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ON A FOUR-DIMENSIONAL BERWALD SPACE WITH VANISHING h-CONNECTION VECTOR k_i

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Abstract. M. Matsumoto and R. Miron [2][1)] constructed an orthonormal frame for an n-dimensional Finsler space and the frame was called 'Miron frame'. T. N. Pandey and D. K. Diwedi [3] and the present authors [4] studied four-dimensional Finsler spaces in terms of scalars. In the present paper, we study a four-dimensional Berwald space with vanishing h-connection vector k_i.

1. Orthonormal Frame and Connection Vectors

Let $L(x,y)$ be the fundamental function and $g_{ij}(x,y)$ be the fundamental metric tensor of a four-dimensional Finsler space F^4. Let δ^{ijkl}_{pqrs} be generalized Kronecker delta, and $\gamma_{ijkl} = \delta^{1234}_{ijkl}$ and $\gamma^{ijkl} = \delta^{ijkl}_{1234}$, then the components of $\in$-tensor are defined by

$$\in_{ijkl} = \sqrt{|g|}\gamma_{ijkl} \quad \text{and} \quad \in^{ijkl} = (\sqrt{|g|})^{-1}\gamma^{ijkl}; \quad \text{where } g = |g_{ij}|.$$

$\in_{ijkl}$ is also called the Levi-Civita permutation symbol.

The Miron frame for a four-dimensional Finsler space is constructed by the unit vectors (l^i, m^i, n^i, p^i), where l^i is the normalized supporting element and m^i is the normalized torsion vector.

In the orthonormal frame, an arbitrary tensor $T = (T^i_j)$ is expressed in terms of scalar components as follows:

$$T^i_j = T_{\alpha\beta} e^i_{\alpha)} e_{\beta)j}, \tag{1.1}$$

where $e^i_{1)} = l^i$, $e^i_{2)} = m^i$, $e^i_{3)} = n^i$, $e^i_{4)} = p^i$ and the summation convention is applied to Greek indices also.

The scalar components of the fundamental tensor g_{ij} and $\in$-tensor $\in_{ijkl}$ are given by $\delta_{\alpha\beta}$ and $\gamma_{\alpha\beta\gamma\delta}$ respectively.

Let $H_{\alpha)\beta\gamma}$ and $\frac{1}{L}V_{\alpha)\beta\gamma}$ be scalar components of the h- and v-covariant derivatives $e^i_{\alpha)|j}$ and $e^i_{\alpha)}|_j$ respectively of the vectors $e_{\alpha)}$, i.e.

$$\begin{aligned} &\text{a)} \quad e^i_{\alpha)|j} = H_{\alpha)\beta\gamma}\, e^i_{\beta)}\, e_{\gamma)j}, \\ &\text{b)} \quad Le^i_{\alpha)}|_j = V_{\alpha)\beta\gamma}\, e^i_{\beta)}\, e_{\gamma)j}. \end{aligned} \tag{1.2}$$

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[1)] Numbers in square brackets refer to the references at the end of the paper.

$H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ are called h- and v-connection scalars respectively and are $(0)\,p$-homogeneous[2)]. From the orthogonality of the frame, we have

$$H_{\alpha)\beta\gamma} = -H_{\beta)\alpha\gamma}, \quad V_{\alpha)\beta\gamma} = -V_{\beta)\alpha\gamma}. \tag{1.3}$$

Also, we have

$$H_{1)\beta\gamma} = 0, \quad V_{1)\beta\gamma} = \delta_{\beta\gamma} - \delta^1_\beta\,\delta^1_\gamma. \tag{1.4}$$

We now define vector fields:

$$h_i = H_{2)3\gamma}\, e_{\gamma)i}, \quad j_i = H_{4)2\gamma}\, e_{\gamma)i}, \quad k_i = H_{3)4\gamma}\, e_{\gamma)i}, \tag{1.5}$$

and

$$u_i = V_{2)3\gamma}\, e_{\gamma)i}, \quad v_i = V_{4)2\gamma}\, e_{\gamma)i}, \quad w_i = V_{3)4\gamma}\, e_{\gamma)i}. \tag{1.6}$$

From (1.2), we get

$$\begin{aligned}
&\text{a)}\quad e^i_{1)|j} = l^i_{|j} = 0,\\
&\text{b)}\quad e^i_{2)|j} = m^i_{|j} = n^i h_j - p^i j_j,\\
&\text{c)}\quad e^i_{3)|j} = n^i_{|j} = p^i k_j - m^i h_j,\\
&\text{d)}\quad e^i_{4)|j} = p^i_{|j} = m^i j_j - n^i k_j,
\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}
&\text{a)}\quad Le^i_{1)}|_j = Ll^i|_j = m^i m_j + n^i n_j + p^i p_j = h^i_j,\\
&\text{b)}\quad Le^i_{2)}|_j = Lm^i|_j = -l^i m_j + n^i u_j - p^i v_j,\\
&\text{c)}\quad Le^i_{3)}|_j = Ln^i|_j = -l^i n_j - m^i u_j + p^i w_j,\\
&\text{d)}\quad Le^i_{4)}|_j = Lp^i|_j = -l^i p_j + m^i v_j - n^i w_j.
\end{aligned} \tag{1.8}$$

The Finsler vector fields h_i, j_i, k_i are called h-connection vectors and the vector fields u_i, v_i, w_i are called v-connection vectors.

The scalars $H_{2)3\gamma}$, $H_{4)2\gamma}$, $H_{3)4\gamma}$ and $V_{2)3\gamma}$, $V_{4)2\gamma}$, $V_{3)4\gamma}$ are considered as the scalar components h_γ, j_γ, k_γ and u_γ, v_γ, w_γ of the h- and v-connection vectors respectively. Because of $(0)\,p$-homogeneity of $e^i_{\alpha)}$, (1.8) gives

$$\begin{aligned}
Lm^i|_j l^j &= 0 = n^i u_j l^j - p^i v_j l^j,\\
Ln^i|_j l^j &= 0 = -m^i u_j l^j + p^i w_j l^j,
\end{aligned}$$

so that $u_1 = u_j l^j = 0$, $v_1 = v_j l^j = 0$, $w_1 = w_j l^j = 0$.

Consequently, we have:

Proposition 1.1. *The first scalar components u_1, v_1, w_1 of v-connection vectors u_i, v_i, w_i vanish identically.*

2) "$(0)\,p$-homogeneous" is an abbreviation of "positively homogeneous of degree 0 in y".

2. Main scalars

Let $\frac{1}{L}C_{\alpha\beta\gamma}$ be scalar components of C_{ijk} with respect to the Miron frame, i.e.

$$L\,C_{ijk} = C_{\alpha\beta\gamma}e_{\alpha)i}e_{\beta)j}e_{\gamma)k}. \tag{2.1}$$

M. Matsumoto [1] showed that

(i) $C_{\alpha\beta\gamma}$ are completely symmetric,

(ii) $C_{1\beta\gamma} = 0$,

(iii) $C_{2\mu\mu} = L\boldsymbol{C} = W$, $C_{3\mu\mu} = C_{4\mu\mu} = \cdots\cdots C_{n\mu\mu} = 0$ for $n \geq 3$, where $\boldsymbol{C}$ is the length of C^i and $W = L\boldsymbol{C}$ is called the unified main scalar.

Therefore in a four-dimensional Finsler space, we have

$$\begin{cases} C_{1\beta\gamma} = 0, \\ C_{222} + C_{233} + C_{244} = L\boldsymbol{C} = W, \\ C_{322} + C_{333} + C_{344} = 0, \\ C_{422} + C_{433} + C_{444} = 0, \\ C_{234} \neq 0 \text{ in general.} \end{cases} \tag{2.2}$$

Thus putting

$$\begin{aligned} &C_{222} = A, \quad C_{233} = B, \quad C_{244} = C, \quad C_{322} = D, \\ &C_{333} = E, \quad C_{422} = F, \quad C_{433} = G, \quad C_{234} = H, \end{aligned} \tag{2.3}$$

we have

$$C_{344} = -(D+E), \quad C_{444} = -(F+G).$$

Eight scalars $A, B, \ldots\ldots G, H$ given by (2.3) are called the main scalars of a four-dimensional Finsler space.

3. Scalar derivatives

Taking h-covariant differentiation of (1.1), we get

$$T^i_{j|k} = (\delta_k T_{\alpha\beta})e^i_{\alpha)}e_{\beta)j} + T_{\alpha\beta}e^i_{\alpha)|k}e_{\beta)j} + T_{\alpha\beta}e^i_{\alpha)}e_{\beta)j|k}. \tag{3.1}$$

If $T_{\alpha\beta,\,\gamma}$ are scalar components of $T^i_{j|k}$, i.e.

$$T^i_{j|k} = T_{\alpha\beta,\gamma}\; e^i_{\alpha)}e_{\beta)j}e_{\gamma)k}, \tag{3.2}$$

then we obtain

$$T_{\alpha\beta,\gamma} = (\delta_k T_{\alpha\beta})e^k_{\gamma)} + T_{\mu\beta}H_{\mu)\alpha\gamma} + T_{\alpha\mu}H_{\mu)\beta\gamma}. \tag{3.3}$$

Similarly the scalar components $T_{\alpha\beta;\gamma}$ of $LT^i_j|k$ are given by

$$T_{\alpha\beta;\gamma} = L(\dot{\partial}_k T_{\alpha\beta})e^k_{\gamma)} + T_{\mu\beta}V_{\mu)\alpha\gamma} + T_{\alpha\mu}V_{\mu)\beta\gamma}. \tag{3.4}$$

The scalar components $T_{\alpha\beta,\gamma}$ and $T_{\alpha\beta;\gamma}$ respectively are called h- and v- scalar derivatives of scalar components $T_{\alpha\beta}$ of T.

4. Berwald space

A Berwald space is characterized by $C_{hij|k} = 0$, which is given by $C_{\alpha\beta\gamma,\delta} = 0$ in terms of scalars.

We are concerned with the tensor $C_{hij|k}$. From (2.1) and (3.2), it follows that

$$L\, C_{hij|k} = C_{\alpha\beta\gamma,\delta}\, e_{\alpha)h} e_{\beta)i} e_{\gamma)j} e_{\delta)k}. \tag{4.1}$$

According to the formula (3.3), $C_{\alpha\beta\gamma,\delta}$ are given by

$$C_{\alpha\beta\gamma,\delta} = \delta_k C_{\alpha\beta\gamma} e^k_{\delta)} + C_{\mu\beta\gamma} H_{\mu)\alpha\delta} + C_{\alpha\mu\gamma} H_{\mu)\beta\delta} + C_{\alpha\beta\mu} H_{\mu)\gamma\delta}.$$

The explicit form of $C_{\alpha\beta\gamma,\delta}$ is obtained as follows:

$$\begin{aligned} C_{222,\delta} &= (\delta_k C_{222}) e^k_{\delta)} + 3C_{\mu 22} H_{\mu)2\delta} \\ &= (\delta_k A) e^k_{\delta)} + 3C_{322} H_{3)2\delta} + 3C_{422} H_{4)2\delta} \\ &= A_{,\delta} - 3Dh_\delta + 3F j_\delta; \end{aligned} \tag{4.2a}$$

where $A_{,\delta} = (\delta_k A) e^k_{\delta)}$.

Remark. As we put $C_{222} = A$, we should notice the difference between $A_{,\delta}$ and $C_{222,\delta}$.

Similarly, we get

$$C_{233,\delta} = B_{,\delta} + (2D - E) h_\delta + G j_\delta - 2H k_\delta, \tag{4.2b}$$
$$C_{244,\delta} = C_{,\delta} + (D + E) h_\delta - (3F + G) j_\delta + 2H k_\delta, \tag{4.2c}$$
$$C_{322,\delta} = D_{,\delta} + (A - 2B) h_\delta + 2H j_\delta - F k_\delta, \tag{4.2d}$$
$$C_{333,\delta} = E_{,\delta} + 3B h_\delta - 3G k_\delta, \tag{4.2e}$$
$$C_{422,\delta} = F_{,\delta} - 2H h_\delta - (A - 2C) j_\delta + D k_\delta, \tag{4.2f}$$
$$C_{433,\delta} = G_{,\delta} + 2H h_\delta - B j_\delta + (2D + 3E) k_\delta, \tag{4.2g}$$
$$C_{234,\delta} = H_{,\delta} + (F - G) h_\delta - (2D + E) j_\delta + (B - C) k_\delta, \tag{4.2h}$$
$$C_{344,\delta} = -D_{,\delta} - E_{,\delta} + C h_\delta - 2H j_\delta + (F + 3G) k_\delta, \tag{4.2i}$$
$$C_{444,\delta} = -F_{,\delta} - G_{,\delta} - 3C j_\delta - (3D + 3E) k_\delta, \tag{4.2j}$$
$$C_{1\beta\gamma,\delta} = 0. \tag{4.2k}$$

Adding (4.2d), (4.2e), (4.2i) and using (2.2), we get

$$C_{322,\delta} + C_{333,\delta} + C_{344,\delta} = (A + B + C) h_\delta = L\mathbf{C} h_\delta = W h_\delta. \tag{4.3}$$

Adding (4.2f), (4.2g), (4.2j) and using (2.2), we get

$$C_{422,\delta}+C_{433,\delta}+C_{444,\delta}=-(A+B+C)\,j_\delta=-L\boldsymbol{C}\,j_\delta=-W\,j_\delta. \tag{4.4}$$

Adding (4.2a), (4.2b), (4.2c) and using (2.2), we get

$$C_{222,\delta}+C_{233,\delta}+C_{244,\delta}=A_{,\delta}+B_{,\delta}+C_{,\delta}=(A+B+C)_{,\delta}=W_{,\delta}. \tag{4.5}$$

Thus, from (4.2), (4.3), (4.4) and (4.5), we have

Theorem 4.1. *In a four dimensional Berwald space, the h-connection vectors h_i and j_i vanish identically. Also main scalar A and the unified main scalar $W = L\boldsymbol{C}$ are h-covariant constants. Furthermore, if h-connection vector k_i vanishes then all the main scalars are h-covariant constants.*

5. Ricci identities

Now, we are concerned with the tensors $e^i_{\alpha|j|k}$, $e^i_{\alpha)|j}|_k$ and $e^i_{\alpha)}|_{j|k}$. From (1.2), we have

$$e^i_{\alpha)|j|k}=H_{\alpha)\beta\gamma,\delta}\,e^i_{\beta)}\,e_{\gamma)j}\,e_{\delta)k}, \tag{5.1}$$

$$Le^i_{\alpha)|j}|_k=H_{\alpha)\beta\gamma;\delta}\,e^i_{\beta)}\,e_{\gamma)j}\,e_{\delta)k}, \tag{5.2}$$

$$Le^i_{\alpha)}|_{j|k}=V_{\alpha)\beta\gamma,\delta}\,e^i_{\beta)}\,e_{\gamma)j}\,e_{\delta)k}. \tag{5.3}$$

According to the formulae (3.3) and (3.4), $H_{\alpha)\beta\gamma,\delta}$, $H_{\alpha)\beta\gamma;\delta}$, $V_{\alpha)\beta\gamma,\delta}$ are given by

$$\begin{aligned}
H_{\alpha)\beta\gamma,\delta} &= (\delta_k H_{\alpha)\beta\gamma})e^k_{\delta)}+H_{\alpha)\mu\gamma}H_{\mu)\beta\delta}+H_{\alpha)\beta\mu}H_{\mu)\gamma\delta},\\
H_{\alpha)\beta\gamma;\delta} &= L(\dot{\partial}_k H_{\alpha)\beta\gamma})e^k_{\delta)}+H_{\alpha)\mu\gamma}V_{\mu)\beta\delta}+H_{\alpha)\beta\mu}V_{\mu)\gamma\delta},\\
V_{\alpha)\beta\gamma,\delta} &= (\delta_k V_{\alpha)\beta\gamma})e^k_{\delta)}+V_{\alpha)\mu\gamma}H_{\mu)\beta\delta}+V_{\alpha)\beta\mu}H_{\mu)\gamma\delta}.
\end{aligned}$$

The explicit forms of these are obtained as follows:

$$\begin{aligned}
H_{2)3\gamma,\delta} &= (\delta_k H_{2)3\gamma})e^k_{\delta)}+H_{2)\mu\gamma}H_{\mu)3\delta}+H_{2)3\mu}H_{\mu)\gamma\delta}\\
&= (\delta_k h_\gamma)e^k_{\delta)}+H_{2)4\gamma}H_{4)3\delta}+h_\mu H_{\mu)\gamma\delta}\\
&= h_{\gamma,\delta}+j_\gamma k_\delta;
\end{aligned}$$

where $h_{\gamma,\delta}=(\delta_k h_\gamma)e^k_{\delta)}+h_\mu H_{\mu)\gamma\delta}$.
Similarly, we get

$$\begin{aligned}
H_{4)2\gamma,\delta} &= j_{\gamma,\delta}+k_\gamma h_\delta,\\
H_{3)4\gamma,\delta} &= k_{\gamma,\delta}+h_\gamma j_\delta,\\
H_{2)3\gamma;\delta} &= h_{\gamma;\delta}+j_\gamma w_\delta,\\
H_{4)2\gamma;\delta} &= j_{\gamma;\delta}+k_\gamma u_\delta,\\
H_{3)4\gamma;\delta} &= k_{\gamma;\delta}+h_\gamma v_\delta,
\end{aligned}$$

and

$$V_{2)3\gamma,\delta} = u_{\gamma,\delta} + v_\gamma k_\delta,$$
$$V_{4)2\gamma,\delta} = v_{\gamma,\delta} + w_\gamma h_\delta,$$
$$V_{3)4\gamma,\delta} = w_{\gamma,\delta} + u_\gamma j_\delta.$$

In terms of scalar components, the Ricci identity

$$e^i_{\alpha)|j}|_k - e^i_{\alpha)}|_{k|j} = e^r_{\alpha)} P^i_{rjk} - e^i_{\alpha)|r} C^r_{jk} - e^i_{\alpha)}|_r C^r_{jk|0}, \tag{5.4}$$

is expressed as

$$H_{\alpha)\beta\gamma;\delta} - V_{\alpha)\beta\delta,\gamma} = P_{\alpha\beta\gamma\delta} - H_{\alpha)\beta\mu} C_{\mu\gamma\delta} - V_{\alpha)\beta\mu} P_{\mu\gamma\delta}. \tag{5.5}$$

For Berwald space $P_{hijk} = 0$, therefore (5.5) becomes

$$H_{\alpha)\beta\gamma;\delta} - V_{\alpha)\beta\delta,\gamma} = -H_{\alpha)\beta\mu} C_{\mu\gamma\delta};$$

which is explicitly written as

$$(h_{\gamma;\delta} + j_\gamma w_\delta) - (u_{\delta,\gamma} + v_\delta k_\gamma) = -h_\mu C_{\mu\gamma\delta},$$
$$(j_{\gamma;\delta} + k_\gamma u_\delta) - (v_{\delta,\gamma} + w_\delta h_\gamma) = -j_\mu C_{\mu\gamma\delta},$$
$$(k_{\gamma;\delta} + h_\gamma v_\delta) - (w_{\delta,\gamma} + u_\delta j_\gamma) = -k_\mu C_{\mu\gamma\delta}.$$

From Theorem 4.1, we see that in a Berwald space $h_i = j_i = 0$. If we take $k_i = 0$, then above equations become $u_{\delta,\gamma} = v_{\delta,\gamma} = w_{\delta,\gamma} = 0$.

Thus we have:

Theorem 5.1. *In a four-dimensional Berwald space with vanishing h-connection vector k_i, the v-connection vectors u_i, v_i, w_i are h-covariant constants.*

From the Ricci identity

$$T^i_{j|k|h} - T^i_{j|h|k} = T^r_j R^i_{rkh} - T^i_r R^r_{jkh} - T^i_j|_r R^r_{kh}, \tag{5.6}$$

we have

$$e^i_{\alpha)|j|k} - e^i_{\alpha)|k|j} = e^r_{\alpha)} R^i_{rjk} - e^i_{\alpha)}|_r R^r_{jk}; \tag{5.7}$$

which is expressed as

$$H_{\alpha)\beta\gamma,\delta} - H_{\alpha)\beta\delta,\gamma} = R_{\alpha\beta\gamma\delta} - V_{\alpha)\beta\pi} R_{1\pi\gamma\delta}. \tag{5.8}$$

Now, we propose:

Proposition 5.1. *Let T_{ij} be a skew-symmetric tensor of a four-dimensional Finsler space. If we put ${}^*T^{ij} = \frac{1}{4} \epsilon^{ijkl}\, T_{kl}$, then we obtain $T_{pq} = \epsilon_{pqij}\, {}^*T^{ij}$.*

Proof. ${}^*T^{ij} = \frac{1}{4} \epsilon^{ijkl}\, T_{kl}$ implies

$${}^*T^{ij}\, \epsilon_{pqij} = \frac{1}{4} \epsilon^{ijkl} \epsilon_{pqij}\, T_{kl} = \frac{1}{4} \delta^{ijkl}_{pqij} T_{kl} = T_{pq}.$$

This completes the proof.

Since R_{hijk} is skew-symmetric in h and i as well as in j and k, in view of Proposition 5.1, R_{hijk} may be written as

$$R_{hijk} = \epsilon_{hirs}\epsilon_{jkpq}{}^{*}R^{rspq}, \tag{5.9}$$

where we put

$${}^{*}R^{rspq} = \frac{1}{16}\epsilon^{rshi}\epsilon^{pqlm} R_{hilm}. \tag{5.10}$$

The scalar components $R_{\alpha\beta\gamma\delta}$ of R_{hijk} are written as

$$R_{\alpha\beta\gamma\delta} = \gamma_{\alpha\beta\mu\lambda}\gamma_{\gamma\delta\theta\tau}{}^{*}R^{\mu\lambda\theta\tau} \tag{5.11}$$

in terms of scalar components ${}^{*}R_{\mu\lambda\theta\tau}$ of ${}^{*}R^{rspq}$.

The scalar components $R_{\beta\gamma\delta}$ of $\dfrac{1}{L}R_{ijk}$ are given by

$$R_{\beta\gamma\delta} = \gamma_{1\beta\mu\lambda}\gamma_{\gamma\delta\theta\tau}{}^{*}R_{\mu\lambda\theta\tau}. \tag{5.12}$$

Therefore (5.8) may be written as

$$H_{\alpha)\beta\gamma,\delta} - H_{\alpha)\beta\delta,\gamma} = (\gamma_{\alpha\beta\mu\lambda} - V_{\alpha)\beta\pi}\gamma_{1\pi\mu\lambda})\gamma_{\gamma\delta\theta\tau}{}^{*}R_{\mu\lambda\theta\tau}.$$

For different values of α, β this gives only three equations:

$$\begin{aligned}
(h_{\gamma,\delta} + j_\gamma k_\delta) - (h_{\delta,\gamma} + j_\delta k_\gamma) &= (\delta^{14}_{\mu\lambda} - u_2\delta^{34}_{\mu\lambda} - u_3\delta^{42}_{\mu\lambda} - u_4\delta^{23}_{\mu\lambda})\gamma_{\gamma\delta\theta\tau}{}^{*}R_{\mu\lambda\theta\tau},\\
(j_{\gamma,\delta} + k_\gamma h_\delta) - (j_{\delta,\gamma} + k_\delta h_\gamma) &= (\delta^{13}_{\mu\lambda} - v_2\delta^{34}_{\mu\lambda} - v_3\delta^{42}_{\mu\lambda} - v_4\delta^{23}_{\mu\lambda})\gamma_{\gamma\delta\theta\tau}{}^{*}R_{\mu\lambda\theta\tau},\\
(k_{\gamma,\delta} + h_\gamma j_\delta) - (k_{\delta,\gamma} + h_\delta j_\gamma) &= (\delta^{12}_{\mu\lambda} - w_2\delta^{34}_{\mu\lambda} - w_3\delta^{42}_{\mu\lambda} - w_4\delta^{23}_{\mu\lambda})\gamma_{\gamma\delta\theta\tau}{}^{*}R_{\mu\lambda\theta\tau}.
\end{aligned}$$

For Berwald space with $k_i = 0$, above equations become

$$\begin{cases}
(\delta^{14}_{\mu\lambda} - u_2\delta^{34}_{\mu\lambda} - u_3\delta^{42}_{\mu\lambda} - u_4\delta^{23}_{\mu\lambda})^{*}R_{\mu\lambda\theta\tau} = 0,\\
(\delta^{13}_{\mu\lambda} - v_2\delta^{34}_{\mu\lambda} - v_3\delta^{42}_{\mu\lambda} - v_4\delta^{23}_{\mu\lambda})^{*}R_{\mu\lambda\theta\tau} = 0,\\
(\delta^{12}_{\mu\lambda} - w_2\delta^{34}_{\mu\lambda} - w_3\delta^{42}_{\mu\lambda} - w_4\delta^{23}_{\mu\lambda})^{*}R_{\mu\lambda\theta\tau} = 0.
\end{cases} \tag{5.13}$$

Now applying the Ricci identity (5.6) to v-connection vectors $v^{(p)}_i$, we have

$$v^{(p)}_{i|j|k} - v^{(p)}_{i|k|j} = -v^{(p)}_r R^r_{ijk} - v^{(p)}_i|_r R^r_{jk}; \tag{5.14}$$

where $(v^{(1)}_i, v^{(2)}_i, v^{(3)}_i) = (u_i, v_i, w_i)$.

In terms of scalars, (5.14) may be written as:

$$v^{(p)}_{\beta,\gamma,\delta} - v^{(p)}_{\beta,\delta,\gamma} = -(v^{(p)}_\pi\gamma_{\beta\pi\mu\lambda} + v^{(p)}_{\beta;\pi}\gamma_{1\pi\mu\lambda})\gamma_{\gamma\delta\theta\tau}{}^{*}R_{\mu\lambda\theta\tau}.$$

We have shown in Theorem 5.1 that in a Berwald space with $k_i = 0$, the v-connection vectors are h-covariant constants, therefore above equation becomes

$$(v_{\pi}^{(p)}\gamma_{\beta\pi\mu\lambda} + v_{\beta;\pi}^{(p)}\gamma_{1\pi\mu\lambda})^{*}R_{\mu\lambda\theta\tau} = 0. \tag{5.15}$$

Because of $v_{1;\pi}^{(p)} = -v_{\pi}^{(p)}$, the above is trivial for $\beta = 1$ and thus from the above we obtain only

$$\left(v_3^{(p)}\delta_{\mu\lambda}^{14} + v_4^{(p)}\delta_{\mu\lambda}^{31} + v_{2;2}^{(p)}\delta_{\mu\lambda}^{34} + v_{2;3}^{(p)}\delta_{\mu\lambda}^{42} + v_{2;4}^{(p)}\delta_{\mu\lambda}^{23}\right)^{*}R_{\mu\lambda\theta\tau} = 0,$$
$$\left(v_2^{(p)}\delta_{\mu\lambda}^{41} + v_4^{(p)}\delta_{\mu\lambda}^{12} + v_{3;2}^{(p)}\delta_{\mu\lambda}^{34} + v_{3;3}^{(p)}\delta_{\mu\lambda}^{42} + v_{3;4}^{(p)}\delta_{\mu\lambda}^{23}\right)^{*}R_{\mu\lambda\theta\tau} = 0,$$
$$\left(v_2^{(p)}\delta_{\mu\lambda}^{13} + v_3^{(p)}\delta_{\mu\lambda}^{21} + v_{4;2}^{(p)}\delta_{\mu\lambda}^{34} + v_{4;3}^{(p)}\delta_{\mu\lambda}^{42} + v_{4;4}^{(p)}\delta_{\mu\lambda}^{23}\right)^{*}R_{\mu\lambda\theta\tau} = 0.$$

In view of (5.13), these equations take the forms:

$$\begin{cases} \left\{(v_{2;2}^{(p)} + v_3^{(p)}u_2 - v_4^{(p)}v_2)\delta_{\mu\lambda}^{34} + (v_{2;3}^{(p)} + v_3^{(p)}u_3 - v_4^{(p)}v_3)\delta_{\mu\lambda}^{42} \right. \\ \qquad \left. + (v_{2;4}^{(p)} + v_3^{(p)}u_4 - v_4^{(p)}v_4)\delta_{\mu\lambda}^{23}\right\}^{*}R_{\mu\lambda\theta\tau} = 0, \\ \left\{(v_{3;2}^{(p)} - v_2^{(p)}u_2 + v_4^{(p)}w_2)\delta_{\mu\lambda}^{34} + (v_{3;3}^{(p)} - v_2^{(p)}u_3 + v_4^{(p)}w_3)\delta_{\mu\lambda}^{42} \right. \\ \qquad \left. + (v_{3;4}^{(p)} - v_2^{(p)}u_4 + v_4^{(p)}w_4)\delta_{\mu\lambda}^{23}\right\}^{*}R_{\mu\lambda\theta\tau} = 0, \\ \left\{(v_{4;2}^{(p)} + v_2^{(p)}v_2 - v_3^{(p)}w_2)\delta_{\mu\lambda}^{34} + (v_{4;3}^{(p)} + v_2^{(p)}v_3 - v_3^{(p)}w_3)\delta_{\mu\lambda}^{42} \right. \\ \qquad \left. + (v_{4;4}^{(p)} + v_2^{(p)}v_4 - v_3^{(p)}w_4)\delta_{\mu\lambda}^{23}\right\}^{*}R_{\mu\lambda\theta\tau} = 0. \end{cases} \tag{5.16}$$

Put $v_{\alpha\beta}^{(p)} = v_{\alpha;\beta}^{(p)} + v_{\mu}^{(p)}V_{\alpha)\mu\beta}$, then equations (5.16) become

$$\begin{cases} \left(v_{22}^{(p)}\delta_{\mu\lambda}^{34} + v_{23}^{(p)}\delta_{\mu\lambda}^{42} + v_{24}^{(p)}\delta_{\mu\lambda}^{23}\right)^{*}R_{\mu\lambda\theta\tau} = 0, \\ \left(v_{32}^{(p)}\delta_{\mu\lambda}^{34} + v_{33}^{(p)}\delta_{\mu\lambda}^{42} + v_{34}^{(p)}\delta_{\mu\lambda}^{23}\right)^{*}R_{\mu\lambda\theta\tau} = 0, \\ \left(v_{42}^{(p)}\delta_{\mu\lambda}^{34} + v_{43}^{(p)}\delta_{\mu\lambda}^{42} + v_{44}^{(p)}\delta_{\mu\lambda}^{23}\right)^{*}R_{\mu\lambda\theta\tau} = 0. \end{cases} \tag{5.17}$$

Again, applying the Ricci identity (5.6) to the main scalars $A^{(q)}$, we have

$$A_{|j|k}^{(q)} - A_{|k|j}^{(q)} = -A^{(q)}|_r R_{jk}^r; \tag{5.18}$$

where $(A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}, A^{(5)}, A^{(6)}, A^{(7)}, A^{(8)}) = (A, B, C, D, E, F, G, H)$.

In terms of scalars, (5.18) assumes the form:

$$A_{,\gamma,\delta}^{(q)} - A_{,\delta,\gamma}^{(q)} = -A_{;\pi}^{(q)}\gamma_{1\pi\mu\lambda}\gamma_{\gamma\delta\theta\tau}{}^{*}R_{\mu\lambda\theta\tau} = 0.$$

We have seen in Theorem 4.1 that all the main scalars are h-covariant constants in a Berwald space with $k_i = 0$. Therefore above equation becomes

$$(A_{;2}^{(q)}\delta_{\mu\lambda}^{34} + A_{;3}^{(q)}\delta_{\mu\lambda}^{42} + A_{;4}^{(q)}\delta_{\mu\lambda}^{23})^{*}R_{\mu\lambda\theta\tau} = 0. \tag{5.19}$$

We now discuss Berwald space with vanishing h-connection vectors, considering the rank ρ of the matrix $({}^{*}R_{\mu\lambda\theta\tau})$, where $(\mu\lambda)$ and $(\theta\tau)$ show the number of rows and columns respectively. From (5.13), it is clear that the rank ρ is less than four,

(i) if $\rho = 0$ then ${}^*R_{\mu\lambda\theta\tau} = 0$. This means ${}^*R_{hijk} = 0$ and therefore the space is locally Minkowskian.

(ii) if $\rho = 1$, then from (5.17) ad (5.19), we have

$$A^{(q)}_{;2} : A^{(q)}_{;3} : A^{(q)}_{;4} = v^{(p)}_{22} : v^{(p)}_{23} : v^{(p)}_{24} = v^{(p)}_{32} : v^{(p)}_{33} : v^{(p)}_{34} = v^{(p)}_{42} : v^{(p)}_{43} : v^{(p)}_{44} \quad (p = 1,2,3;\ q = 1,2,\dots,8) \tag{5.20}$$

(iii) if $\rho = 2$, then from (5.17), $\begin{vmatrix} v^{(p)}_{22} & v^{(p)}_{23} & v^{(p)}_{24} \\ v^{(p)}_{32} & v^{(p)}_{33} & v^{(p)}_{34} \\ v^{(p)}_{42} & v^{(p)}_{43} & v^{(p)}_{44} \end{vmatrix} = 0$ such that conditions (5.20) do not hold.

(iv) if $\rho = 3$, then from (5.17) and (5.19), $v^{(p)}_{\alpha\beta} = 0$; $\alpha, \beta = 2,3,4$ and $A^{(q)}_{;2} = A^{(q)}_{;3} = A^{(q)}_{;4} = 0$ so that all the main scalars are v-covariant constants and therefore they are constants.

Summarizing the above, we conclude:

Theorem 5.2. *In a four-dimensional Berwald space with vanishing h-connection vector k_i, the rank ρ of the matrix (R_{hijk}), where (hi) and (jk) show the number of rows and columns respectively, is less than four. Further*

(i) *if $\rho = 0$, the space is locally Minkowskian.*

(ii) *if $\rho = 1$, we have the conditions* (5.20).

(iii) *if $\rho = 2$,* $\begin{vmatrix} v^{(p)}_{22} & v^{(p)}_{23} & v^{(p)}_{24} \\ v^{(p)}_{32} & v^{(p)}_{33} & v^{(p)}_{34} \\ v^{(p)}_{42} & v^{(p)}_{43} & v^{(p)}_{44} \end{vmatrix} = 0$ *and conditions* (5.20) *do not hold.*

(iv) *if $\rho = 3$, all the main scalars are constants and $v^{(p)}_{\alpha\beta} = 0$,* $(p = 1,2,3;\ \alpha, \beta = 2,3,4)$.

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