive if and only if the following conditions are satisfied:*

1. $|S| > 2$ *is even,* $G_3 = \langle S \rangle$ *and* $S = S^{-1}$,
2. $S = T \cup T^{-1}$, *where T is an orbit of $Aut(G_3, S)$ and*

$$T \subseteq \{a^lb, a^kb^4, a^tb^7 \mid 1 \leq l, t, k \leq p-1\ ;\ l \neq t\ ; l \neq k\ ;\ t \neq k\},$$
$$T \subseteq \{a^lb^3, a^kb^3 \mid 1 \leq l, k \leq p-1\ ;\ l \neq k\}.$$

Moreover, if $\Gamma_3 = Cay(G_3, S)$ *is normal* $\frac{1}{2}$*−arc transitive and* $|S| = 2d$ *then* $d|3p(p-1)$.

Proof. Since each orbit of $Aut(G_3, S)$ on S is a subset of an orbit of $Aut(G_3)$ under its natural action on G_3 and there is no orbit containing elements of the form x and x^{-1}, a similar argument like Theorem 2.1 leaded us to the proof of this theorem. □

Example 2.6. Set $S = \{a^l b, a^l b^4, a^l b^7, (a^l b)^{-1}, (a^l b^4)^{-1}, (a^l b^7)^{-1}\}$ and $T = \{a^l b, a^l b^4, a^l b^7\}$. Since,

$$\begin{array}{ll}
\sigma_{1,0,1}(a^l b) = a^l b^4 & \sigma_{1,0,1}(a^l b^4) = a^l b^7 \\
\sigma_{1,0,1}(a^l b^7) = a^l b & \sigma_{1,0,1}(a^{-l(r^2+1)} b^8) = a^{-l(r^2+1)} b^5 \\
\sigma_{1,0,1}(a^{-l(r^2+1)} b^5) = a^{-l(r^2+1)} b^2 & \sigma_{1,0,1}(a^{-l(r^2+1)} b^2) = a^{-l(r^2+1)} b^8 \\
\sigma_{1,0,2}(a^l b) = a^l b^7 & \sigma_{1,0,2}(a^l b^4) = a^l b \\
\sigma_{1,0,2}(a^l b^7) = a^l b^4 & \sigma_{1,0,2}(a^{-l(r^2+1)} b^8) = a^{-l(r^2+1)} b^2 \\
\sigma_{1,0,2}(a^{-l(r^2+1)} b^5) = a^{-l(r^2+1)} b^8 & \sigma_{1,0,2}(a^{-l(r^2+1)} b^2) = a^{-l(r^2+1)} b^5
\end{array}$$

by consdiering $S = T \cup T^{-1}$, $\Gamma_3 = Cay(G_3, S)$ is normal $\frac{1}{2}$−arc transitive Cayley graph.

Example 2.7. Set $S = \{a^i b^3, a^{-i} b^3, a^i b^6, a^{-i} b^6\}$ and $T = \{a^i b^3, a^{-i} b^3\}$. Since $S = T \cup T^{-1}$, $\sigma_{-1,0,1}(a^i b^3) = a^{-i} b^3$, $\sigma_{-1,0,1}(a^{-i} b^3) = a^i b^3$, $\sigma_{-1,0,1}(a^i b^6) = a^{-i} b^6$ and $\sigma_{-1,0,1}(a^{-i} b^6) = a^i b^6$, $\Gamma_3 = Cay(G_3, S)$ is normal $\frac{1}{2}$−arc transitive Cayley graph.

Using a similar argument as Theorems 2.1, it is possible to investigate the normal $\frac{1}{2}$−arc transitive Cayley graphs constructed by groups G_4 and G_5. We mention here these results without proof.

Theorem 2.8. *The Cayley graph* $\Gamma_4 = Cay(G_4, S)$ *is normal* $\frac{1}{2}$*−arc transitive if and only if the following conditions are satisfied:*

1. $|S| > 2$ *is even,* $G_4 = \langle S \rangle$ *and* $S = S^{-1}$,
2. $S = T \cup T^{-1}$, *where* T *is an orbit of* $Aut(G_4, S)$ *and for a fix positive integer* j, $1 \le j \le 9$, *we have* $T \subseteq \{a^l b^j \mid 1 \le l \le p-1\}$.

Moreover, if $\Gamma_4 = Cay(G_4, S)$ *is normal* $\frac{1}{2}$*−arc transitive and* $|S| = 2d$ *then* $d = p$ *or* $d|p-1$.

Theorem 2.9. *The Cayley graph* $\Gamma_5 = Cay(G_5, S)$ *is normal* $\frac{1}{2}$*−arc transitive if and only if the following conditions are satisfied:*

1. $|S| > 2$ *is even,* $G_5 = \langle S \rangle$ *and* $S = S^{-1}$,
2. $S = T \cup T^{-1}$, *where* T *is an orbit of* $Aut(G_5, S)$ *and* $T \subseteq \{a^i b^j c \mid 1 \le i \le p-1\,;\, 1 \le j \le 2\}$

Table 2: The Orbits of $Aut(H_2)$.

$\{1\},\ \{a^i \mid 1 \le i \le p-1\}$,	$\{cb, cba, \cdots, cba^{p-1}\} \cup \{c^2b, c^2ba, \cdots, c^2ba^{p-1}\}$
$\{c, c^2\},\ \{ca^i, c^2a^i \mid 1 \le i \le p-1\}$,	$\{cb^2, cb^2a, \cdots, cb^2a^{p-1}\} \cup \{c^2b^2, c^2b^2a, \cdots, c^2b^2a^{p-1}\}$
$\{b, ba, ba^2, \cdots, ba^{p-1}\}$,	$\{cb^3, cb^3a, \cdots, cb^3a^{p-1}\} \cup \{c^2b^3, c^2b^3a, \cdots, c^2b^3a^{p-1}\}$
$\vdots$	$\vdots$
$\{b^{q-1}, b^{q-1}a, b^{q-1}a^2, \cdots, b^{q-1}a^{p-1}\}$,	$\{c^ib^{q-1}, c^ib^{q-1}a, \cdots, c^ib^{q-1}a^{p-1} \mid i = 1, 2\}$

Moreover, if $\Gamma_5 = Cay(G_5, S)$ *is normal* $\frac{1}{2}$*−arc transitive and* $|S| = 2d$ *then* $d|6p(p-1)$.

3. Cayley graphs on groups of odd order $3pq$, p and q are distinct primes

In this section, the connected Cayley graphs of groups of odd order $3pq$, p and q are distinct primes, are investigated. All Cayley graphs considered here are assumed to be undirected.

Apply Theorem 1.4 to compute the orbits of $Aut(H_i)$ under natural action on H_i, $3 \le i \le 7$. Suppose n_i, $3 \le i \le 7$, denote the number of orbits of $Aut(H_i)$ on H_i. Then by a tedious calculation, one can see that $n_3 = 7$, $n_4 = 3q+2$, $n_5 = 8$, $n_6 = n_7 = 6$. Moreover, we assume that Ω_i^j, $3 \le j \le 7$ and $1 \le i \le n_j$, denote the i^{th} orbit of $Aut(H_j)$ on H_j. Our calculations are recorded in Table 4.

Theorem 3.1. *The Cayley graph* $\Delta_2 = Cay(H_2, S)$ *is normal* $\frac{1}{2}$*−arc transitive if and only if the following conditions are satisfied:*

1. $|S| > 2$ *is even,* $H_2 = \langle S \rangle$ *and* $S = S^{-1}$,
2. $S = T \cup T^{-1}$*, where* T *is an orbit of* $Aut(H_2, S)$ *and for a fixed* j, $T \subseteq \{c^ib^ja^k \mid 1 \le i \le 2\,;\, 0 \le k \le p-1\}$.

Moreover, if $\Delta_2 = Cay(H_2, S)$ *is normal* $\frac{1}{2}$*−arc transitive and* $|S| = 2d$ *then* $d = p$ *or* $d|2(p-1)$.

Proof. It is clear that each orbit of $Aut(H_2, S)$ under its natural action on S is a subset of an orbit of $Aut(H_2)$ on H_2. Note that the orbits in the second column of Table 2, is generated the group H_2.

Since for each orbit O in the second column, $O \cap O^{-1} = \emptyset$, S can be written as $T \cup T^{-1}$, where T is an orbit of $Aut(H_2, S)$. Thus, Δ_2 is normal $\frac{1}{2}$−arc transitive. To prove (2), we notice that $|Aut(H_2)| = 2p(p-1)$ and by a similar argument as Theorem 2.1(2), $d = p$ or $d|2(p-1)$.□

Since each orbit of $Aut(H_i, S_i)$ under natural action on S_i is a subset of the orbits of $Aut(H_i)$ on H_i and Tables 3 and 4, we have the following theorem:

Table 3: The Orders of Elements in $H_i, 1 \le i \le 7$.

Orders	H_2	H_3	H_4	H_5	H_6	H_7
$O(a^i)$	p	p	p	p	p	p
$O(b^j)$	q	q	$\frac{3q}{(3q,j)}$	pq	q	q
$O(c^k)$	3	3	–	3	3	3
$O(a^ib^j)$	q	–	–	pq	pq	pq
$O(a^ic^k)$	$3p$	–	–	$3p$	–	–
$O(b^jc^k)$	$3q$	–	–	–	–	–
$O(b^ja^i)$	–	pq	$\begin{cases} 3p & (j,3q) = q \\ 3q & (j,3q) = 1 \\ q & (j,3q) = 3 \end{cases}$	–	–	–
$O(c^ka^i)$	–	3	–	–	3	3
$O(c^kb^ja^i)$	–	$3q$	–	–	3	3
$O(a^ib^jc^k)$	$3q$	–	–	–	–	
$O(a^ic^kb^j)$	–	–	–	$3p$	–	–

Theorem 3.2. *The Cayley graph $\Delta_i = Cay(H_i, S)$, $3 \le i \le 7$, is normal $\frac{1}{2}$−arc−transitive if and only if the following conditions are satisfied:*

1. $|S| > 2$ *is even,* $H_i = \langle S\rangle$ *and* $S = S^{-1}$,
2. $S = T_i \cup T_i^{-1}$, *where* T_i *is an orbit of* $Aut(H_i, S)$, $T_3 \subseteq \{cb^ia^j \mid 1 \le j \le p-1\}$, $1 \le i \le q-1$, $T_4 \subseteq \{b^ia^j \mid 1 \le j \le p-1\}$, $1 \le i \le 3q-1$, $T_5 \subseteq \{ca^ib^j \mid 1 \le j \le q-1\}$, $1 \le i \le p-1$, *and* $T_6, T_7 \subseteq \{ca^ib^j \mid 1 \le i \le p-1 \& 1 \le j \le q-1\}$.

Moreover, if $|S_i| = 2d_i$ *and* $\Delta_i = Cay(H_i, S_i)$, $3 \le i \le 7$, *is normal* $\frac{1}{2}$*−arc−transitive then* $d_3|p(p-1)(q-1)$, $d_4|p(p-1)$, $d_5|q(p-1)(q-1)$ *and* $d_6, d_7|pq(p-1)(q-1)$.

Proposition 3.3. *Suppose* $S = \{c^iba^k, c^iba^l, (c^iba^k)^{-1}, (c^iba^l)^{-1}\}$, $l \ne k$. *Then* $Cay(H_2, S)$ *is normal* $\frac{1}{2}$*−arc−transitive and* $Aut(H_2, S)$ *is a cyclic group of order* 2.

Proof. It is clear that,

$$\sigma_{-1,l+k,1}(c^iba^k) = c^iba^l,$$
$$\sigma_{-1,l+k,1}(c^iba^l) = c^iba^k,$$
$$\sigma_{-1,l+k,1}(c^{3-i}b^{q-1}a^{-k(u^q-1)}) = c^{3-i}b^{q-1}a^{-l(u^q-1)},$$
$$\sigma_{-1,l+k,1}(c^{3-i}b^{q-1}a^{-l(u^q-1)}) = c^{3-i}b^{q-1}a^{-k(u^q-1)}.$$

There is no automorphism $\alpha \in Aut(H_2)$ with this property that for an element $t \in S$, $\alpha(t) = t^{-1}$. Thus, if $T = \{c^iba^k, c^iba^l\}$ then $S = T \cup T^{-1}$ where T is an orbit of $Aut(H_2, S)$ and

Table 4: The Orbits of $Aut(H_i)$ on H_i under Natural Group Action, $3 \le i \le 7$.

$\Omega_1^3 = \Omega_1^4 = \Omega_1^5 = \Omega_1^6 = \Omega_1^7 = \{1\}$,
$\Omega_2^3 = \{c^2 a^i \mid 0 \le i \le p-1\}$, $\Omega_3^3 = \{ca^i \mid 0 \le i \le p-1\}$,
$\Omega_4^3 = \{b^i a^j \mid 1 \le i \le q-1, 1 \le j \le p-1\}$,
$\Omega_5^3 = \{cb^i a^j \mid 1 \le i \le q-1, 1 \le j \le p-1\}$,
$\Omega_6^3 = \{c^2 b^i a^j \mid 1 \le i \le q-1, 1 \le j \le p-1\}$, $\Omega_2^4 = \{a^i \mid 1 \le i \le p-1\}$,
$\Omega_3^4 = \{ba^i \mid 0 \le i \le p-1\}$, $\ldots$, $\Omega_{3q+2}^4 = \{b^{3q-1} a^i \mid 0 \le i \le p-1\}$,
$\Omega_2^5 = \{a^i \mid 1 \le i \le p-1\}$, $\Omega_3^5 = \{b^i \mid 1 \le i \le q-1\}$,
$\Omega_4^5 = \{cb^j \mid 0 \le j \le q-1\}$, $\Omega_5^5 = \{c^2 b^j \mid 0 \le j \le q-1\}$,
$\Omega_6^5 = \{a^i b^j \mid 1 \le i \le p-1 \& 1 \le j \le q-1\}$, $\Omega_7^5 = \{ca^i b^j \mid 1 \le i \le p-1 \& 0 \le j \le q-1\}$,
$\Omega_8^5 = \{c^2 a^i b^j \mid 1 \le i \le p-1 \& 0 \le j \le q-1\}$,
$\Omega_2^6 = \{a^i \mid 1 \le i \le p-1\}$, $\Omega_3^6 = \{b^j \mid 1 \le j \le q-1\}$,
$\Omega_4^6 = \{a^i b^j \mid 1 \le i \le p-1 \ \& \ 1 \le j \le q-1\}$, $\Omega_5^6 = \{ca^i b^j \mid 0 \le i \le p-1 \ \& \ 0 \le j \le q-1\}$,
$\Omega_6^6 = \{c^2 a^i b^j \mid 0 \le i \le p-1 \ \& \ 0 \le j \le q-1\}$,
$\Omega_2^7 = \{a^i \mid 1 \le i \le p-1\}$, $\Omega_3^7 = \{b^j \mid 1 \le j \le q-1\}$,
$\Omega_4^7 = \{a^i b^j \mid 1 \le i \le p-1 \ \& \ 1 \le j \le q-1\}$, $\Omega_5^7 = \{ca^i b^j \mid 0 \le i \le p-1 \ \& \ 0 \le j \le q-1\}$,
$\Omega_6^7 = \{c^2 a^i b^j \mid 0 \le i \le p-1 \ \& \ 0 \le j \le q-1\}$.

so $Cay(H_2, S)$ is normal $\frac{1}{2}$−arc−transitive. To prove $Aut(H_2, S) \cong Z_2$, we notice that $H_2 = \langle S \rangle$ and $Aut(H_2, S)$ has a faithful action on S. This implies that $Aut(H_2, S)$ is isomorphic to a subgroup of H_2. We first prove that $Aut(H_2, S)$ does not have an element of order 3 and 4. If $\sigma \in Aut(H_2, S)$ has order 3, then the automorphism σ is fixed an element $y \in S$. This implies that y^{-1} is another fixed element of σ, which is impossible. We now assume that $\sigma \in Aut(H_2, S)$ has order 4, $x = c^i ba^l$ and $y = c^i ba^k$. Then σ has the forms $g = (xy^{-1}x^{-1}y)$ or $h = (xyx^{-1}y^{-1})$. Next $\sigma \in Aut(H_2, S) \subseteq Aut(H_2)$ and so there exist r, s, t, $1 \le r \le p-1$, $0 \le s \le p-1$ and $1 \le t \le 2$ such that $\sigma = \sigma_{r,s,t}$. If $\sigma = g$ then $\sigma(x) = y^{-1}$ which implies that $\sigma_{r,s,t}(c^i ba^l) = c^{3-i} b^{q-1} a^{-k(u^q-1)}$. But $\sigma_{r,s,t}(c^i ba^l) = c^{it} ba^{s+lr}$ and so $c^{it} ba^{s+lr} = c^{3-i} b^{q-1} a^{-k(u^q-1)}$. Thus, we have $c^{3-i-it} = b^{2-q} a^{s+lr+k(u^q-1)}$ and therefore $b^{2-q} a^{s+lr+k(u^q-1)} \neq e$. Othewise, $b^{2-q} = a^{s+lr+k(u^q-1)}$. Since $q > 3$, $b^{q-2} \neq e$ and $O(b^{2-q}) \neq O(a^{s+lr+k(u^q-1)})$, $c^{3-i-it} \neq e$. On the other hand, $O(c^{3-i-it}) = 3$ and $O(b^{2-q} a^{s+lr+ku^{q-1}}) = q$ that leaded us to another contradiction. If $h = \sigma_{r,s,t}$ then a similar argument as above gives a contradiction. This proves that there is no automorphism of order 4.

We now prove that $Aut(H_2, S) = \langle \sigma_{-1,l+k,1} \rangle$. Suppose $\sigma_{r,s,t}$ is an arbitrary element of $Aut(H_2, S)$. Since there is no automorphism α in $Aut(H_2)$ such that α maps $c^i b^j a^k$ to $c^{i'} b^{j'} a^{k'}$,

$j \neq j'$, $\sigma_{r,s,t}(c^i ba^l) = c^i ba^k$ and $\sigma_{r,s,t}(c^i ba^k) = c^i ba^l$. Therefore,

$$\begin{aligned}
\sigma_{r,s,t}(c^{3-i}b^{q-1}a^{-k(u^q-1)}) &= c^{3-i}b^{q-1}a^{-l(u^q-1)},\\
\sigma_{r,s,t}(c^{3-i}b^{q-1}a^{-l(u^q-1)}) &= c^{3-i}b^{q-1}a^{-k(u^{q-1})},\\
\sigma_{r,s,t}(c^i ba^l) &= c^{it}ba^{s+lr},\\
\sigma_{r,s,t}(c^i ba^k) &= c^{it}ba^{s+kr},\\
\sigma_{r,s,t}(c^{3-i}b^{q-1}a^{-l(u^q-1)}) &= c^{t(3-i)}b^{q-1}a^{s(u^{q-2}+\cdots+u+1)-lru^{q-1}},\\
\sigma_{r,s,t}(c^{3-i}b^{q-1}a^{-k(u^q-1)}) &= c^{t(3-i)}b^{q-1}a^{s(u^{q-2}+\cdots+u+1)-kru^{q-1}}.
\end{aligned}$$

Thus, the following equalities are satisfied:

$$\begin{aligned}
c^i ba^l = c^{it}ba^{s+lr} \; &and \; c^i ba^l = c^{it}ba^{s+lr},\\
c^{3-i}b^{q-1}a^{-k(u^q-1)} &= c^{t(3-i)}b^{q-1}a^{s(u^{q-2}+\cdots+u+1)-kru^{q-1}},\\
c^{3-i}b^{q-1}a^{-l(u^q-1)} &= c^{t(3-i)}b^{q-1}a^{s(u^{q-2}+\cdots+u+1)-lru^{q-1}}.
\end{aligned}$$

This shows that $c^{i(1-t)} = a^{s+kr-l}$. Therefore, $t \equiv 1 \bmod 3)$ and $s+kr-l \equiv 0 \bmod p)$. On the other hand, $a^{s(u^{q-2}+\cdots+u+1)-kru^{q-1}} = a^{-l(u^{q-1})}$ and $a^{s(u^{q-2}+\cdots+u+1)-lru^{q-1}} = a^{-k(u^{q-1})}$. These congruences and above equalities imply that $r \equiv -1 \bmod p)$ and $s \equiv k+l \bmod p)$ which completes our proof. □

By a similar argument as Proposition 3.3, one can prove the following result:

Proposition 3.4. *Define*

$$\begin{aligned}
S &= \{cb^j a^k, cb^j a^l, (cb^j a^k)^{-1}, (cb^j a^l)^{-1}\},\\
S' &= \{ba^l, ba^k, (ba^l)^{-1}, (ba^k)^{-1}\},
\end{aligned}$$

where $l \neq k$. Then $Cay(H_3, S)$ and $Cay(H_4, S')$ are normal $\frac{1}{2}$−arc−transitive. Moreover, $Aut(H_3, S)$ and $Aut(H_4, S')$ are cyclic groups of order 2.

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