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GENERALIZATION OF FIXED POINT THEOREMS RELATING TO THE DIAMETER OF ORBITS BY USING A CONTROL FUNCTION

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Abstract. The main purpose of this paper is to obtain fixed points for a selfmap T of a metric space which is T-orbitally complete under a more general contraction type condition by using a certain continuous control function. Further generalization relating to the diameter of orbits is given.

1. Introduction

The study of fixed point theory has been increased due to its wide applications in proving the existence and uniqueness of solutions of differential, integral, integro-differential and impulsive differential equations and in obtaining solutions of optimization problems; and hence its development is in the following three main directions:

(i) generalization of conditions which ensure the existence, and if possible, uniqueness of fixed points;

(ii) investigation of the character of the sequence of iterates $\{T^n x\}_{n=0}^{\infty}$ where $T : X \to X$, X a metric or complete metric space, $x \in X$ is the map under consideration; and

(iii) study of the topological properties of the set of fixed points, whenever T has more than one fixed point.

This paper deals with the directions mentioned in (i) and (ii) above.

A new technique of generalization of conditions and obtaining fixed point theorems is known as *altering distances between the points of the underlying space* by using a certain function. Delbosco [3] and Skof [9] initiated this technique with the use of a function $\psi : R^+ \to R^+$ ($R^+ = [0, \infty)$) satisfying the following properties:

(1) ψ is continuous
(2) ψ is strictly increasing in R^+
(3) $\psi(t) = 0$ if and only if $t = 0$
(4) $\psi(t) \geq M t^{\mu}$ for every $t > 0$, where $M > 0$, $\mu > 0$ are constants,

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and established fixed point theorems in complete metric spaces.

Park [6, 7] established some fixed point theorems in this direction and it became famous by Khan, Swaleh and Sessa [4]. Sastry and Babu [8] discussed in detail and established the existence of fixed points in metric spaces for a single selfmap and pair of selfmaps by using a function $\psi : R^+ \to R^+$ satisfying (1) and (3) only, with several examples. The benefit of using a control function is that it unifies and generalizes many known results.

The purpose of this paper is to obtain fixed points for a selfmap T of a metric space which is T-orbitally complete under a more general contraction type condition (2.2.1) by using a certain continuous control function (Theorem 3.1). Further generalization relating to the diameter of orbits is obtained (Theorem 3.6).

Notation, some definitions and known results are given in Section 2. In Section 3, we present main results. Examples are given in Section 4.

2. Notation, Definition and Some Known Results

Let (X, d) be a metric space, T a selfmap of X. For any $x \in X$, $O(x) = \{x, Tx, T^2x, \ldots\}$ denotes the *orbit of* x. Throughout this paper, we assume $O(x)$ is bounded.

For any subset A of X, $\delta[A] = \sup\{d(x, y) : x, y \in A\}$ is the *diameter of* A.

R^+ denotes the set of all nonnegative reals and N the set of all natural numbers. We write

$$\Psi = \left\{ \psi : R^+ \to R^+ \Big/ \begin{array}{l} \text{(i) } \psi \text{ is continuous, (ii) } \psi \text{ is nondecreasing} \\ \text{and (iii) } \psi(t) = 0 \text{ if and only if } t = 0 \end{array} \right\}.$$

We used to call an element $\psi \in \Psi$, *a continuous control function.*

We observe that if ψ is an element of Ψ and d is a metric on X then the composition of ψ and d, namely $\psi \circ d$ defined on $X \times X$ by $(\psi \circ d)(x, y) = \psi(d(x, y))$, $x, y \in X$ need not be a metric; for example, let $X = R$, the real line and define ψ on R^+ by $\psi(t) = t^2$. Then, for $x, y \in R$, $(\psi \circ d)(x, y) = \psi(|x - y|) = |x - y|^2$, and $\psi \circ d$ does not satisfy the triangle inequality of the definition of metric when $x = 0$, $y = 5$ and $z = 1$,

$$(\psi \circ d)(x, y) = 25 \not\leq (\psi \circ d)(x, z) + (\psi \circ d)(z, y) = 1 + 16 = 17.$$

For any $A \subset X$ and $\psi \in \Psi$, $\psi(A)$ is defined as $\psi(A) = \{\psi(d(x, y)) : x, y \in A\}$. We write $\delta[\psi(A)] = \sup\{\psi(d(x, y)) : x, y \in A\}$.

Note that if A is bounded then $\delta[\psi(A)]$ is finite.

Definition 2.1. (Ciric, [2]). A metric space (X, d) is said to be T-*orbitally complete* iff for every Cauchy sequence which is contained in $O(x)$ for some $x \in X$ converges in X.

Definition 2.2. A selfmap T of X is said to be ψ-*quasi-contraction* where $\psi \in \Psi$ iff there is a number $q \in [0, 1)$ such that

$$\psi(d(Tx, Ty)) \leq q \max\{\psi(d(x, y)), \psi(d(x, Tx)), \psi(d(y, Ty)), \psi(d(y, Tx)), \psi(d(x, Ty))\} \tag{2.2.1}$$

holds for every x, y in X.

Example 2.3. Let $X = [0, 4^{-1}] \times [0, 4^{-1}]$ with ℓ_2-norm, $\|.\|_2$ being defined by $\|(u,v)\|_2 = \sqrt{u^2+v^2}$ for $(u,v) \in X$. We defined a metric d^* on X by $d^*(x,y) = \|x-y\|_2$ for x, $y \in X$. We define $\psi : R^+ \to R^+$ by $\psi(t) = t^2$, $t \geq 0$ so that $\psi \in \Psi$. We define $T : X \to X$ by

$$T(u,v) = (u^2, v^2) \quad \text{for} \quad (u,v) \in X. \tag{2.3.1}$$

Then, for $x = (x_1, y_1) \in X$ and $y = (x_2, y_2) \in X$, we have

$$\psi(d^*(x,y)) = \|x-y\|_2^2 = (x_1-x_2)^2 + (y_1-y_2)^2.$$

Now

$$\begin{aligned}
\psi(d^*(Tx,Ty)) &= \psi(d^*((x_1^2, y_1^2), (x_2^2, y_2^2))) \\
&= \|(x_1^2, y_1^2) - (x_2^2, y_2^2)\|_2^2 \\
&= (x_1^2 - x_2^2)^2 + (y_1^2 - y_2^2)^2 \\
&= (x_1+x_2)^2(x_1-x_2)^2 + (y_1+y_2)^2(y_1-y_2)^2 \\
&\leq 4^{-1}[(x_1-x_2)^2 + (y_1-y_2)^2] \\
&= 4^{-1}\psi(d^*(x,y)) \\
&\leq 4^{-1}\max\{\psi(d^*(x,y)), \psi(d^*(x,Tx)), \psi(d^*(y,Ty)), \psi(d^*(x,Ty)), \\
&\qquad \psi(d^*(y,Tx))\}
\end{aligned}$$

so that T is a ψ-quasi-contraction with $q = 4^{-1}$.

When ψ is the identity mapping on R^+ in Definition 2.2, then we say that T *is a quasi-contraction* (Definition 1 of Circ [2]).

Ciric proved the following theorem.

Theorem 2.4. (Circ [2], Theorem 1, p.270). *Let T be as quasi-contraction map on a metric space (X,d) and let X be T-orbitally complete. Then, for any $x \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ is Cauchy,* $\lim_{n\to\infty} T^n x = z$ *for some z in X and z is the unique fixed point of T.*

Ohta and Nikaido [5] established the following theorem relating to the diameter of orbits.

Theorem 2.5. (Ohta and Nikaido [5], Theorem 3, p.288). *Let (X,d) be a bounded complete metric space and assume that T is a continuous selfmap of X with the following property: there exists a nonnegative integer k and* $q \in [0,1)$ *such that*

$$d(T^k x, T^k y) \leq q\delta[O(x) \cup O(y)] \quad \textit{for all } x,\ y \textit{ in } X.$$

Then T has a unique fixed point z in X and $\lim_{n\to\infty} T^n x = z$ *for any x in X.*

Definition 2.6. A selfmap T on a metric space X is said to be *T-orbitally continuous at a point z* of X if for any sequence $\{x_n\} \subset O(x)$ with $x_n \to z$ as $n \to \infty$ implies $Tx_n \to Tz$ as $n \to \infty$.

Definition 2.7. (Browder and Petryshyn [1]). A selfmap T of a metric space X is said to be *asymptotically regular at a point x in X* if $d(T^n x, T^{n+1} x) \to 0$ as $n \to \infty$. If T is asymptotically regular at each point x of X, we say that T is *asymptotically regular on X*.

3. Main Results

Theorem 3.1. *Let T be a selfmap of a metric space (X, d). Assume that (X, d) is T-orbitally complete. If T is a ψ-quasi-contaction for some $\psi \in \Psi$ then for each $x \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ is Cauchy, $\lim_{n\to\infty} T^n x = z$ for some z in X and z is the unique fixed point of T.*

To prove Theorem 3.1, first we prove the following two Lemmas (Lemma 3.2 and Lemma 3.4).

Lemma 3.2. *Let T be a ψ-quasi-contraction on X for some $\psi \in \Psi$. Then for each $x \in X$,*

$$\delta[\psi(O(T^n x))] \leq q^n \delta[\psi(O(x))], \quad n = 1, 2, \ldots. \tag{3.2.1}$$

and hence

$$\lim_{n\to\infty} \delta[\psi(O(T^n x))] = 0. \tag{3.2.2}$$

Proof. Let x be an arbitary point of X. Then for any i, j in N,

$$\begin{aligned}
\psi(d(T^i x, T^j x)) &= \psi(d(T(T^{i-1}x)), T(T^{j-1}x))) \\
&\leq q \max\{\psi(d(T^{i-1}x, T^{j-1}x)), \psi(d(T^{i-1}x, T^i x)), \psi(d(T^{j-1}x, T^j x)), \\
&\qquad \psi(d(T^{i-1}x, T^j x)), \psi(d(T^i x, T^{j-1}x))\} \\
&\leq q\delta[\psi(O(x))].
\end{aligned}$$

This implies

$$\delta[\psi(O(Tx))] \leq q\delta[\psi(O(x))], \quad \text{so that (3.2.1) is true for } n = 1.$$

Assume that the inequality (3.2.1) is true for $n = m$. Hence

$$\delta[\psi(O(T^m x))] \leq q^m \delta[\psi(O(x))]. \tag{3.2.3}$$

Now, for $i = 1, 2, \ldots$ and $j = 1, 2, \ldots$ consider

$$\begin{aligned}
\psi(d(T^{m+i}x, T^{m+j}x)) &= \psi(d(T(T^{m+i-1}x), T(T^{m+j-1}x))) \\
&\leq q \max\{\psi(d(T^{m+i-1}x, T^{m+j-1}x)), \psi(d(T^{m+i-1}x, T^{m+i}x)), \\
&\qquad \psi(d(T^{m+j-1}x, T^{m+j}x)), \psi(d(T^{m+i-1}x, T^{m+j}x)), \\
&\qquad \psi(d(T^{m+i}x, T^{m+j-1}x))\}.
\end{aligned}$$

Since $T^{m+i-1}x$, $T^{m+j-1}x$, $T^{m+i}x$, $T^{m+j}x \in O(T^m x)$
for each $i = 1, 2, \dots$ and $j = 1, 2, \dots$, it follows by (3.2.3) that

$$\psi(d(T^{m+i}x, T^{m+j}x)) \leq q\delta[\psi(O(T^m x))] \leq q^{m+1}\delta[\psi(O(x))].$$

This implies

$$\delta[\psi(O(T^{m+1}x))] \leq q^{m+1}\delta[\psi(O(x))].$$

Hence by mathematical induction, (3.2.1) follows, so that (3.2.2) holds.

Proposition 3.3. *Under the assumptions of Lemma* 3.2*, T is asymptotically regular on X.*

Proof. From (3.2.1) it follows that

$$\psi(d(T^n x, T^{n+1}x)) \leq \delta[\psi(O(T^n x))] \leq q^n \delta[\psi(O(x))]$$

and hence $\lim_{n\to\infty} \psi(d(T^n x, T^{n+1}x)) = 0$.

Now by using the properties (i) and (iii) of ψ, we have T is asymptotically regular at x. As x was an arbitrary point of X, Proposition 3.3 follows.

Lemma 3.4. *Under the hypothesis of Lemma* 3.2*, for each $x \in X$, the sequence $\{T^n x\}_{n=1}^{\infty}$ is Cauchy in X.*

Proof. If $\{T^n x\}_{n=1}^{\infty}$ is not Cauchy in X, there exists an $\epsilon > 0$ and a sequence of positive integers $m(k)$ and $n(k)$ such that $m(k) < n(k)$ with

$$d(T^{n(k)}x, T^{m(k)}x) \geq \epsilon \quad \text{and} \quad d(T^{n(k)-1}x, T^{m(k)}x) < \epsilon. \tag{3.4.1}$$

For this $\epsilon > 0$,

$$\begin{aligned}
\psi(\epsilon) &\leq \psi(d(T^{n(k)}x, T^{m(k)}x)) \\
&= \psi(d(T(T^{n(k)-1}x), T(T^{m(k)-1}x))) \\
&\leq q \max\{\psi(d(T^{n(k)-1}x, T^{m(k)-1}x)), \psi(d(T^{n(k)-1}x, T^{n(k)}(x)), \psi(d(T^{m(k)-1}x, T^{m(k)}x)), \\
&\qquad \psi(d(T^{n(k)-1}x, T^{m(k)}x)), \psi(d(T^{n(k)}x, T^{m(k)-1}x))\}. \qquad (3.4.2)
\end{aligned}$$

We now prove the following *four* equalities:

(i) $\lim_{k\to\infty} d(T^{n(k)}x, T^{m(k)}x) = \epsilon$ (ii) $\lim_{k\to\infty} d(T^{n(k)-1}x, T^{m(k)-1}x) = \epsilon$

(iii) $\lim_{k\to\infty} d(T^{n(k)-1}x, T^{m(k)}x) = \epsilon$ (iv) $\lim_{k\to\infty} d(T^{m(k)-1}x, T^{n(k)}x) = \epsilon$.

We have, by Proposition 3.3,

$$\begin{aligned}
\epsilon \leq d(T^{n(k)}x, T^{m(k)}x) &\leq d(T^{n(k)}x, T^{n(k)-1}x) + d(T^{n(k)-1}x, T^{m(k)}x) \\
&< d(T^{n(k)}x, T^{n(k)-1}x) + \epsilon
\end{aligned}$$

and by taking limits as $k \to \infty$,

$$\epsilon = \lim_{k\to\infty} d(T^{n(k)}x, T^{m(k)}x),$$

so that (i) follows.

To prove (ii), we have

$$\begin{aligned}\epsilon &\le d(T^{n(k)}x, T^{m(k)}x)\\ &\le d(T^{n(k)}x, T^{n(k)-1}x) + d(T^{n(k)-1}x, T^{m(k)-1}x) + d(T^{m(k)-1}x, T^{m(k)}x).\end{aligned}$$

Now by Proposition 3.3, letting $k \to \infty$,

$$\epsilon \le \lim_{k\to\infty} d(T^{n(k)-1}x, T^{m(k)-1}x). \tag{3.4.3}$$

Also

$$\begin{aligned}d(T^{n(k)-1}x, T^{m(k)-1}x) &\le d(T^{n(k)-1}x, T^{m(k)}x) + d(T^{m(k)}x, T^{m(k)-1}x)\\ &< \epsilon + d(T^{m(k)}x, T^{m(k)-1}x).\end{aligned}$$

Hence

$$\lim_{k\to\infty} d(T^{n(k)-1}x, T^{m(k)-1}x) \le \epsilon. \tag{3.4.4}$$

The inequalities (3.4.3) and (3.4.4) prove (ii).

To prove (iii),

$$\begin{aligned}\epsilon \le d(T^{n(k)}x, T^{m(k)}x) &\le d(T^{n(k)}x, T^{n(k)-1}x) + d(T^{n(k)-1}x, T^{m(k)}x)\\ &< d(T^{n(k)}x, T^{n(k)-1}x) + \epsilon.\end{aligned}$$

Taking limits as $k \to \infty$,

$$\epsilon \le \lim_{k\to\infty} d(T^{n(k)}x, T^{m(k)}x) \le \epsilon$$

so that (iii) follows.

We now prove (iv).

$$\epsilon \le d(T^{n(k)}x, T^{m(k)}x) \le d(T^{n(k)}x, T^{m(k)-1}x) + d(T^{m(k)-1}x, T^{m(k)}x)$$

and hence

$$\epsilon \le \lim_{k\to\infty} d(T^{n(k)}x, T^{m(k)-1}x). \tag{3.4.5}$$

Now

$$d(T^{n(k)}x, T^{m(k)-1}x) \le d(T^{n(k)}x, T^{m(k)}x) + d(T^{m(k)}x, T^{m(k)-1}x).$$

Using (i), we get

$$\lim_{k\to\infty} d(T^{n(k)}x, T^{m(k)-1}x) \le \lim_{k\to\infty} d(T^{n(k)}x, T^{m(k)}x) = \epsilon. \tag{3.4.6}$$

From (3.4.5) and (3.4.6), we get

$$\epsilon = \lim_{k\to\infty} d(T^{n(k)}x, T^{m(k)-1}x).$$

Hence (iv) follows.

By using (i)-(iv) in (3.4.2), we get

$$\psi(\epsilon) \le q \max\{\psi(\epsilon), 0, 0, \psi(\epsilon), \psi(\epsilon)\} < \psi(\epsilon), \quad \text{a contradiction.}$$

This proves that $\{T^n x\}$ is Cauchy in X and hence Lemma 3.4 follows.

Proof of Theorem 3.1. Let $x \in X$. From Lemma 3.2 and Lemma 3.4, it follows that $\{T^n x\}$ is Cauchy and since X is T-orbitally complete, $\{T^n x\}$ has a limit, z (say) in X.

We now show that this z is a fixed point of T. Otherwise, i.e., if $Tz \neq z$, consider

$$\begin{aligned}\psi(d(T^{n+1}x, Tz)) &= \psi(d(T(T^n x), Tz))\\ &\le q \max\{\psi(d(T^n x, z)), \psi(d(T^n x, T^{n+1}x)), \psi(d(z, Tz)),\\ &\qquad \psi(d(T^{n+1}x, z)), \psi(d(T^n x, Tz))\}.\end{aligned}$$

Taking limits as $k \to \infty$,

$$\begin{aligned}\psi(d(z, Tz)) &= \lim_{n\to\infty} \psi(d(T^{n+1}x), Tz))\\ &\le q \max\{\psi(d(z,z)), \psi(d(z,z)), \psi(d(z,Tz)), \psi(d(z,z)), \psi(d(z,Tz))\}\\ &= q\psi(d(Tz,z)) < \psi(d(Tz,z)),\end{aligned}$$

a contradiction. Hence $\psi(d(Tz,z)) = 0$ so that $d(z,Tz) = 0$.

This proves that $Tz = z$.

Uniqueness of the fixed point z follows trivially from the ψ-quasi-contraction of T.

This proves Theorem 3.1.

Note. In Theorem 3.1, we were not assumed any continuity of T; by taking $\psi(t) = t$, $t \ge 0$, Theorem 2.4 follows as a corollary to Theorem 3.1.

Theorem 3.5. *Let (X,d) be a metric space, which is T-orbitally complete. Assume that there is a positive integer k and $q \in [0,1)$ such that T^k is ψ-quasi-contraction for some $\psi \in \Psi$. Then T has a unique fixed point z in X. Also, for $x \in X$, $\lim_{n\to\infty} T^n x = z$ provided T is continuous.*

Proof. Since T^k is ψ-quasi-contraction, by Lemma 3.4, $\{(T^k)^m x\}$ is Cauchy and its limit

$$\lim_{n\to\infty} (T^k)^m x = z \quad \text{(say)}, \quad z \in X. \tag{3.5.1}$$

By Theorem 3.1, z is the unique fixed point of T^k.

Hence $T^k z = z$. Also $T^k(Tz) = T(T^k z) = Tz$, so that it follows that

$$Tz = z. \tag{3.5.2}$$

Let n be any positive integer, $n = mk + j$, $0 \leq j < k$, m nonnegative integer. Let $x \in X$. Then

$$T^n x = T^{mk+j} x = (T^k)^m T^j x = T^j (T^k)^m x.$$

Letting $m \to \infty$ and using the continuity of T, we have by (3.5.1) and (3.5.2), it follows that

$$\begin{aligned}\lim_{n\to\infty} T^n x = \lim_{m\to\infty} T^j (T^k)^m x &= T^j \lim_{m\to\infty} (T^k)^m x \\ &= T^j z = z.\end{aligned}$$

This completes the proof of the theorem.

In fact, we have the following generalization of Theorem 3.5.

Theorem 3.6. *Let (X, d) be a metric space and let X be T-orbitally complete. Assume that there is a $\psi \in \Psi$ and for some positive integer k, and $q \in [0, 1)$*

$$\psi(d(T^k x, T^k y)) \leq q\delta[\psi(O(x) \cup O(y))] \tag{3.6.1}$$

for all x, y in X. Then for each $x \in X$, the sequence $\{T^n x\}$ is Cauchy in X, $\lim_{n\to\infty} T^n x = z$ for some z in X and z is the unique fixed point of T provided T is orbitally continuous at z.

Proof. Let $x \in X$. Let $m > n \geq k$. Then, by (3.6.1), we have

$$\begin{aligned}\psi(d(T^m x, T^n x)) &= \psi(d(T^k(T^{m-k} x), T^k(T^{n-k} x))) \\ &\leq q\delta[\psi(O(T^{m-k} x) \cup O(T^{n-k} x))] \\ &= q\delta[\psi(O(T^{n-k} x))] \\ &\leq q\delta[\psi(O(x))].\end{aligned}$$

Hence

$$\delta[\psi(O(T^k x))] \leq q\delta[\psi(O(x))].$$

In general, it is trivial to see that, for $n = 1, 2, \ldots$.

$$\delta[\psi(T^{nk} x))] \leq q^n \delta[\psi(O(x))] \tag{3.6.2}$$

holds.

We now show that the sequence $\{T^n x\}$ is Cauchy. Otherwise, there exists an $\epsilon > 0$ and a sequence of positive integers $\{n(k)\}$ and $\{m(k)\}$ such that $m(k) < n(k)$ with

$$d(T^{n(k)} x, T^{m(k)} x) \geq \epsilon \quad \text{and} \quad d(T^{n(k)-1} x, T^{m(k)} x) < \epsilon. \tag{3.6.3}$$

From (3.6.2), there is a positive integer N_1 such that

$$q^n \delta[\psi O(x))] < \psi(\epsilon) \quad \text{for all } n \geq N_1. \tag{3.6.4}$$

Choose a positive integer ℓ so large such that $\ell > N_1 k$. Then for $m(\ell) > N_1 k$, by using (3.6.3) and (3.6.4),

$$\psi(\epsilon) \leq \psi(d(T^{n(\ell)}x, T^{m(\ell)}x)) \leq \delta[\psi(O(T^{N_1 k}x))] \leq q^{N_1}\delta[\psi(O(x))] < \psi(\epsilon),$$

a contradiction.

Therefore $\{T^n x\}$ is Cauchy in X, and hence converges, say, to z in X. Hence by the orbital continuity of T at z, it follows that

$$z = \lim_{n\to\infty} T^{n+1}x = Tz.$$

Hence Theorem 3.6 follows.

Remark 3.7. By taking $\psi(t) = t$ in Theorem 3.6, we obtain Theorem 2.5. In fact, in Theorem 2.5 the orbital continuity of T at z is sufficient instead of the continuity of T on X.

Remark 3.8. If we omit the orbital continuity of T at z in Theorem 3.6, then T may not have a fixed point in X. Examples 4.2 and 4.3 of Section 4 illustrate this fact.

4. Examples

Example 4.1. From Example 2.3, the mapping T defined on $X = [0, 4^{-1}] \times [0, 4^{-1}]$ by (2.3.1) satisfies all the conditions of Theorem 3.1 with $\psi(t) = t^2$ and $q = 4^{-1}$; also (X, d^*) is T-orbitally complete; and T has the unique fixed point $(0, 0)$ in X.

Example 4.2. Let $X = \{1, 0, 2^{-1}, 2^{-2}, 2^{-3}, \ldots\}$ with the usual metric. Define $\psi : R^+ \to R^+$ by $\psi(t) = t$; and define $T : X \to X$ by $T(0) = 1$ and $T(2^{-n}) = 2^{-(n+1)}$ for $n = 1, 2, 3, \ldots$. Then T satisfies the inequality (3.6.1) for all x, y in X, with $k = 1$, $q = 2^{-1}$ and T is not orbitally continuous at any point of X and T has no fixed point in X.

Example 4.3. Let X be as in Example 4.2. Define $T : X \to X$ by $T(0) = 0$ and $T(2^{-n}) = 2^{-(n+1)}$ for $n = 0, 1, 2, \ldots$. Then T satisfies all the conditions of Theorem 3.6 with $\psi(t) = t^2$, $k = 1$, $q = 4^{-1}$ and T is orbitally continuous at 0 and T has the unique fixed point 0.

An open problem: Is it possible that the *boundedness* of the orbit of x, $O(x)$ for each x in X, can be relaxed in Theorems 3.1, 3.5 and 3.6?

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References

[1] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl. **20**(1967), 197-228.

[2] Lj. B. Ciric, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45**(1974), 267-273.

[3] D. Delbosco, *On estensione di un teorema sul punto fisso di S. Reich*, Rend. Sem. Mat. Univers Politecn. Torino **35**(1976-1977), 233-238.

[4] M. S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Austra. Math. Soc. **30**(1984), 1-9.

[5] M. Ohta and G. Nikaido, *Remarks on fixed point theorems in complete metric spaces*, Math. Japonica **39**(1994), 287-290.

[6] S. Park, *A unified approach to fixed points of contractive maps*, J. Korean Math. Soc. **16**(1980), 95-105.

[7] S. Park, *Fixed points and periodic points of contractive pairs of maps*, Proc. College Nat. Sci., Seoul Nat. University **5**(1980), 9-22.

[8] K. P. R. Sastry and G. V. R. Babu, *Some fixed point theorems by altering distances between the points*, Indian Jour. Pure and Appl. Math. **30**(1999), 641-647.

[9] F. Skof, *Teorema di punti fisso per applicazioni negli spazi metrici*, Atti. Accad. Sci., Toino **111**(1977), 323-329.

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