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LIAR'S DOMINATION IN GRAPHS UNDER SOME OPERATIONS

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Abstract. A set $S \subseteq V(G)$ is a *liar's dominating set* (lds) of graph G if $|N_G[v] \cap S| \geq 2$ for every $v \in V(G)$ and $|(N_G[u] \cup N_G[v]) \cap S| \geq 3$ for any two distinct vertices $u, v \in V(G)$. The *liar's domination number* of G, denoted by $\gamma_{LR}(G)$, is the smallest cardinality of a liar's dominating set of G. In this paper we study the concept of liar's domination in the join, corona, and lexicographic product of graphs.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph. The open neighborhood of $v \in V(G)$ is the set $N_G(v) = \{x \in V(G) : xv \in E(G)\}$ and its closed neighborhood is $N_G[v] = N_G(v) \cup \{v\}$. If $S \subseteq V(G)$, then the open neighborhood of S is the set $N_G(S) = \cup_{v \in S} N_G(v)$ and the closed neighborhood of S is the set $N_G[S] = S \cup N_G(S)$ of a graph G. A vertex z is an *external private neighbor* (*epn*) of $v \in S$ if $z \in V(G) \setminus S$ and $N(z) \cap S = \{v\}$. The set of all external private neighbors of v is denoted by $epn_G(v; S)$.

A set $S \subseteq V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. It is a *total dominating set* of G if $N_G(S) = V(G)$. S is a 2-*dominating set* of G if for every $v \in V(G) \setminus S$, $|N_G(v) \cap S| \geq 2$. A total dominating set S of G is a *double dominating set* of G if it is also 2-dominating. The *domination* (resp. *total domination*, 2-*domination*, and *double domination*) number of G, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$, $\gamma_2(G)$, and $\gamma_{\times 2}(G)$), is the smallest cardinality of a dominating (resp. total dominating, 2-dominating , and double dominating) set of G. Any dominating (resp. total dominating, 2-dominating, double dominating) set of G of cardinality $\gamma(G)$ (resp. $\gamma_t(G)$, $\gamma_2(G)$, and $\gamma_{\times 2}(G)$) is referred to as a γ-set (resp. γ_t-set, γ_2-set, and $\gamma_{\times 2}$-set) of G.

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A subset S of $V(G)$ is called an *almost dominating set* of G if $|V(G) \setminus N[S]| \leq 1$. The *a-domination number* of G, denoted by $\gamma_a(G)$, is the smallest cardinality of an almost dominating set of G. An almost dominating set of G with cardinality $\gamma_a(G)$ is referred to as a γ_a-set of G. Since every dominating set of G is an almost dominating set, it follows that $\gamma_a(G) \leq \gamma(G)$. Moreover, if $\gamma_a(G) < \gamma(G)$, then $\gamma_a(G) = \gamma(G) - 1$. Domination in graphs as well as some of its variations can be found in [1].

In 2009, P.J. Slater et al. in [3] and [4] introduced the concept of liar‘s dominating set in the following way: Consider a network (modelled by a graph, say, G) where each vertex indicates a location and an intruder might be present in any of the locations in the given network. In some locations of the network are monitors or sensors which are responsible for reporting on the presence and location of the possible intruders in their respective closed neighborhoods. It is assumed that in any point of time at most one intruder can occur in the network. Further, if $S \subseteq V(G)$ is the set of monitors (or the locations of the monitors), it is assumed that at most one monitor $x \in S$ fails to report the existence of the intruder in its closed neighborhood or gives a wrong location y of the intruder when the intruder is at v, where $y, v \in N_G[x]$.

Slater defined a set $S \subseteq V(G)$ to be a *liar's dominating set* of G if for any designated vertex $v \in V(G)$ (the location of an intruder) all or all but one of the vertices in $N_G[v] \cap S$ report vertex v, and at most one vertex $w \in N_G[v] \cap S$ reports a vertex $y \in N_G[w] \setminus \{v\}$ or fails to report any vertex, then the vertex v can be correctly identified as the designated vertex. Slater et al. in [3] characterized this concept as follows: A subset S of $V(G)$ is a liar's dominating set of G if and only if $|N_G[v] \cap S| \geq 2$ for every $v \in V(G)$ (that is, S is a double dominating set of G), and $|(N_G[u] \cup N_G[v]) \cap S| \geq 3$ for any two distinct vertices $u, v \in V(G)$. The *liar's domination number* of G, denoted by $\gamma_{LR}(G)$, is the smallest cardinality of a liar's dominating set of G. Any liar's dominating set of G with cardinality $\gamma_{LR}(G)$ is called a γ_{LR}-set of G. Liar's domination and its variants are also investigated by Nikodem in [2].

2. Liar's domination in the join of graphs

The *join* of two graphs G and H is the graph $G+H$ with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 2.1. *Let G be a connected graph of order $n \geq 3$ and $K_1 = \langle\{v\}\rangle$. Then $S \subseteq V(K_1 + G)$ is a liar's dominating set of $K_1 + G$ if and only if one of the following holds:*

(i) *S is a liar's dominating set of G;*

(ii) *$S = S_1 \cup \{v\}$, where S_1 is a dominating set of G such that for each $x \in S_1$, either $|epn_G(x; S_1)| = 0$, or $|epn_G(x; S_1)| = 1$ and $x \in N_G(S_1)$.*

Proof. Suppose that S is an lds of K_1+G. If $v \notin S$, then, clearly, S is an lds of G. Thus (i) holds. Suppose that $v \in S$. Then $S = S_1 \cup \{v\}$, where $\varnothing \neq S_1 \subseteq V(G)$. Since S is a double dominating set of K_1+G, it follows that S_1 is a dominating set of G. Let $x \in S_1$. Suppose that $|epn_G(x;S_1)| > 1$. Then there exist $y, z \in epn_G(x;S_1)$ such that $y \neq z$. Thus, $|(N_G[y] \cup N_G[z]) \cap S_1| = 1$. Hence, $|(N_{K_1+G}[y] \cup N_{K_1+G}[z]) \cap S| = |\{v\}| + |(N_G[y] \cup N_G[z]) \cap S_1| = 2$. This implies that S is not an lds of K_1+G, contrary to our assumption. Hence, $|epn_G(x;S_1)| \leq 1$. Suppose now that $|epn_G(x;S_1)| = 1$, say $z \in epn_G(x;S_1)$. Since S is an lds of $G+K_1$, $|(N_{K_1+G}[z] \cup N_{K_1+G}[x]) \cap S| = 1 + |(N_G[z] \cup N_G[x]) \cap S_1| = 1 + 1 + |N_G(x) \cap S_1| \geq 3$. This implies that $|N_G(x) \cap S_1| \geq 1$, that is, $x \in N_G(S_1)$. This proves (ii).

For the converse, suppose first that (i) holds. Then, clearly, S is an lds of K_1+G. Next, suppose that (ii) holds. Then $S = S_1 \cup \{v\}$, where S_1 is a dominating set of G satisfying the given conditions in (ii). Let $x \in V(K_1+G)$. If $x \in V(G) \setminus S_1$, then there exists $y \in S_1 \cap N_G(x)$. Since $v \in N_{K_1+G}(x)$, it follows that $|N_{G+K_1}[x] \cap S| = |\{v\} \cup (N_G[x] \cap S_1)| \geq 2$. If $x = v$, then $|N_{K_1+G}[x] \cap S| = |\{x\} \cup S_1| \geq 2$. If $x \in S_1$, then $|N_{K_1+G}[x] \cap S| = |N_G[x] \cap S_1| + 1 \geq 2$. This shows that S is a double dominating set of K_1+G.

Next, let $x, y \in V(K_1+G)$ such that $x \neq y$. Consider the following cases:

Case 1. $x = v$ and $y \in S_1$.

If $epn_G(y;S_1) = \varnothing$, then $|S_1| \geq 2$. Hence, $|N_{K_1+G}[x] \cap S| = 1 + |S_1| \geq 3$. This implies that $|(N_{K_1+G}[v] \cup N_{K_1+G}[y]) \cap S| \geq 3$. Suppose $|epn_G(y;S_1)| = 1$. Then, by assumption, there exists $z \in S_1 \cap N_G(y)$. Hence, $|(N_{K_1+G}[v] \cup N_{K_1+G}[y]) \cap S| \geq 3$.

Case 2. $x = v$ and $y \in V(G) \setminus S_1$.

Since S_1 is a dominating set of G, there exists $w \in N_G(y) \cap S_1$. If $y \notin epn_G(w;S_1)$, then there exists $p \in S_1 \cap N_G(y)$ with $p \neq w$. Hence, $|S_1| \geq 2$ and $|(N_{K_1+G}[v] \cup N_{K_1+G}[y]) \cap S| \geq 3$. If $y \in epn_G(w;S_1)$, then $w \in N_G(S_1)$. Hence, $|S_1| \geq 2$ and $|(N_{K_1+G}[v] \cup N_{K_1+G}[y]) \cap S| \geq 3$.

Case 3. $x, y \in S_1$.

Then $x, y, v \in (N_{K_1+G}[x] \cup N_{K_1+G}[y]) \cap S$. Thus, $|(N_{K_1+G}[x] \cup N_{K_1+G}[y]) \cap S| \geq 3$.

Case 4. $x, y \in V(G) \setminus S_1$.

Since S_1 is a dominating set of G, there exist $z_1, z_2 \in S_1$ such that $xz_1, yz_2 \in E(G)$. If $z_1 \neq z_2$, then $|(N_{G+K_1}[x] \cup N_{G+K_1}[y]) \cap S| \geq 3$ since $v \in N_{K_1+G}(x) \cap N_{K_1+G}(y)$. Suppose $z_1 = z_2$. Since $|epn_G(z_1;S_1)| \leq 1$, it follows that one of x and y, say x, is not in $epn(z_1;S_1)$. Thus, $|(N_{G+K_1}[x] \cup N_{G+K_1}[y]) \cap S| = 1 + |(N_G(x) \cup N_G(y)) \cap S_1| \geq 3$. Therefore, S is an lds of $G+K_1$.

Case 5. $x \in S_1$ and $y \in V(G) \setminus S_1$

Since S_1 is a dominating set of G, there exists $z \in S_1 \cap N_G(y)$. If $x \neq z$, then $|(N_{K_1+G}[x] \cup N_{K_1+G}[y]) \cap S| \geq 3$. If $x = z$, then, by the additional property of S_1, $|(N_{K_1+G}[x] \cup N_{K_1+G}[y]) \cap S| \geq 3$.

Accordingly, S is an lds of $K_1 + G$. □

Throughout the remaining sections we let $\Omega_G = \{D : D$ is a dominating set of G such that for each $x \in D$, either $|epn_G(x;D)| = 1$ and $x \in N_G(D)$ or $|epn_G(x;D)| = 0\}$, and $\gamma^\star(G) = \min \{|D| : D \in \Omega_G\}$. Any set $S \in \Omega_G$ with $\gamma^\star(G) = |S|$ will be called a $\gamma^\star$-set of G.

Lemma 2.2. *Let G be connected graph of order $n \geq 3$. Then $\gamma^\star(G) + 1 \leq \gamma_{LR}(G)$.*

Proof. Let D be a γ_{LR}-set of G and let $v \in D$. Since D is a total dominating set of G, there exists $y \in D \cap N_G(v)$. Let $D^\star = D \setminus \{y\}$. Since D is a 2-dominating set of G, it follows that $D^\star$ is a dominating set of G. Let $x \in D^\star$ and suppose that $|epn_G(x;D^\star)| \geq 2$. Let $p, q \in epn_G(x;D^\star)$, where $p \neq q$. If one of p and q is y, say $p = y$, then $pq \in E(G)$ since D is a 2-dominating set of G. Hence, $|(N_G[p] \cup N_G[q]) \cap D| = 2$. If $p \neq y$ and $q \neq y$, then $py, qy \in E(G)$ since D is a 2-dominating set of G. Again, this implies that $|N_G[p] \cup N_G(q)) \cap D| = 2$. Thus, in both cases, we find that D is not a liar's dominating set of G, contrary to our assumption. Therefore $|epn_G(x;D^\star)| \leq 1$ for each $x \in D^\star$.

Next, let $x \in D^\star$ with $|epn_G(x;D^\star)| = 1$, say $\{z\} = epn_G(x;D^\star)$. Suppose further that $x \notin N_G(D^\star)$. Then y is the only neighbor of x in D. Hence, $(N_G[z] \cup N_G[x]) \cap D = \{x, y\}$, contradicting the assumption that D is an lds. Hence, $x \in N_G(D^\star)$. Since x was arbitrarily chosen, it follows that $D^\star \in \Omega_G$. Consequently, $\gamma^\star(G) \leq |D^\star| = \gamma_{LR}(G) - 1$. □

The next result is a direct consequence of Theorem 2.1 and Lemma 2.2.

Corollary 2.3. *Let G be connected graph of order $n \geq 3$. Then*

$$\gamma_{LR}(K_1 + G) = \gamma^\star(G) + 1.$$

Example 2.4. Consider the graphs $K_1 + G$ and $K_1 + K_4$ in *Figure* 1.

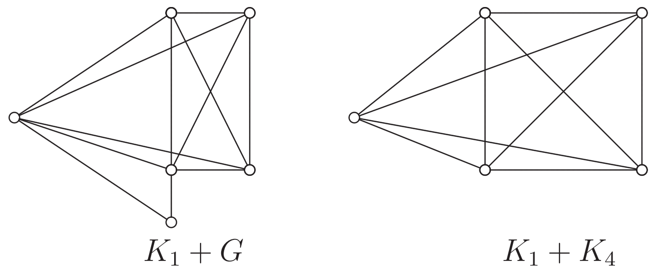

Figure 1: The graphs $K_1 + G$ and $K_1 + K_4$

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Clearly, $1+\gamma^\star(G)=1+2=3<4=\gamma_{LR}(G)$. Hence, $\gamma_{LR}(K_1+G)=3=1+\gamma^\star(G)$. Also, it can easily be verified that $\gamma_{LR}(K_1+K_4)=1+\gamma^\star(K_4)=1+2=3=\gamma_{LR}(K_4)$.

Theorem 2.5. *Let G and H be non-trivial connected graphs. Then $S\subseteq V(G+H)$ is a liar's dominating set of $G+H$ if and only if one of the following holds:*

(i) *S is a liar's dominating set of G.*

(ii) *S is a liar's dominating set of H.*

(iii) $|S\cap V(G)|\geq 3$ *and* $|S\cap V(H)|\geq 3$.

(iv) *$S=S_1\cup S_2$, where $S_1\subseteq V(G)$ and $S_2\subseteq V(H)$ and satisfy the following:*

  (a) *S_1 is a dominating set of G such that $|S_1|\geq 2$ and $S_1\in\Omega_G$, and*

  (b) *$|S_2|=1$, where S_2 is an almost dominating set of H whenever $|S_1|=2$.*

(v) *$S=S_1\cup S_2$, where $S_1\subseteq V(G)$ and $S_2\subseteq V(H)$ and satisfy the following:*

  (a) *S_2 is a dominating set of H such that $|S_2|\geq 2$ and $S_2\in\Omega_H$, and*

  (b) *$|S_1|=1$, where S_1 is an almost dominating set of G whenever $|S_2|=2$.*

(vi) *$S=S_1\cup S_2$, where $S_1\subseteq V(G)$ and $S_2\subseteq V(H)$ and satisfy the following:*

  (a) *$|S_1|=2$ and S_1 is an almost dominating set of G, and*

  (b) *$|S_2|=2$ and S_2 is an almost dominating set of H.*

(vii) *$S=S_1\cup S_2$, where $S_1\subseteq V(G)$ and $|S_1|=2$ and $S_2\subseteq V(H)$, $|S_2|\geq 3$ and S_2 is an almost dominating set of H.*

(viii) *$S=S_1\cup S_2$, where $S_2\subseteq V(H)$ and $|S_2|=2$ and $S_1\subseteq V(G)$, $|S_1|\geq 3$ and S_1 is an almost dominating set of G.*

Proof. Suppose that S is an lds of $G+H$. If $S\cap V(H)=\varnothing$ or $S\cap V(G)=\varnothing$, then S is an lds of G or H. Thus, (i) or (ii) holds. Now, suppose that $S_1=S\cap V(G)\neq\varnothing$ and $S_2=S\cap V(H)\neq\varnothing$. If $|S_1|\geq 3$ and $|S_2|\geq 3$, then (iii) holds. Consider the following cases:

Case 1. $|S_1|\geq 2$ and $|S_2|=1$ or $|S_1|=1$ and $|S_2|\geq 2$.

Suppose that $|S_1|\geq 2$ and $|S_2|=1$. Since S is a double dominating set of $G+H$, it follows that S_1 is a dominating set of G. Let $x\in S_1$. Suppose further that $|epn_G(x;S_1)|\geq 2$, say $y,z\in epn_G(x;S_1)$, where $y\neq z$. Then $N_{G+H}[y]\cap S=N_{G+H}[z]\cap S=\{x\}\cup S_2$. This implies that S is not a liar's dominating set of $G+H$, contrary to our assumption. Thus, $|epn_G(x;S_1)|\leq 1$. Suppose now that $|epn_G(x;S_1)|=1$, say $z\in epn_G(x;S_1)$. Since S is an lds of $G+H$, $|(N_{G+H}[z]\cup N_{G+H}[x])\cap S|=1+|(N_G[z]\cup N_G[x])\cap S_1|=1+1+|N_G(x)\cap S_1|\geq 3$. This implies that $|N_G(x)\cap S_1|\geq 1$, that is, $x\in N_G(S_1)$. Hence, $S_1\in\Omega_G$. Next, suppose that $|S_1|=2$ and suppose further that $|V(H)\setminus N_H[S_2]|\geq 2$. Let $a,b\in V(H)\setminus N_H[S_2]$ with $a\neq b$. Then $|(N_{G+H}[a]\cup N_{G+H}[b])\cap S|=$

$|S_1| = 2$, contrary to the assumption that S is an lds of $G+H$. Thus, S_2 is an almost dominating set of H. This shows that (iv) holds. Similarly, (v) holds if $|S_1| = 1$ and $|S_2| \geq 2$.

Case 2. $|S_1| = 2$ and $|S_2| = 2$.

Suppose that $|V(G) \setminus N[S_1]| \geq 2$. Then there exist $x, y \in V(G)$ such that $x, y \notin N[S_1]$. This implies that $|(N_{G+H}[x] \cup N_{G+H}[y]) \cap S| = 2$. Thus, S is not a liar's dominating set of $G+H$, contrary to our assumption. Hence, $|V(G) \setminus N[S_1]| \leq 1$. Similarly, $|V(H) \setminus N[S_2]| \leq 1$. Thus, (vi) holds.

Case 3. $|S_1| = 2$ and $|S_2| \geq 3$ or $|S_1| \geq 3$ and $|S_2| = 2$.

Suppose that $|S_1| = 2$ and $|S_2| \geq 3$. Suppose further that $|V(H) \setminus N[S_2]| \geq 2$, say $x, y \in V(H) \setminus N[S_2]$, where $x \neq y$. Then, $|(N_{G+H}[x] \cup N_{G+H}[y]) \cap S| = |S_1| = 2$. This implies that S is not a liar's dominating set of $G+H$, contrary to our assumption. Thus, $|V(H) \setminus N[S_2]| \leq 1$. This shows that (vii) holds. Similarly, (viii) holds if $|S_1| \geq 3$ and $|S_1| = 2$.

The converse is clear. □

It is immediate from Theorem 2.5 that $3 \leq \gamma_{LR}(G+H) \leq 6$ for any non-trivial connected graphs G and H. The next results are also consequences of Theorem 2.5.

Corollary 2.6. *Let G and H be non-trivial connected graphs. Then $\gamma_{LR}(G+H) = 3$ if and only if at least one of the following holds:*

(i) $\gamma_{LR}(G) = 3$;

(ii) $\gamma_{LR}(H) = 3$;

(iii) $\gamma_a(H) = 1$ *and* $\gamma^{\star}(G) \leq 2$; *or*

(iv) $\gamma_a(G) = 1$ *and* $\gamma^{\star}(H) \leq 2$.

Corollary 2.7. *Let G and H be non-trivial connected graphs such that $\gamma_{LR}(G+H) \neq 3$. Then $\gamma_{LR}(G+H) = 4$ if and only if at least one of the following holds:*

(i) $\gamma_{LR}(G) = 4$;

(ii) $\gamma_{LR}(H) = 4$;

(iii) $\gamma_a(G) \leq 2$ *and* $\gamma_a(H) \leq 2$;

(iv) $\gamma^{\star}(H) = 3$; *or*

(v) $\gamma^{\star}(G) = 3$.

Corollary 2.8. *Let G and H be non-trivial connected graphs such that $\gamma_{LR}(G+H) > 4$. Then $\gamma_{pLR}(G+H) = 5$ if and only if at least one of the following holds:*

(i) $\gamma_{LR}(G) = 5$;

(ii) $\gamma_{LR}(H) = 5$;

(iii) $\gamma_a(H) = 3$;

(iv) $\gamma_a(G) = 3$;

(v) $\gamma^\star(G) = 4$; *or*

(vi) $\gamma^\star(H) = 4$.

3. Liar's domination in the corona of graphs

The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then forming the join $\langle\{v\}\rangle + H^v = v + H^v$, where H^v is a copy of H, for each $v \in V(G)$.

Theorem 3.1. *Let G be any graph and H be any non-trivial graph. Then $C \subseteq V(G \circ H)$ is a liar's dominating set of $G \circ H$ if and only if $C = A \cup (\cup_{v\in A} S_v) \cup \left(\cup_{u \notin A} D_u\right)$, where $A \subseteq V(G)$, $S_v \in \Omega_{H^v}$ for each $v \in A$, and D_u is a liar's dominating set of H^u for each $u \notin A$.*

Proof. Suppose that C is an lds of $G \circ H$. Because H is a non-trivial graph, $C \cap V(v + H^v)$ is a liar's dominating set of $v + H^v$ for each $v \in V(G)$. Now let $A = C \cap V(G)$. Let $v \in A$ and set $S_v = C \cap V(H^v)$. By Theorem 2.1(ii), $S_v \in \Omega_{H^v}$. Next, let $u \notin A$ and set $D_u = C \cap V(H^u)$. By Theorem 2.1(i), D_u is a liar's dominating set of H^u.

For the converse, suppose that C has the given form and the given properties. Let $z \in V(G \circ H)$ and let $w \in V(G)$ such that $z \in V(w + H^w)$. If $w \in A$, then $E_w = S_w \cup \{w\}$ is an lds of $w + H^w$ by assumption and Theorem 2.1. Hence, $|N_{G\circ H}[z] \cap C| \geq |N_{w+H^w}[z] \cap E_w| \geq 2$. If $w \notin A$, then $E_w = D_w$ is an lds of $w + H^w$ by assumption and Theorem 2.1. Hence, $|N_{G\circ H}[z] \cap C| \geq |N_{w+H^w}[z] \cap D_w| \geq 2$. Therefore, C is a double dominating set of $G \circ H$.

Next, let $a, b \in V(G \circ H)$ such that $a \neq b$. Let $u, v \in V(G)$ such that $a \in V(u + H^u)$ and $b \in V(v + H^v)$. If $u = v$, then $|(N_{G\circ H}[a] \cup N_{G\circ H}[b]) \cap C| \geq |(N_{u+H^u}[a] \cup N_{u+H^u}[b]) \cap E_u| \geq 3$ since E_u is an lds of $u + H^u$, where $E_u = S_u \cup \{u\}$ if $u \in A$ and $E_u = D_u$ if $u \notin A$. If $u \neq v$, then $|(N_{G\circ H}[a] \cup N_{G\circ H}[b]) \cap C| \geq |N_{u+H^u}[a] \cup E_u| + |N_{v+H^v}[b] \cap E_v| \geq 2 + 2 = 4$ since E_u and E_v are double dominating sets of $u + H^u$ and $v + H^v$, respectively, where $E_u = S_u \cup \{u\}$ if $u \in A$ and $E_u = D_u$ if $u \notin A$, and $E_v = S_v \cup \{v\}$ if $v \in A$ and $E_v = D_v$ if $v \notin A$. Therefore, C is an lds of $G \circ H$. □

Corollary 3.2. *Let G be any graph and H be any non-trivial graph. Then*
$\gamma_{LR}(G \circ H) = |V(G)|(\gamma^\star(H) + 1)$.

Proof. Let S be a $\gamma^\star$-set of H. For each $v \in V(G)$, let $S_v \subseteq V(H^v)$ such that $\langle S_v\rangle \cong \langle S\rangle$. By Theorem 3.1, $C = V(G) \cup (\cup_{v\in V(G)} S_v)$ is an lds of $G \circ H$. Hence, $\gamma_{LR}(G \circ H) \leq |C| = |V(G)|(\gamma^\star(H) + 1)$.

Next, suppose that C' is a γ_{LR}-set of $G\circ H$. Then $C' = A\cup(\cup_{v\in A}S_v)\cup\big(\cup_{u\notin A}D_u\big)$, where $A\subseteq V(G)$ and the sets S_v's and D_u's satisfy the properties given in Theorem 3.1. Thus, $\gamma_{LR}(G\circ H) = |C'| = |A| + \sum_{v\in A}|S_v| + \sum_{v\notin A}|D_u| \geq |A| + |A|\gamma^\star(H) + (|V(G)| - |A|)\gamma_{LR}(H)$. By Lemma 2.2, it follows that $\gamma_{LR}(G\circ H) \geq |V(G)|(\gamma^\star(H)+1)$. This proves the desired equality. □

4. Liar's domination in the lexicographic product of graphs

The *lexicographic product* of two graphs G and H is the graph $G[H]$ with vertex-set $V(G[H]) = V(G)\times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(u_1,u_2)(v_1,v_2)\in E(G[H])$ if and only if either $u_1v_1\in E(G)$ or $u_1 = v_1$ and $u_2v_2\in E(H)$.

Observe that a non-empty subset C of $V(G[H]) = V(G)\times V(H)$ can be written as $C = \bigcup\limits_{x\in S}(\{x\}\times T_x)$, where $S\subseteq V(G)$ and $T_x\subseteq V(H)$ for every $x\in S$. Henceforth, we shall use this form to denote any non-empty subset C of $V(G[H])$.

Theorem 4.1. *Let G and H be connected graphs of orders $n\geq 2$ and $m\geq 3$, respectively. A non-empty subset $C = \bigcup_{x\in S}(\{x\}\times T_x)$ of $V(G[H])$, where $S\subseteq V(G)$ and $\varnothing\neq T_x\subseteq V(H)$ for each $x\in S$, is a liar's dominating set of $G[H]$ if and only if S is a dominating set of G and satisfies each of the following:*

(i) *T_x is a liar's dominating set of H for each $x\in S\setminus N_G(S)$;*

(ii) *for each $x\in S\cap N_G(S)$ such that $N_G(x)\cap S = \{y\}$, one of the following holds:*

  (a) *$|T_y|\geq 3$;*

  (b) *T_x is an almost dominating set and $|T_y| = 2$;*

  (c) *$|T_y| = 1$ and T_x is a dominating set such that $T_x\in\Omega_H$;*

(iii) *T_x is an almost dominating set of H for each $x\in S\cap N_G(S)$ such that $|N_G(x)\cap S| = 2$ and $|T_y| = |T_z| = 1$, where $N_G(x)\cap S = \{y,z\}$; and*

(iv) *for each $x\in V(G)\setminus S$,*

  (a) *$|T_y|\geq 3$ whenever $N_G(x)\cap S = \{y\}$; and*

  (b) *$|T_y|\geq 2$ or $|T_z|\geq 2$ whenever $N_G(x)\cap S = \{y,z\}$.*

Proof. Suppose that C is an lds of $G[H]$. Since C is a (double) dominating set of $G[H]$, it follows that S is a dominating set of G.

Let $x\in S\setminus N_G(S)$ and let $q\in V(H)$. Since C is a double dominating set of $G[H]$, there exist distinct $(z,c),(w,p)\in C\cap N_{G[H]}[(x,q)]$. Since $x\notin N_G(S)$, it follows that $z = w = x$. Thus, $c,p\in T_x\cap N_H[q]$. Since (z,c) and (w,p) are distinct, $c\neq p$. Thus, $|N_H[q]\cap T_x|\geq 2$. Next, let $q,r\in V(H)$ such that $q\neq r$. Then $(x,q)\neq(x,r)$. Since C is an lds of $G[H]$ and $x\notin N_G(S)$, it follows that $|(N_{G[H]}[(x,q)] \cup N_{G[H]}[(x,r)]) \cap C| = |(N_H[q] \cup N_H[r]) \cap T_x| \geq 3$. Therefore, T_x is a liar's dominating set of H and shows that (i) holds.

Let $x \in S \cap N_G(S)$ with $|N_G(x) \cap S| = 1$, say $N_G(x) \cap S = \{y\}$. Suppose that $|T_y| \leq 2$. First, suppose that $|T_y| = 2$ and suppose further that T_x is not an almost dominating set of H. Then there exist two distinct vertices $a, b \in V(H) \setminus T_x$ such that $a, b \notin N_H(T_x)$. Thus, $|(N_{G[H]}(x,a) \cup N_{G[H]}((x,b))) \cap C| = |\{y\} \times T_y| = |T_y| = 2$, contrary to our assumption that C is an lds of $G[H]$. Hence, T_x is an almost dominating set of H, showing that (b) holds.

Next, suppose that $|T_y| = 1$. Since C is a double dominating set of $G[H]$, T_x is a dominating set of H. Let $a \in T_x$ and suppose that $|epn_H(a;T_x| \geq 2$, say $l, m \in epn_H(a;T_x)$ $(l \neq m)$. Then $(x,l),(x,m) \notin C$ and $|(N_{G[H]}((x,l)) \cup N_{G[H]}((x,m))) \cap C| = |\{(x,a)\}| + |T_y| = 2$, contrary to our assumption that C is an lds of $G[H]$. Thus, $|epn_H(a;T_x)| \leq 1$. Suppose now that $|epn_H(a;T_x)| = 1$, say $d \in epn_H(a;T_x)$. Since C is an lds of $G[H]$, $|(N_{G[H]}[(x,d)] \cup N_{G[H]}[(x,a)]) \cap C| = 1 + |T_y| + |N_H(a) \cap T_x| \geq 3$. This implies that $|N_H(a) \cap T_x| \geq 1$, that is, $a \in N_H(T_x)$, showing that $T_x \in \Omega_H$. Therefore, (ii) holds.

Let $x \in S \cap N_G(S)$ with $N_G(x) \cap S = \{y,z\}$ and $|T_y| = |T_z| = 1$. Let $T_y = \{r\}$ and $T_x = \{s\}$. Suppose that $|V(H) \setminus N[T_x]| \geq 2$. Then there exist $p, q \in V(H) \setminus T_x$ such that $N_{G[H]}((x,p)) \cap C = \{(y,r),(z,s)\} = N_{G[H]}((x,q)) \cap C$. This means that C is not an lds of $G[H]$, contrary to our assumption. Thus, T_x is an almost dominating set of H. This shows that (iii) holds.

Finally, let $x \in V(G) \setminus S$ with $|N_G(x) \cap S| = 1$, say $N_G(x) \cap S = \{y\}$. Let $a, b \in V(H)$, where $a \neq b$. Then $(x,a),(x,b) \notin C$. Since C is liar's dominating set of $G[H]$, there exist at least three distinct vertices $(v,b),(w,c),(z,d) \in C \cap (N_{G[H]}((x,a)) \cup N_{G[H]}((x,b)))$. This implies that $v = w = z = y$, and $b, c, d \in T_y$. Hence, $|T_y| \geq 3$, showing that (a) holds. Suppose now that $N_G(x) \cap S = \{y,z\}$. Let $r, s \in V(H)$, where $r \neq s$. Then $(x,r),(x,s) \notin C$. Since C is an lds of $G[H]$, it follows that $|(N_{G[H]}[(x,r)] \cup N_{G[H]}[(x,s)]) \cap C| = |\{y\} \times T_y| + |\{z\} \times T_z| = |T_y| + |T_z| \geq 3$. Hence, $|T_y| \geq 2$ or $|T_z| \geq 2$. This shows that (iv) holds.

For the converse, suppose that S is a dominating set of G and satisfies properties (i), (ii), (iii) and (iv). Let $(x,a) \in V(G[H])$ and consider the following cases:

Case 1. $x \notin S$

Since S is a dominating set of G, it follows that $|N_G(x) \cap S| \geq 1$. Clearly, $|N_{G[H]}[(x,a)] \cap C| \geq 2$ if $|N_G(x) \cap S| \geq 2$. So suppose that $|N_G(x) \cap S| = 1$, say $N_G(x) \cap S = \{y\}$. Then, by (iv)(a), $|N_{G[H]}[(x,a)] \cap C| = |T_y| \geq 3$.

Case 2. $x \in S$

If $x \in S \setminus N_G(S)$, then T_x is an lds of H by (i). Hence, $|N_{G[H]}[(x,a)] \cap C| = |N_H[a] \cap T_x| \geq 2$. Suppose that $x \in S \cap N_G(S)$. Suppose first that $|N_G(x) \cap S| = 1$. Then by (a), (b), and (c) of (ii), we have $|N_{G[H]}[(x,a)] \cap C| \geq 2$. Suppose that $|N_G(x) \cap S| = 2$. Then $|N_{G[H]}[(x,a)] \cap C| \geq 2$ by (iii).

Therefore, C is a double dominating set of $G[H]$.

Next, let $(x,a),(v,b) \in V(G[H])$, where $(x,a) \neq (v,b)$. If $x \neq v$, then, by (i), (ii), (iii), and (iv), it can be easily shown that $|(N_{G[H]}[(x,a)] \cup N_{G[H]}[(v,b)]) \cap C| \geq 3$. Suppose that $x = v$. Then $a \neq b$. Consider the following cases:

Case 1. $x \notin S$

Then, by (iv), $|(N_{G[H]}[(x,a)] \cup N_{G[H]}[(v,b)]) \cap C| \geq 3$.

Case 2. $x \in S$

Sub-case 1. $x \notin N_G(S)$

Then, by (i), T_x is a liar's dominating set of H. Hence, $|(N_{G[H]}[(x,a)] \cup N_{G[H]}[(v,b)]) \cap C| = |(N_H[a] \cup N_H[b]) \cap T_x)| \geq 3$.

Sub-case 2. $x \in N_G(S)$

Then, by (ii) and (iii), $|(N_{G[H]}[(x,a)] \cup N_{G[H]}[(v,b)]) \cap C| \geq 3$.

Accordingly, C is a liar's dominating set of $G[H]$. □

Corollary 4.2. *Let G and H be connected graphs of orders $n \geq 2$ and $m \geq 3$, respectively. Then* $\gamma_{LR}(G[H]) \leq \min\{\gamma(G)\gamma_{LR}(H), 3\gamma_t(G)\}$.

Proof. Let S_1 and S_2 be γ-set and γ_t-set of G, respectively, and let D be a γ_{LR}-set of H. By Theorem 4.1, $C_1 = \cup_{x \in S_1}[\{x\} \times T_x]$ and $C_2 = \cup_{y \in S_2}[\{y\} \times E_y]$, where $T_x = D$ for each $x \in S_1$ and $E_y = \{a,b,c\} \subseteq V(H)$ for each $y \in S_2$, are lds of $G[H]$. Thus, $\gamma_{LR}(G[H]) \leq \min\{|C_1|,|C_2|\} = min\{\gamma(G)\gamma_{LR}(H), 3\gamma_t(G)\}$. □

Remark 4.3. Both the upper bound and the strict inequality in Corollary 4.2 can be attained.

To see this, consider $P_3[P_3]$, $P_4[P_7]$ in Figure 2, and $G[C_3]$ in Figure 3.

It can be verified that $\gamma_{LR}(P_3[P_3]) = 3 = \gamma(P_3)\gamma_{LR}(P_3) < 6 = 3\gamma_t(P_3)$, $\gamma_{LR}(P_4[P_7]) = 6 = 3\gamma_t(P_4) < 12 = \gamma(P_4)\gamma_{LR}(P_7)$, and $\gamma_{LR}(G[C_3]) = 4 < 6 = min\{\gamma(G)\gamma_{LR}(C_3), 3\gamma_t(G)\}$.

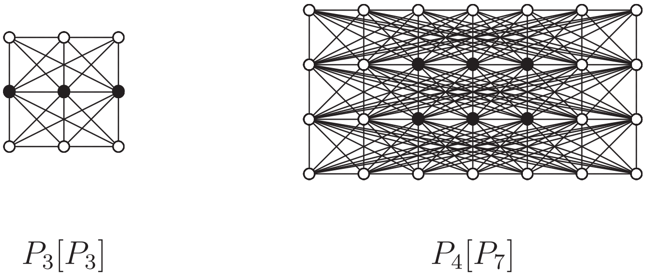

Figure 2: The graphs $P_3[P_3]$ and $P_4[P_7]$.

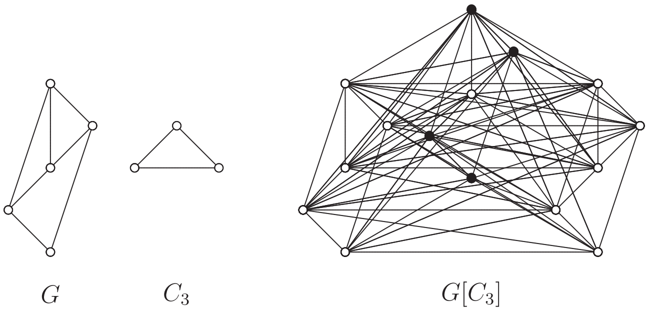

Figure 3: The graphs G, C_3, and $G[C_3]$.

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