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ON WEYL FRACTIONAL INTEGRAL OPERATORS

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Abstract. In this paper, we first establish an interesting theorem exhibiting a relationship existing between the Laplace transform and Weyl fractional integral operator of related functions. This theorem is sufficiently general in nature as it contains n series involving arbitrary complex numbers $\Omega(r_1,\dots r_n)$. We have obtained here as applications of the theorem, the Weyl fractional integral operators of Kampé de Fériet function, Appell's functions F_1, F_4, Humbert's function Ψ_1 and Lauricella's, triple hypergeometric series F_E. References of known results which follow as special cases of our theorem are also cited. Finally, we obtain some transformations of $F^{(3)}$ and Kampé de Fériet function with the application of our main theorem .

1. Introduction

The Weyl fractional integral is defined by

$$W^{\mu}[f(t);s]=\frac{1}{\Gamma(\mu)}\int_s^{\infty}(t-s)^{\mu-1}f(t)dt,\quad Re(\mu)>0. \tag{1.1}$$

The Laplace transform is defined by

$$L[f(t);s]=\int_0^{\infty}e^{-st}f(t)dt,\quad Re(s)>0. \tag{1.2}$$

In this paper, we establish a theorem exhibiting a relationship between (1.1) and (1.2). This theorem provides more efficient tools which allow the straightforward derivation of the Weyl fractional integral operators associated with hypergeometric functions of Appell, Humbert, Kampé de Fériet and Lauricella.

2. Theorem

For bounded complex coefficients $\Omega(r_1,\dots,r_n)$ $(r_j\in N_0,\ j=1,\dots,r)$, let

$$f(t;x_1,\dots,x_n)=\sum_{r_1,\dots,r_n=0}^{\infty}\Omega(r_1,\dots,r_n)\frac{x_1^{r_1}}{r_1!}\cdots\frac{x_n^{r_n}}{r_n!}\ t^{R+\sigma-1} \tag{2.1}$$

and

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$$g(s;x_1,\dots,x_n) = L[f(t;x_1,\dots,x_n);s] \tag{2.2}$$

Then

$$\begin{aligned} &W^{\mu}[t^{-\lambda}g(t+\delta);x_1,\dots,x_n);s] \\ &= \frac{s^{\mu-\lambda}}{(\delta+s)^{\sigma}}\frac{\Gamma(\sigma)\Gamma(\sigma+\lambda-\mu)}{\Gamma(\sigma+\lambda)}\sum_{r_1,\dots,r_n=0}^{\infty}\frac{(\sigma)_R(\sigma+\lambda-\mu)_R}{(\sigma+\lambda)_R} \\ &\times\frac{\Omega(r_1,\dots,r_n)}{r_1!\dots r_n!}\left(\frac{x_1}{\delta+s}\right)^{r_1}\dots\left(\frac{x_n}{\delta+s}\right)^{r_n} {}_2F_1\left(\begin{matrix}\sigma+R,\mu\ ; \\ \\ \sigma+\lambda+R\ ;\end{matrix}\ \frac{\delta}{\delta+s}\right) \end{aligned} \tag{2.3}$$

where $R=\sum\limits_{i=1}^{n} r_i$, $\operatorname{Re}(\sigma+\lambda-\mu)>0$, $\operatorname{Re}(\delta+s)>0$, Weyl fractional integral involved in (2.3) exists and the series involved in its R.H.S is absolutely convergent.

Proof. We have [1, p.137(1)]

$$\begin{aligned} L[f(t;x_1,\dots,x_n);s] &= g(s;x_1,\dots,x_n) \\ &= \frac{\Gamma(\sigma)}{s^{\sigma}}\sum_{r_1,\dots,r_n=0}^{\infty}(\sigma)_R\frac{\Omega(r_1,\dots,r_n)}{r_1!\dots r_n!}\left(\frac{x_1}{s}\right)^{r_1}\dots\left(\frac{x_n}{s}\right)^{r_n}. \end{aligned} \tag{2.4}$$

Also

$$g(s+\delta;x_1,\dots,x_n) = L[e^{-\delta t}f(t;x_1,\dots,x_n);s] \tag{2.5}$$

and by virtue of a known result [1, p.294(6)]

$$L[t^{\mu-1}(t+a)^{-\lambda};s] = \Gamma(\mu)\ s^{\frac{(\lambda-\mu-1)}{2}}a^{\frac{(\lambda-\mu-1)}{2}}e^{as/2}W_{k,m}(as) = \psi(as)\quad \text{(say)} \tag{2.6}$$

where $k=\frac{1-\mu-\lambda}{2}$, $m=\frac{\mu-\lambda}{2}$, $\operatorname{Re}(\mu)>0$, $\mid \arg a \mid < \pi$ and $W_{k,m}$ is the usual Whittaker function [2].

Again

$$e^{-as}\psi(as) = L[(t-a)^{\mu-1}t^{-\lambda}H(t-a);s] \tag{2.7}$$

where $H(t)$ is Heaviside's function.

Now applying operational pairs (2.5) and (2.7) in the Goldstein theorem for the Laplace transform, using (2.4) and changing the order of integration and summation, we get

$$\begin{aligned} &\int_0^{\infty} t^{-\lambda}(t-a)^{\mu-1}g(t+\delta;x_1,\dots,x_n)H(t-a)dt \\ &= \sum_{r_1,\dots,r_n=0}^{\infty}\Omega(r_1,\dots,r_n)\frac{{x_1}^{r_1}}{r_1!}\dots\frac{{x_n}^{r_n}}{r_n!}\int_0^{\infty} t^{R+\delta-1}e^{-(a+\delta)t}\ \psi(at)dt \end{aligned} \tag{2.8}$$

where $R=\sum\limits_{i=1}^{n} r_i$, $\operatorname{Re}(\sigma+\lambda-\mu)>0$, $\operatorname{Re}(\delta+a)>0$ and provided that the inversion of summation and integration is permissible under the hypothesis of the theorem.

On substituting the value of $\psi(at)$ from (2.6) in the R.H.S of (2.8), performing the integration involved with the help of a known result [1, p.216(16)] and interpreating the L.H.S of (2.8) in terms of Weyl fractional integral defined by (1.1), we arrive at the required result (2.3) on replacing a by s.

3. Special Cases and Applications

If we take $n = 1$ and $\Omega(r) = \dfrac{1}{\Gamma(\sigma + r)}$ in the main theorem, we arrive at a known result [2, p.203(15)] after taking $\delta \to 0$.

If we take $n = 2$ and $\delta \to 0$, the main theorem reduces to a known theorem [3, p.52].

Again, if we take $n = 1$ and $\Omega(r) = \dfrac{\prod_{j=1}^{P}(a_j)_r}{\prod_{j=1}^{Q}(b_j)_r}$ in the main theorem we get a known result [2, p.212(77)] after taking $\delta \to 0$.

Next, we choose $n = 2$ and $\Omega(r_1, r_2) = \dfrac{\prod_{j=1}^{p}(a_j)_{r_1+r_2}\prod_{j=1}^{q}(b_j)_{r_1}\prod_{j=1}^{k}(c_j)_{r_2}}{\prod_{j=1}^{l}(\alpha_j)_{r_1+r_2}\prod_{j=1}^{m}(\beta_j)_{r_1}\prod_{j=1}^{n}(\gamma_j)_{r_2}}$ in the main theorem to get the following result.

If

$$f(t; x_1, x_2) = t^{\sigma-1} F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p) : (b_q) \ ; (c_k) \ ; \\ (\alpha_l) : (\beta_m) \ ; (\gamma_n) \ ;\end{matrix} \; x_1 t, x_2 t\right] \tag{3.1}$$

and

$$g(s; x_1, x_2) = \frac{\Gamma(\sigma)}{s^{\sigma}} F^{p+1:q;k}_{l:m;n}\left[\begin{matrix}\sigma, (a_p) : (b_q) \ ; (c_k) \ ; \\ (\alpha_l) \quad : (\beta_m) \ ; (\gamma_n) \ ;\end{matrix} \; \frac{x_1}{s}, \frac{x_2}{s}\right] \tag{3.2}$$

then

$$W^{\mu}\left[t^{-\lambda}(t+\delta)^{-\sigma} F^{p+1:q;k}_{l:m;n}\left[\begin{matrix}\sigma, (a_p) \ : (b_q) \ ; (c_k) \ ; \\ (\alpha_l) : (\beta_m) \ ; (\gamma_n) \ ;\end{matrix} \; \frac{x_1}{t+\delta}, \frac{x_2}{t+\delta}\right] ; s\right]$$

$$= \frac{\Gamma(\sigma + \lambda - \mu) s^{\mu-\lambda}}{\Gamma(\sigma+\lambda)(\delta+s)^{\sigma}}$$

$$F^{(3)}\left[\begin{matrix}\sigma \quad :: (a_p), (\sigma+\lambda-\mu) \ ; \_ \ ; \_ : (b_q) \ ; (c_k) \ ; \mu \ ; \\ \sigma+\lambda :: (\alpha_l) \quad ; \_ \ ; \_ : (\beta_m) \ ; (\gamma_n) \ ; \_ \ ;\end{matrix} \; \frac{x_1}{\delta+s}, \frac{x_2}{\delta+s}, \frac{\delta}{\delta+s}\right] \tag{3.3}$$

where $F^{p:q;k}_{l:m;n}$ is generalized Kampé de Fériet series [4, p.27(28)], $F^{(3)}$ stands for Srivastava's triple hypergeometric series [4, p.44(14)], min $\text{Re}(\sigma, \mu, s) > 0$ and $\text{Re}(\delta + s) > 0$.

The result (3.3) can be specialized to four Appell series F_1, F_2, F_3 and F_4, and Humbert series $\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \Xi_1$ and Ξ_2. We list here only the following special cases of (3.3) involving F_1, F_4 and ψ_1

$$\begin{aligned}&W^{\mu}[t^{-\lambda}(t+\delta)^{-\sigma}F_1(\sigma,b,c;\alpha;\frac{x_1}{t+\delta},\frac{x_2}{t+\delta};s]\\&=\frac{\Gamma(\sigma+\lambda-\mu)s^{\mu-\lambda}}{\Gamma(\sigma+\lambda)(\delta+s)^{\sigma}}\\&F^{(3)}\left[\begin{matrix}\sigma :: \sigma+\lambda-\mu\,;\_\,;\_:b\;;c\;;\mu\,;\\ \qquad\qquad\frac{x_1}{\delta+s},\frac{x_2}{\delta+s},\frac{\delta}{\delta+s}\\ \sigma+\lambda :: \alpha\;;\_\,;\_:\_\,;\_\,;\_\,;\end{matrix}\right]\end{aligned}\tag{3.4}$$

$$\begin{aligned}&W^{\mu}[t^{-\lambda}(t+\delta)^{-\sigma}F_4\left(\sigma,a;\beta,\gamma;\frac{x_1}{t+\delta},\frac{x_2}{t+\delta}\right);s]\\&=\frac{\Gamma(\sigma+\lambda-\mu)s^{\mu-\lambda}}{\Gamma(\sigma+\lambda)(\delta+s)^{\sigma}}\\&F^{(3)}\left[\begin{matrix}\sigma : (\sigma+\lambda-\mu),a\;;\_\,;\_\,;\_\,;\_\,;\mu\,;\\ \qquad\qquad\frac{x_1}{\delta+s},\frac{x_2}{\delta+s},\frac{\delta}{\delta+s}\\ \sigma+\lambda : \_\_\_\_\;;\_\,;\_\,;\beta\;;\gamma\;;\_\,;\end{matrix}\right]\end{aligned}\tag{3.5}$$

$$\begin{aligned}&W^{\mu}[t^{-\lambda}(t+\delta)^{-\sigma}\Psi_1(\sigma,b;\beta,\gamma;\frac{x_1}{t+\delta},\frac{x_2}{t+\delta};s]\\&=\frac{\Gamma(\sigma+\lambda-\mu)s^{\mu-\lambda}}{\Gamma(\sigma+\lambda)(\delta+s)^{\sigma}}\\&F^{(3)}\left[\begin{matrix}\sigma :: \_\_\;;\_\,;\_\,;b\;;\_\,;\mu\,;\\ \qquad\qquad\frac{x_1}{\delta+s},\frac{x_2}{\delta+s},\frac{\delta}{\delta+s}\\ \sigma+\lambda :: \_\_\;;\_\,;\_\,;\beta\;;\gamma\;;\_\,;\end{matrix}\right]\end{aligned}\tag{3.6}$$

Next, we set $n=3$ and $\Omega(r_1,r_2,r_3)=\frac{(\beta_1)_{r_1}(\beta_2)_{r_2+r_3}}{(\gamma_1)_{r_1}(\gamma_2)_{r_2}(\gamma_3)_{r_3}}$ in the main theorem to get

$$\begin{aligned}&W^{\mu}\left[t^{-\lambda}(t+\delta)^{-\sigma}F_E\left(\sigma,\sigma,\sigma,\beta_1,\beta_2,\beta_2;\gamma_1,\gamma_2,\gamma_3;\frac{x_1}{t+\delta},\frac{x_2}{t+\delta},\frac{x_3}{t+\delta}\right);s\right]\\&=\frac{\Gamma(\sigma+\lambda-\mu)\;s^{\mu-\lambda}}{\Gamma(\sigma+\lambda)\;(\delta+s)^{\sigma}}\sum_{r_1,r_2,r_3=0}^{\infty}\frac{(\sigma)_{r_1+r_2+r_3}(\sigma+\lambda-\mu)_{r_1+r_2+r_3}(\beta_2)_{r_2+r_3}(\beta_1)_{r_1}}{(\sigma+\lambda)_{r_1+r_2+r_3}(\gamma_1)_{r_1}(\gamma_2)_{r_2}(\gamma_3)_{r_3}r_1!\;r_2!\;r_3!}\\&\left(\frac{x_1}{\delta+s}\right)^{r_1}\left(\frac{x_2}{\delta+s}\right)^{r_2}\left(\frac{x_3}{\delta+s}\right)^{r_3}{}_2F_1\left(\begin{matrix}\sigma+r_1+r_2+r_3,\mu\;;\\ \qquad\frac{\delta}{\delta+s}\\ \sigma+\lambda+r_1+r_2+r_3\;;\end{matrix}\right)\end{aligned}\tag{3.7}$$

where $\mathrm{Re}(\sigma,\mu,\delta+s)>0$ and F_E is Lauricella's function [4, p.42(1)].

It should be remarked in passing that our main theorem is capable of yielding a number of transformations of hypergeometric functions of several variables, whenever the integral on the left hand side of (2.3) can be evaluated, e.g. expanding $F_{l:m;n}^{p+1:q;k}$ involved in (3.3) into double series, evaluating the integral with the help of the result [2, p.201(8)] and using the definition of $F^{(3)}$, we get

$$F^{(3)}\left[\begin{array}{l} a,b :: (a_p)\,;\_\_\,;\_\_ : (b_q)\;\; ;(c_k)\;;\_\_\,; \\ \qquad\qquad\qquad\qquad\qquad\qquad x_1, x_2, -\delta \\ c \quad :: (\alpha_l)\,;\_\_\,;\_\_ : (\beta_m)\,;(\gamma_n)\,;\_\_\,; \end{array}\right]$$

$$= (1+\delta)^{-a}F^{(3)}\left[\begin{array}{l} a :: (a_p), b\,;\_\_\,;\_\_ : (b_q)\;\;;(c_k)\;;c-b\,; \\ \qquad\qquad\qquad\qquad\qquad\qquad \dfrac{x_1}{1+\delta}, \dfrac{x_2}{1+\delta}, \dfrac{\delta}{1+\delta} \\ c :: (\alpha_l) \quad ;\_\_\,;\_\_ : (\beta_m)\,;(\gamma_n)\,;\_\_ \quad ; \end{array}\right] \tag{3.8}$$

For $x_2 \to 0$, (3.8) reduces to

$$F_{1:l+m;0}^{2:p+q;0}\left[\begin{array}{l} a,b : (a_p),(b_q)\;\;;\_\_\,; \\ \qquad\qquad\qquad\qquad x_1, -\delta \\ \quad c : (\alpha_l),(\beta_m)\,;\_\_\,; \end{array}\right]$$

$$= (1+\delta)^{-a}F_{1:l+m;0}^{1:p+q+1;1}\left[\begin{array}{l} a : (a_p), b, (b_q)\,; c-b\,; \\ \qquad\qquad\qquad\qquad \dfrac{x_1}{1+\delta}, \dfrac{\delta}{1+\delta} \\ c : (\alpha_l),(\beta_m) \;\;;\_\_ \quad ; \end{array}\right] \tag{3.9}$$

Some straightforward generalizations of the results like [cf. Srivastava and Karlsson (4, p.304(99) and p. 307(119)] follow from (3.8) and (3.9). Thus if we employ $p = q = l = m = 0$ in (3.8), we get (4, p.304(99)) and if $q = 0$, $l = 1$ then we get (4, p.307(119)). There are numerous other transformations and reduction formulas which stem similarly from such formulas as (3.8) and (3.9).

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