ON A FOUR-DIMENSIONAL BERWALD SPACE WITH
VANISHING $h$-CONNECTION VECTOR $k_l$

P. N. PANDEY AND MANISH KUMAR GUPTA

Abstract. M. Matsumoto and R. Miron \cite{1} constructed an orthonormal frame for an $n$-dimensional Finsler space and the frame was called 'Miron frame'. T. N. Pandey and D. K. Diwerdi \cite{2} and the present authors \cite{3} studied four-dimensional Finsler spaces in terms of scalars. In the present paper, we study a four-dimensional Berwald space with vanishing $h$-connection vector $k_l$.

1. Orthonormal Frame and Connection Vectors

Let $L(\alpha, \beta)$ be the fundamental function and $g_{ij}(\alpha, \beta)$ be the fundamental metric tensor of a four-dimensional Finsler space $\mathcal{F}^4$. Let $\delta_{ijkl}^{pqrs}$ be generalized Kronecker delta, and $\gamma_{ijkl} = \delta_{ijkl}^{1234}$ and $\gamma^{ijkl} = \delta_{ijkl}^{1234}$, then the components of $\varepsilon$-tensor are defined by

$$\varepsilon_{ijkl} = \sqrt{|g|} \gamma_{ijkl} \quad \text{and} \quad \varepsilon_{ijkl}^{ij} = (\sqrt{|g|})^{-1} \gamma_{ijkl}; \quad \text{where} \quad g = |g_{ij}|.$$

$\varepsilon_{ijkl}$ is also called the Levi-Civita permutation symbol.

The Miron frame for a four-dimensional Finsler space is constructed by the unit vectors $(l^i, m^i, n^i, p^i)$, where $l^i$ is the normalized supporting element and $m^i$ is the normalized torsion vector.

In the orthonormal frame, an arbitrary tensor $T(T^i_j)$ is expressed in terms of scalar components as follows:

$$T^i_j = T_{\alpha\beta}^i e^\alpha_{a\beta} e^\beta_{b\gamma} j,$$

where $e^a_{b\gamma} = l^i$, $e^a_{c\gamma} = m^i$, $e^a_{d\gamma} = n^i$, $e^a_{e\gamma} = p^i$ and the summation convention is applied to Greek indices also.

The scalar components of the fundamental tensor $g_{ij}$ and $\varepsilon$-tensor $\varepsilon_{ijkl}$ are given by $\delta_{\alpha\beta}$ and $\gamma_{\alpha\beta\gamma\delta}$ respectively.

Let $H_{\alpha\beta\gamma}$ and $\frac{1}{L} V_{\alpha\beta\gamma}$ be scalar components of the $h$- and $\nu$-covariant derivatives $e^a_{\alpha\beta\gamma}$ and $e^a_{\alpha\beta\gamma}$ respectively of the vectors $e_{\alpha\beta\gamma}$, i.e.

$$\begin{align*}
a) \quad e^a_{\alpha\beta\gamma} &= H_{\alpha\beta\gamma} e^\alpha_{\beta\gamma} e^\beta_{\gamma\delta}, \\
b) \quad Le^a_{\alpha\beta\gamma} &= V_{\alpha\beta\gamma} e^\alpha_{\beta\gamma} e^\beta_{\gamma\delta}.
\end{align*}$$

Received October 20, 2006.

2000 Mathematics Subject Classification. 53B40.

Key words and phrases. Finsler space, Berwald space, $h$-connection vector.

1) Numbers in square brackets refer to the references at the end of the paper.
$H_{\alpha \beta \gamma}$ and $V_{\alpha \beta \gamma}$ are called $h$- and $v$-connection scalars respectively and are $(0)p$-homogeneous\(^2\).

From the orthogonality of the frame, we have

$$H_{\alpha \beta \gamma} = -H_{\beta \alpha \gamma}, \quad V_{\alpha \beta \gamma} = -V_{\beta \alpha \gamma}. \quad (1.3)$$

Also, we have

$$H_{1 \beta \gamma} = 0, \quad V_{1 \beta \gamma} = \delta_{\beta \gamma} - \delta_{\beta}^{1} \delta_{\gamma}^{1}. \quad (1.4)$$

We now define vector fields:

$$h_i = H_{2(3 \gamma) \epsilon \gamma}, \quad j_i = H_{3(2 \gamma) \epsilon \gamma}, \quad k_i = H_{3(4 \gamma) \epsilon \gamma}, \quad (1.5)$$

and

$$u_i = V_{2(3 \gamma) \epsilon \gamma}, \quad v_i = V_{3(2 \gamma) \epsilon \gamma}, \quad w_i = V_{3(4 \gamma) \epsilon \gamma}. \quad (1.6)$$

From (1.2), we get

\begin{align*}
a) & \quad e_{1}^{i} | j = h_{j}^{i}, \\
b) & \quad e_{2}^{i} | j = m_{j}^{i} = n_{j}^{i} h_{j} - p_{j}^{i} j, \\
c) & \quad e_{3}^{i} | j = n_{j}^{i} = p_{j}^{i} k_{j} - m_{j}^{i} h_{j}, \\
d) & \quad e_{4}^{i} | j = u_{j}^{i} = -v_{j}^{i} u_{j} - n_{j}^{i} k_{j},
\end{align*} \quad (1.7)

and

\begin{align*}
a) & \quad L e_{1}^{i} | j = L h_{j}^{i} = m_{j}^{i} + n_{j}^{i} h_{j} + p_{j}^{i} j, \\
b) & \quad L e_{2}^{i} | j = L m_{j}^{i} = -n_{j}^{i} h_{j}, \\
c) & \quad L e_{3}^{i} | j = L n_{j}^{i} = -m_{j}^{i} h_{j}, \\
d) & \quad L e_{4}^{i} | j = L p_{j}^{i} = n_{j}^{i} h_{j} + p_{j}^{i} w_{j}, \quad (1.8)
\end{align*}

The Finsler vector fields $h_{i}, j_{i}, k_{i}$ are called $h$-connection vectors and the vector fields $u_{i}, v_{i}, w_{i}$ are called $v$-connection vectors.

The scalars $H_{23 \gamma}, H_{42 \gamma}, H_{34 \gamma}$ and $V_{23 \gamma}, V_{42 \gamma}, V_{34 \gamma}$ are considered as the scalar components $h_{\gamma}, j_{\gamma}, k_{\gamma}$ and $u_{\gamma}, v_{\gamma}, w_{\gamma}$ of the $h$- and $v$-connection vectors respectively. Because of $(0)p$-homogeneity of $e_{\alpha}^{i}$, (1.8) gives

$$L m_{j}^{i} l^{j} = 0 = n_{j}^{i} u_{j} l^{i} - p_{j}^{i} v_{j} l^{i},$$

$$L n_{j}^{i} l^{j} = 0 = -m_{j}^{i} u_{j} l^{i} + p_{j}^{i} w_{j} l^{i},$$

so that $u_{1} = u_{j} l^{j} = 0$, $v_{1} = v_{j} l^{j} = 0$, $w_{1} = w_{j} l^{j} = 0$.

Consequently, we have:

**Proposition 1.1.** The first scalar components $u_{1}, v_{1}, w_{1}$ of $v$-connection vectors $u_{i}, v_{i}, w_{i}$ vanish identically.

\(^2\) “$(0)p$-homogeneous” is an abbreviation of “positively homogeneous of degree 0 in $\gamma$.”
2. Main scalars

Let $\frac{1}{L}C_{\alpha\beta\gamma}$ be scalar components of $C_{ijk}$ with respect to the Miron frame, i.e.

$$L C_{ijk} = C_{\alpha\beta\gamma} e_{\alpha} e_{\beta} e_{\gamma}.$$  

(2.1)

M. Matsumoto [1] showed that

(i) $C_{\alpha\beta\gamma}$ are completely symmetric,

(ii) $C_{1\beta\gamma} = 0$,

(iii) $C_{2\mu\mu} = LC = W$, $C_{3\mu\mu} = C_{4\mu\mu} = \cdots C_{n\mu\mu} = 0$ for $n \geq 3$, where $C$ is the length of $C^i$ and $W = LC$ is called the unified main scalar.

Therefore in a four-dimensional Finsler space, we have

$$\begin{align*}
C_{1\beta\gamma} &= 0, \\
C_{222} + C_{233} + C_{244} &= LC = W, \\
C_{322} + C_{333} + C_{344} &= 0, \\
C_{422} + C_{433} + C_{444} &= 0, \\
C_{234} &\neq 0 \text{ in general.}
\end{align*}$$  

(2.2)

Thus putting

$$\begin{align*}
C_{222} &= A, & C_{233} &= B, & C_{244} &= C, & C_{322} &= D, \\
C_{333} &= E, & C_{422} &= F, & C_{433} &= G, & C_{234} &= H, \\
\end{align*}$$  

(2.3)

we have

$$C_{344} = -(D + E), \quad C_{444} = -(F + G).$$

Eight scalars $A, B, \ldots, G, H$ given by (2.3) are called the main scalars of a four-dimensional Finsler space.

3. Scalar derivatives

Taking $h$-covariant differentiation of (1.1), we get

$$T^i_{j|k} = (\delta_k T_{a\beta}) e_{a}^i e_{\beta} e_{j} e_{k}.  
(3.1)$$

If $T_{a\beta,\gamma}$ are scalar components of $T^i_{j|k}$, i.e.

$$T^i_{j|k} = T_{a\beta,\gamma} e_{a}^i e_{\beta} e_{\gamma} e_{k},  
(3.2)$$

then we obtain

$$T_{a\beta,\gamma} = (\delta_k T_{a\beta}) e_{a}^k e_{\gamma} + T_{a\mu} H_{\mu a\gamma} + T_{a\gamma} H_{\mu a\beta}.  
(3.3)$$

Similarly the scalar components $T_{a\beta,\gamma}$ of $LT^i_{j|k}$ are given by

$$T_{a\beta,\gamma} = L (\delta_k T_{a\beta}) e_{a}^k + T_{a\mu} V_{\mu a\gamma} + T_{a\gamma} V_{\mu a\beta}.  
(3.4)$$
The scalar components \( T_{\alpha\beta\gamma} \) and \( T_{\alpha\beta\gamma} \) respectively are called \( h\)- and \( v\)- scalar derivatives of scalar components \( T_{\alpha\beta} \) of \( T \).

4. Berwald space

A Berwald space is characterized by \( C_{hi j|k} = 0 \), which is given by \( C_{\alpha\beta\gamma|\delta} = 0 \) in terms of scalars.

We are concerned with the tensor \( C_{hi j|k} \). From (2.1) and (3.2), it follows that

\[
L C_{hi j|k} = C_{\alpha\beta\gamma|\delta} e^{\alpha}_i e^{\beta}_j e^{\gamma}_k e^{\delta}_l.
\]

According to the formula (3.3), \( C_{\alpha\beta\gamma|\delta} \) are given by

\[
C_{\alpha\beta\gamma|\delta} = \delta^k_\delta C_{\alpha\beta\gamma} e^k_\delta + C_{\mu\beta\gamma} H_{\mu\alpha\delta} + C_{\alpha\mu\gamma} H_{\mu\beta\delta} + C_{\alpha\beta\mu} H_{\mu\gamma\delta}.
\]

The explicit form of \( C_{\alpha\beta\gamma|\delta} \) is obtained as follows:

\[
C_{222,\delta} = (\delta^k_\delta C_{222}) e^k_\delta + 3 C_{\mu 22} H_{\mu 2\delta}
\]

\[
= (\delta^k_\delta A) e^k_\delta + 3 C_{322} H_{3 2\delta} + 3 C_{422} H_{4 2\delta}
\]

\[
= A_{\delta} - 3 D h_\delta + 3 F j_\delta;
\]

\[
(4.2a)
\]

Similarly, we get

\[
C_{233,\delta} = B_{\delta} + (2D - E) h_\delta + G j_\delta - 2 H k_\delta,
\]

\[
C_{244,\delta} = C_{\delta} + (D + E) h_\delta - (3F + G) j_\delta + 2 H k_\delta,
\]

\[
C_{322,\delta} = D_{\delta} + (A - 2 B) h_\delta + 2 H j_\delta - F k_\delta,
\]

\[
C_{333,\delta} = E_{\delta} + 3 B h_\delta - 3 G k_\delta,
\]

\[
C_{422,\delta} = F_{\delta} - 2 H h_\delta - (A - 2 C) j_\delta + D k_\delta,
\]

\[
C_{433,\delta} = G_{\delta} + 2 H h_\delta - B j_\delta + (2 D + 3 E) k_\delta,
\]

\[
C_{234,\delta} = H_{\delta} + (F - G) h_\delta - (2 D + E) j_\delta + (B - C) k_\delta,
\]

\[
C_{344,\delta} = - D_{\delta} - E_{\delta} + C h_\delta - 2 H j_\delta + (F + 3 G) k_\delta,
\]

\[
C_{444,\delta} = - F_{\delta} - G_{\delta} - 3 C j_\delta - (3 D + 3 E) k_\delta,
\]

\[
C_{1\beta\gamma,\delta} = 0.
\]

(4.2b)(4.2c)(4.2f)(4.2g)(4.2h)(4.2i)(4.2j)(4.2k)

Adding (4.2d), (4.2e), (4.2i) and using (2.2), we get

\[
C_{322,\delta} + C_{333,\delta} + C_{344,\delta} = (A + B + C) h_\delta = L C h_\delta = W h_\delta.
\]

(4.3)
According to the formulae (3.3) and (3.4),

$$H_{222,\delta} + C_{233,\delta} + C_{244,\delta} = -(A + B + C)\delta = -LC\delta = -W\delta.$$  

Adding (4.2f), (4.2g), (4.2j) and using (2.2), we get

$$H_{222,\delta} + C_{233,\delta} + C_{244,\delta} = A\delta + B\delta + C\delta = (A + B + C)\delta = W\delta.$$  

Thus, from (4.2), (4.3), (4.4) and (4.5), we have

**Theorem 4.1.** In a four dimensional Berwald space, the h-connection vectors $h_i$ and $j_i$ vanish identically. Also main scalar $A$ and the unified main scalar $W = LC$ are h-covariant constants. Furthermore, if h-connection vector $k_i$ vanishes then all the main scalars are h-covariant constants.

### 5. Ricci identities

Now, we are concerned with the tensors $e^i_{a|j|k}$, $e^i_{a|j|l}$ and $e^i_{a|j|k}$. From (1.2), we have

$$e^i_{a|j|k} = H_{a|j} \delta^i_{\delta} e^\delta_{\delta} k,$$

$$L e^i_{a|j|l} = H_{a|j} \delta^i_{\delta} e^\delta_{\delta} j,$$

$$L e^i_{a|j|k} = V_{a|j} \delta^i_{\delta} e^\delta_{\delta} k.$$  

According to the formulae (3.3) and (3.4), $H_{a|j} \delta^i_{\delta}$, $H_{a|j} \delta^i_{\delta}$, $V_{a|j} \delta^i_{\delta}$ are given by

$$H_{a|j} \delta^i_{\delta} = (\delta_k H_{a|j} \delta^i_{\delta} e^k_{\delta} + H_{a|j} \mu \gamma \lambda \mu \gamma \lambda \delta,$$

$$H_{a|j} \delta^i_{\delta} = L(\delta_k H_{a|j} \delta^i_{\delta} e^k_{\delta} + H_{a|j} \mu \gamma \lambda \mu \gamma \lambda \delta, + H_{a|j} \mu \gamma \lambda \mu \gamma \lambda \delta,$$

$$V_{a|j} \delta^i_{\delta} = (\delta_k V_{a|j} \delta^i_{\delta} e^k_{\delta} + V_{a|j} \mu \gamma \lambda \mu \gamma \lambda \delta, + V_{a|j} \mu \gamma \lambda \mu \gamma \lambda \delta.$$  

The explicit forms of these are obtained as follows:

$$H_{2|j} \gamma \delta = (\delta_k H_{2|j} \gamma \delta e^k_{\delta} + H_{2|j} \mu \gamma \lambda \mu \gamma \lambda \delta,$$

$$= (\delta_k h_{\gamma \delta} e^k_{\delta} + H_{2|j} \gamma \delta H_{a|j} \delta + h_{\mu} H_{\mu} \gamma \delta,$$

$$= h_{\gamma \delta} + j_\gamma k_\delta,$$

where $h_{\gamma \delta} = (\delta_k h_{\gamma} e^k_{\delta} + h_{\mu} H_{\mu} \gamma \delta$. 

Similarly, we get

$$H_{a|2|j} \gamma \delta = j_{\gamma \delta} + k_\gamma h_{\delta},$$

$$H_{a|3|j} \gamma \delta = k_\gamma \delta + h_{\gamma} j_\delta,$$

$$H_{a|4|j} \gamma \delta = h_{\gamma \delta} + j_\gamma w_{\delta},$$

$$H_{a|2|j} \gamma \delta = j_{\gamma \delta} + k_\gamma u_\delta,$$

$$H_{a|3|j} \gamma \delta = k_\gamma \delta + h_{\gamma} v_\delta,$$

and
In terms of scalar components, the Ricci identity
\[ e^l_{a[ij]} - e^l_{a[ij]} = e^l_{a[i]} p^l_{jk} - e^l_{a[i]} r^l_{jk} = e^l_{a[i]} (r^l_{jk} - r^l_{jk}), \] (5.4)
is expressed as
\[ H_{\alpha}^{\beta\gamma\delta} - V_{\alpha}^{\beta\delta,\gamma} = -H_{\alpha}^{\beta\mu} C_{\mu\gamma\delta}; \] (5.5)
For Berwald space \( P_{hkij} = 0 \), therefore (5.5) becomes
\[ H_{\alpha}^{\beta\gamma\delta} - V_{\alpha}^{\beta\delta,\gamma} = -H_{\alpha}^{\beta\mu} C_{\mu\gamma\delta}; \]
which is explicitly written as
\[
\begin{align*}
(h_{\gamma,\delta} + j_{\gamma} w_{\delta}) - (u_{\delta,\gamma} + v_{\delta} k_{\gamma}) &= -h_{\mu} C_{\mu\gamma\delta}, \\
(j_{\gamma,\delta} + k_{\gamma} u_{\delta}) - (v_{\delta,\gamma} + u_{\delta} h_{\gamma}) &= -j_{\mu} C_{\mu\gamma\delta}, \\
(k_{\gamma,\delta} + h_{\gamma} v_{\delta}) - (w_{\delta,\gamma} + u_{\delta} j_{\gamma}) &= -k_{\mu} C_{\mu\gamma\delta}.
\end{align*}
\]
From Theorem 4.1, we see that in a Berwald space \( h_i = j_i = 0 \). If we take \( k_i = 0 \), then above equations become \( u_{\delta,\gamma} = v_{\delta,\gamma} = w_{\delta,\gamma} = 0 \).

Thus we have:

**Theorem 5.1.** In a four-dimensional Berwald space with vanishing h-connection vector \( k_i \), the v-connection vectors \( u_i, v_i, w_i \) are h-covariant constants.

From the Ricci identity
\[ T^l_{ij|k} - T^l_{ij|k} = T^l_{ij} R^l_{kh} - T^l_{ij} R^l_{jk} - T^l_{ij} R^l_{kh}, \] (5.6)
we have
\[ e^l_{a[ij]} - e^l_{a[ij]} = e^l_{a[i]} p^l_{jk} - e^l_{a[i]} r^l_{jk} = e^l_{a[i]} (r^l_{jk} - r^l_{jk}); \] (5.7)
which is expressed as
\[ H_{\alpha}^{\beta\gamma\delta} - H_{\alpha}^{\beta\delta,\gamma} = R_{\alpha}^{\beta\gamma\delta} - V_{\alpha}^{\beta\delta,\gamma} \] (5.8)

Now, we propose:

**Proposition 5.1.** Let \( T_{ij} \) be a skew-symmetric tensor of a four-dimensional Finsler space. If we put \( * T_{ij} = \frac{1}{4} \epsilon^{ijkl} T_{kl} \), then we obtain \( T_{pq} = \epsilon_{pqij} * T_{ij} \).

**Proof.** \( * T_{ij} = \frac{1}{4} \epsilon^{ijkl} T_{kl} \) implies
\[
* T_{ij} \epsilon_{pqij} = \frac{1}{4} \epsilon^{ijkl} \epsilon_{pqij} T_{kl} = \frac{1}{4} \delta^{ijkl} T_{kl} = T_{pq}.
\]
This completes the proof.

Since $R_{ijkl}$ is skew-symmetric in $h$ and $i$ as well as in $j$ and $k$, in view of Proposition 5.1, $R_{ijkl}$ may be written as

$$R_{ijkl} = \varepsilon_{hirs} \varepsilon_{jkpq} R^{rsqp},$$  \hfill (5.9)

where we put

$$\varepsilon^{rsqp} = \frac{1}{16} \varepsilon^{rshi} \varepsilon^{pqlm} R_{hilm}.$$  \hfill (5.10)

The scalar components $R_{ijkl}$ of $R_{ijkl}$ are written as

$$R_{ijkl} = \gamma_{ijkl} \gamma_{k}\delta_{ij} \gamma_{k} R^{\mu\lambda\delta\tau}$$  \hfill (5.11)

in terms of scalar components $\gamma_{ijkl}$ of $R^{rsqp}$.

The scalar components $R_{ijkl}$ of $R^{rsqp}$ are given by

$$R_{ijkl} = \gamma_{ijkl} \gamma_{k}\delta_{ij} \gamma_{k} R^{\mu\lambda\delta\tau}.$$  \hfill (5.12)

Therefore (5.8) may be written as

$$H_{\alpha\beta\gamma, \delta} - H_{\alpha\beta\delta, \gamma} = (\gamma_{\alpha\beta\mu\lambda} - \gamma_{\alpha\beta\gamma\delta}) \gamma_{\gamma\delta} \delta_{ij} \gamma_{k} R^{\mu\lambda\delta\tau}.$$  \hfill (5.13)

For different values of $\alpha$, $\beta$ this gives only three equations:

$$\begin{align*}
(h_{\gamma, \delta} + j_{\delta} k_{\gamma}) - (h_{\delta, \gamma} + j_{\gamma} k_{\delta}) &= (\gamma_{\mu\lambda}^4 - u_2 \delta_{\mu\lambda}^2 - u_3 \delta_{\mu\lambda}^2 - u_4 \delta_{\mu\lambda}^2) \gamma_{\gamma\delta} \delta_{ij} \gamma_{k} R^{\mu\lambda\delta\tau}, \\
(j_{\gamma, \delta} + k_{\delta} h_{\gamma}) - (j_{\delta, \gamma} + k_{\gamma} h_{\delta}) &= (\gamma_{\mu\lambda}^3 - u_2 \delta_{\mu\lambda} - u_3 \delta_{\mu\lambda} - u_4 \delta_{\mu\lambda}) \gamma_{\gamma\delta} \delta_{ij} \gamma_{k} R^{\mu\lambda\delta\tau}, \\
(k_{\gamma, \delta} + h_{\delta} j_{\gamma}) - (k_{\delta, \gamma} + h_{\gamma} j_{\delta}) &= (\gamma_{\mu\lambda}^2 - u_2 \delta_{\mu\lambda} - u_3 \delta_{\mu\lambda} - u_4 \delta_{\mu\lambda}) \gamma_{\gamma\delta} \delta_{ij} \gamma_{k} R^{\mu\lambda\delta\tau}.
\end{align*}$$

For Berwald space with $\kappa = 0$, above equations become

$$\begin{align*}
(\gamma_{\mu\lambda}^4 - u_2 \delta_{\mu\lambda}^2 - u_3 \delta_{\mu\lambda}^2 - u_4 \delta_{\mu\lambda}^2) R_{\mu\lambda\delta\tau} &= 0, \\
(\gamma_{\mu\lambda}^3 - u_2 \delta_{\mu\lambda} - u_3 \delta_{\mu\lambda} - u_4 \delta_{\mu\lambda}) R_{\mu\lambda\delta\tau} &= 0, \\
(\gamma_{\mu\lambda}^2 - u_2 \delta_{\mu\lambda} - u_3 \delta_{\mu\lambda} - u_4 \delta_{\mu\lambda}) R_{\mu\lambda\delta\tau} &= 0.
\end{align*}$$

Now applying the Ricci identity (5.6) to $\nu$-connection vectors $\nu_{\nu}^{(p)}$, we have

$$\nu_{\nu}^{(p)}_{\nu, \nu} - \nu_{\nu}^{(p)}_{\nu, \nu} = - \nu_{\nu}^{(p)} R_{\nu, \nu} - \nu_{\nu}^{(p)} R_{\nu}^{(p)};$$  \hfill (5.14)

where $(\nu_{\nu}^{(1)}, \nu_{\nu}^{(2)}, \nu_{\nu}^{(3)}) = (u_i, v_i, w_i)$.

In terms of scalars, (5.14) may be written as:

$$\nu_{\nu}^{(p)} R_{\nu, \nu} = -(\nu_{\nu}^{(p)} \gamma_{\nu\lambda\mu\lambda} + \nu_{\nu}^{(p)} \gamma_{\nu\lambda\mu\lambda}) R_{\mu\lambda\delta\tau}.$$
We have shown in Theorem 5.1 that in a Berwald space with $k_l = 0$, the $v$-connection vectors are $h$-covariant constants, therefore above equation becomes

$$\left(v^{(p)}_\beta \gamma_{\rho \beta \mu \lambda} + v^{(p)}_{\beta \rho} \gamma_{1 \beta \mu \lambda}\right)^* R_{\mu \lambda \theta \tau} = 0. \quad (5.15)$$

Because of $v^{(p)}_{1;\beta} = -v^{(p)}_\beta$, the above is trivial for $\beta = 1$ and thus from the above we obtain only

$$\left\{ v^{(p)}_3 \delta_{14} + v^{(p)}_4 \delta_{34} + v^{(p)}_{2;3} \delta_{\mu \lambda} + v^{(p)}_{1;4} \delta_{\mu \lambda} \right\}^* R_{\mu \lambda \theta \tau} = 0,$$

$$\left\{ v^{(p)}_2 \delta_{12} + v^{(p)}_4 \delta_{34} + v^{(p)}_{2;3} \delta_{\mu \lambda} + v^{(p)}_{1;4} \delta_{\mu \lambda} \right\}^* R_{\mu \lambda \theta \tau} = 0,$$

$$\left\{ v^{(p)}_2 \delta_{13} + v^{(p)}_3 \delta_{21} + v^{(p)}_{4;2} \delta_{\mu \lambda} + v^{(p)}_{4;3} \delta_{\mu \lambda} + v^{(p)}_{4;4} \delta_{\mu \lambda} \right\}^* R_{\mu \lambda \theta \tau} = 0. \quad (5.16)$$

In view of (5.13), these equations take the forms

$$\left\{ v^{(p)}_{2;2} + v^{(p)}_3 (u_2 - v^{(p)}_2 u_2) \delta_{\mu \lambda} + (v^{(p)}_{2;3} + v^{(p)}_3 u_3 - v^{(p)}_4 (u_3) \delta_{\mu \lambda} + (v^{(p)}_{1;4} + v^{(p)}_4 u_4 - v^{(p)}_4 u_4) \delta_{\mu \lambda} \right\}^* R_{\mu \lambda \theta \tau} = 0, \quad (5.16a)$$

$$\left\{ v^{(p)}_{3;2} + v^{(p)}_2 (u_2 - v^{(p)}_2 u_2) \delta_{\mu \lambda} + (v^{(p)}_{3;3} + v^{(p)}_3 u_3 - v^{(p)}_4 (u_3) \delta_{\mu \lambda} + (v^{(p)}_{1;4} + v^{(p)}_4 u_4 - v^{(p)}_4 u_4) \delta_{\mu \lambda} \right\}^* R_{\mu \lambda \theta \tau} = 0, \quad (5.16b)$$

$$\left\{ v^{(p)}_{4;2} + v^{(p)}_2 (u_2 - v^{(p)}_2 u_2) \delta_{\mu \lambda} + (v^{(p)}_{4;3} + v^{(p)}_3 u_3 - v^{(p)}_4 (u_3) \delta_{\mu \lambda} + (v^{(p)}_{1;4} + v^{(p)}_4 u_4 - v^{(p)}_4 u_4) \delta_{\mu \lambda} \right\}^* R_{\mu \lambda \theta \tau} = 0. \quad (5.16c)$$

Put $v^{(p)}_{\alpha \beta} = v^{(p)}_{\alpha \beta} + v^{(p)}_{\alpha \mu} (A_{\mu \beta})$, then equations (5.16) become

$$\left\{ v^{(p)}_{2;2} \delta_{\mu \lambda} + v^{(p)}_{2;3} \delta_{\mu \lambda} + v^{(p)}_{2;4} \delta_{\mu \lambda} \right\}^* R_{\mu \lambda \theta \tau} = 0,$$

$$\left\{ v^{(p)}_{3;2} \delta_{\mu \lambda} + v^{(p)}_{3;3} \delta_{\mu \lambda} + v^{(p)}_{3;4} \delta_{\mu \lambda} \right\}^* R_{\mu \lambda \theta \tau} = 0,$$

$$\left\{ v^{(p)}_{4;2} \delta_{\mu \lambda} + v^{(p)}_{4;3} \delta_{\mu \lambda} + v^{(p)}_{4;4} \delta_{\mu \lambda} \right\}^* R_{\mu \lambda \theta \tau} = 0. \quad (5.17)$$

Again, applying the Ricci identity (5.6) to the main scalars $A^{(q)}$, we have

$$A^{(q)}_{ijk} - A^{(q)}_{kij} = -A^{(q)}_{ij} R^{r}_{jk}, \quad (5.18)$$

where $(A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}, A^{(5)}, A^{(6)}, A^{(7)}, A^{(8)}) = (A, B, C, D, E, F, G, H)$.

In terms of scalars, (5.18) assumes the form

$$A^{(q)}_{ijk} - A^{(q)}_{kij} = -A^{(q)}_{ij} \gamma_{1 \mu \lambda} \gamma_{1 \theta \tau} R_{\mu \lambda \theta \tau} = 0. \quad (5.19)$$

We have seen in Theorem 4.1 that all the main scalars are $h$-covariant constants in a Berwald space with $k_l = 0$. Therefore above equation becomes

$$A^{(q)}_{2 \delta_{\mu \lambda}} + A^{(q)}_{3 \delta_{\mu \lambda}} + A^{(q)}_{4 \delta_{\mu \lambda}} \right\}^* R_{\mu \lambda \theta \tau} = 0. \quad (5.19)$$

We now discuss Berwald space with vanishing $h$-connection vectors, considering the rank $\rho$ of the matrix $(R_{\mu \lambda \theta \tau})$, where $(\mu \lambda)$ and $(\theta \tau)$ show the number of rows and columns respectively. From (5.13), it is clear that the rank $\rho$ is less than four.
(i) if $\rho = 0$ then $^{*}R_{\mu \theta \tau} = 0$. This means $^{*}R_{hijk} = 0$ and therefore the space is locally Minkowskian.

(ii) if $\rho = 1$, then from (5.17) and (5.19), we have

\[ A_{2}^{(q)} : A_{3}^{(q)} : A_{4}^{(q)} = v_{22}^{(p)} : v_{23}^{(p)} : v_{24}^{(p)} = v_{33}^{(p)} : v_{34}^{(p)} = v_{42}^{(p)} : v_{43}^{(p)} : v_{44}^{(p)} \]

\[(p = 1, 2, 3; q = 1, 2, \ldots, 8) \quad (5.20)\]

(iii) if $\rho = 2$, then from (5.17),

\[ v_{32}^{(p)} v_{33}^{(p)} v_{34}^{(p)} = 0 \text{ such that conditions } (5.20) \text{ do not hold.} \]

(iv) if $\rho = 3$, then from (5.17) and (5.19), $v_{a\beta}^{(p)} = 0$; $\alpha, \beta = 2, 3, 4$ and $A_{2}^{(q)} = A_{3}^{(q)} = A_{4}^{(q)} = 0$ so that all the main scalars are $\nu$-covariant constants and therefore they are constants.

Summarizing the above, we conclude:

**Theorem 5.2.** In a four-dimensional Berwald space with vanishing $h$-connection vector $k_{i}$, the rank $\rho$ of the matrix $(R_{hijk})$, where $(hi)$ and $(jk)$ show the number of rows and columns respectively, is less than four. Further

(i) if $\rho = 0$, the space is locally Minkowskian.

(ii) if $\rho = 1$, we have the conditions (5.20).

(iii) if $\rho = 2$, $v_{22}^{(p)} v_{23}^{(p)} v_{24}^{(p)} v_{32}^{(p)} v_{33}^{(p)} v_{34}^{(p)} = 0$ and conditions (5.20) do not hold.

(iv) if $\rho = 3$, all the main scalars are constants and $v_{a\beta}^{(p)} = 0$, $(p = 1, 2, 3; \alpha, \beta = 2, 3, 4)$.

**References**


Department of Mathematics, University of Allahabad, Allahabad, India.

E-mail: pnpipa@rediffmail.com

Department of Mathematics, University of Allahabad, Allahabad, India.

E-mail: manishkg_1982@yahoo.com