ANTI-INTEGRAL EXTENSIONS \( R[\alpha]/R \) 
AND INVERTIBILITY OF \( \alpha^n - a \)

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Abstract. Let \( R \) be an integral domain and \( \alpha \) an anti-integral element of degree \( d \) over \( R \). In the paper [3] we give a condition for \( \alpha^2 - a \) to be a unit of \( R[\alpha] \). In this paper we will generalize the result to an arbitrary positive integer \( n \) and give a condition, in terms of the ideal \( I[\alpha]D(\sqrt[n]{\eta}) \) of \( R \), for \( \alpha^n - a \) to be a unit of \( R[\alpha] \).

1. Conditions of Invertibility of \( \alpha^n - a \)

Let \( R \) be an integral domain with quotient field \( K \) and \( R[X] \) a polynomial ring over \( R \) in an indeterminate \( X \). Let \( \alpha \) be an element of an algebraic field extension of \( K \) and \( \pi : R[X] \rightarrow R[\alpha] \) the \( R \)-algebra homomorphism defined by \( \pi(X) = \alpha \). Let \( \varphi_{\alpha}(X) \) be the minimal polynomial of \( \alpha \) over \( K \) with \( \deg \varphi_{\alpha}(X) = d \) and write \( \varphi_{\alpha}(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d, (\eta_1, \ldots, \eta_d \in K) \). We will define \( I[\alpha] = \bigcap_{i=1}^{d} (R:R \eta_i) \) and \( J[\alpha] = I[\alpha](1, \eta_1, \ldots, \eta_d) \) where \( (R:R \eta_i) = \{ c \in R; c \eta_i \in R \} \) and \( (1, \eta_1, \ldots, \eta_d) \) is an \( R \)-module generated by \( 1, \eta_1, \ldots, \eta_d \). An element \( \alpha \) is called an anti-integral element of degree \( d \) over \( R \) if

\[
\text{Ker} \pi = I[\alpha] \varphi_{\alpha}(X) R[X].
\]

We say that the extension \( R[\alpha]/R \) is an anti-integral extension if \( \alpha \) is an anti-integral element of degree \( d \) over \( R \).

Our notation is standard and our general reference for unexplained terms is [5].

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We will frequently use the following lemma throughout this paper:

**Lemma 1.1.** Let \( R \subset S \) be a flat extension of integral domains. Let \( \alpha \) be an element of an algebraic field extension of the quotient field of \( S \).

1. Set \( I_{S, \alpha} = \bigcap_{i=1}^{d} (S : S \eta_i) \) and \( J_{S, \alpha} = I_{S, \alpha}(1, \eta_1, \ldots, \eta_d) \) where \( (S : S \eta_i) = \{ b \in S; b \eta_i \in S \} \) and \( (1, \eta_1, \ldots, \eta_d) \) is an \( S \)-module generated by \( 1, \eta_1, \ldots, \eta_d \). Then \( I_{S, \alpha} = I[\alpha]S \) and \( J_{S, \alpha} = J[\alpha]S \).

2. If \( \alpha \) is an anti-integral element of degree \( d \) over \( R \), then \( \alpha \) is also an anti-integral element of degree \( d \) over \( S \).

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Proof. (1) Since $S$ is a flat extension over $R$, we have

$$I_{S, [\alpha]} = \bigcap_{i=1}^{d}(S : \eta_i) = \bigcap_{i=1}^{d}(R : R \eta_i)S = (\bigcap_{i=1}^{d}(R : R \eta_i))S = I_{[\alpha]}S$$

and

$$J_{S, [\alpha]} = I_{S, [\alpha]}(1, \eta_1, \ldots, \eta_d) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)S = J_{[\alpha]}S.$$

(2) By tensoring $S$ to the following exact sequence:

$$0 \rightarrow I_{[\alpha]} \varphi_\alpha(X)R[X] \rightarrow R[X] \rightarrow R[\alpha] \rightarrow 0,$$

we have an exact sequence:

$$0 \rightarrow I_{[\alpha]} \varphi_\alpha(X)S[X] \rightarrow S[X] \rightarrow S[\alpha] \rightarrow 0,$$

By (1), we know that $I_{[\alpha]}S = I_{S, [\alpha]}$. Hence $\alpha$ is also an anti-integral element of degree $d$ over $S$.

We list some facts which will be used later for reference sake:

Lemma 1.2. ([10, Theorem 1] and [1, Theorem 4]) Let $R$ be an integral domain and $\alpha$ an algebraic element over the quotient field of $R$. Let $a$ be an element of $R$ such that $\alpha - a$ is not zero. Then the following hold:

1. $I_{[\alpha-a]-1} = I_{[\alpha]} \varphi_\alpha(a)$.
2. If $\alpha$ is an anti-integral element over $R$, then so is $(\alpha - a)^{-1}$.

Remark 1.3. If $\alpha - a = 0$, then $\varphi_\alpha(X) = X - a$, and so $I_{[\alpha]} \varphi_\alpha(a) = (0)$ and $J_{[\alpha]} = R$. Hence $I_{[\alpha]} \varphi_\alpha(a) \subseteq J_{[\alpha]}$. Especially, $I_{[\alpha]} \varphi_\alpha(a) \neq R$.

Lemma 1.4. ([6, Theorem 2.2] and [8, Lemma 8]) Let $R$ be an integral domain and $\gamma$ an anti-integral element over $R$. Then the following conditions are equivalent:

1. $\gamma$ is integral over $R$.
2. $I_{[\gamma]} = R$.

By making use of Lemmas 1.3 and 1.4 we have the following:

Lemma 1.5. (cf. [7, Theorem 6]) Let $R$ be an integral domain and $\alpha$ an anti-integral element over $R$. Let $a$ be an element of $R$. Then $\alpha - a$ is a unit of $R[\alpha]$ if and only if $I_{[\alpha]} \varphi_\alpha(a) = R$.

Proof. First we shall prove the ‘only if’ part. Since $\alpha - a$ is a unit of $R[\alpha]$, then there exist elements $f(X)$ of $R[X]$ and $g(X)$ of $I_{[\alpha]}R[X]$ such that $f(X)(X - a) - 1 = g(X) \varphi_\alpha(X)$. Hence $g(a)\varphi_\alpha(a) = -1$. This means that $I_{[\alpha]} \varphi_\alpha(a) = R$.

Next we shall prove the ‘if’ part. By Remark 1.3, $\alpha - a$ is not zero. Therefore we get $I_{[\alpha-a]-1} = R$ by Lemma 1.2 (1). Then Lemma 1.4 asserts that $(\alpha - a)^{-1}$ is integral over $R$. Therefore there exist elements $c_1, \ldots, c_n$ of $R$ such that

$$(\alpha - a)^{-1} + c_1((\alpha - a)^{-1})^{n-1} + \cdots + c_n = 0.$$
Hence $1 = -(c_1 + \cdots + c_n(\alpha - a)^{n-1})(\alpha - a)$. This shows that $\alpha - a$ is a unit of $R[\alpha]$.

Let $n$ be a positive integer and $a$ an element of $R$. Assume that the following three conditions hold:

1. $\alpha$ is an anti-integral element of degree $d$ over $R$.
2. $[K(\sqrt[n]{a}) : K] = n$.
3. $[K(\sqrt[n]{a})/K(\sqrt[n]{a}(\alpha)) : K(\sqrt[n]{a})] = d$.

Set $B = R[\sqrt[n]{a}]$. By the condition (3), the minimal polynomial of $\alpha$ over $K(\sqrt[n]{a})$ coincides with $\varphi_{\alpha}(X)$. The condition (2) implies that $B$ is a free $R$-module of rank $n$. Hence $B$ is a flat extension over $R$. Therefore, by Lemma 1.1, $I_{B[\alpha]} = I_{B}(\alpha)B$, $J_{B[\alpha]} = J_{B}(\alpha)B$ and $\alpha$ is also an anti-integral element of degree $d$ over $B$.

We give a condition for the element $\alpha^n - a$ to be a unit of $R[\alpha]$.

Theorem 1.6. Let $R$ be an integral domain with quotient field $K$. Let $n$ be a positive integer and $\omega$ a primitive $n$-th root of unity. Let $a$ be an element of $R$. Assume that the following four conditions hold:

1. $\omega \in R$.
2. $\alpha$ is an anti-integral element of degree $d$ over $R$.
3. $[K(\sqrt[n]{a}) : K] = n$.
4. $[K(\sqrt[n]{a})(\alpha) : K(\sqrt[n]{a})] = d$.

Set $A = R[\alpha]$ and $B = R[\sqrt[n]{a}]$. Then the following conditions are equivalent to each other:

(i) $\alpha^n - a$ is a unit of $A$.
(ii) $I_{B[\alpha]}\varphi_{\alpha}(\sqrt[n]{a}\omega^k)B = B$ for $k = 0, \ldots, n - 1$.

Proof. (i) $\Rightarrow$ (ii). Since

$\alpha^n - a = (\alpha - \sqrt[n]{a})(\alpha - \sqrt[n]{a}\omega)\cdots(\alpha - \sqrt[n]{a}\omega^{n-1}),$

we see that $\alpha - \sqrt[n]{a}\omega^k$ is a unit of $B[\alpha]$ for $k = 0, 1, \ldots, n - 1$. Hence by Lemma 1.5 we get

$I_{B[\alpha]}\varphi_{\alpha}(\sqrt[n]{a}\omega^k)B = B$

for $k = 0, 1, \ldots, n - 1$.

(ii) $\Rightarrow$ (i). By Lemma 1.5 we see that

$\alpha - \sqrt[n]{a}\omega^k$

is a unit of $B[\alpha]$ for $k = 0, 1, \ldots, n - 1$. Hence $\alpha^n - a$ is also a unit of $B[\alpha]$. Since $\alpha^n - a$ is an element of $A$ and $B[\alpha]$ is an integral extension of $A$, we know that $\alpha^n - a$ is a unit of $A$.

Recall that $\varphi_{\alpha}(X) = X^d + \eta_1X^{d-1} + \cdots + \eta_d$. Let $s$ be an integer such that $0 \leq s \leq n - 1$. Then we define

$\varphi_{\alpha,s}(X) = (\sum \eta_i X^{d-i})X^{-s}$
where $\eta_0 = 1$ and the sum is taken over $i$ such that $0 \leq i \leq d$ and that the remainder of $d - i$ divided by $n$ is $s$. Then it is easily verified:

$$\varphi_\alpha(X) = \sum_{s=0}^{n-1} X^s \varphi_{\alpha,s}(X)$$

and

$$\varphi_{\alpha,s}(X) \in R[\eta_1, \ldots, \eta_d][X^n].$$

Set $\beta_s = (\sqrt[n]{a})^{s} \varphi_{\alpha,s}(\sqrt[n]{a})$ for $s = 0, 1, \ldots, n-1$. The cyclic determinant

$$\begin{vmatrix}
\beta_0 & \beta_1 & \cdots & \beta_{n-1} \\
\beta_{n-1} & \beta_0 & \cdots & \beta_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_1 & \beta_2 & \cdots & \beta_0
\end{vmatrix}$$

is denoted by $D(\sqrt[n]{a})$. Then

$$D(\sqrt[n]{a}) = \prod_{k=0}^{n-1} \left( \sum_{s=0}^{n-1} (\sqrt[n]{a} \omega^k)^s \varphi_{\alpha,s}(\sqrt[n]{a}) \right).$$

For cyclic determinants, see [9, p. 91].

**Example 1.7.** Let $R$ be an integral domain with quotient field $K$. Let $\alpha$ be an element of $K$. Let $a$ be an element of $R$ and $n$ a positive integer. Then $D(\sqrt[n]{a}) = (-1)^n (\alpha^n - a)$. Hence, if $I_{[a]} D(\sqrt[n]{a}) = R$, then $\alpha^n - a$ is a unit of $R[a]$.

**Proof.** Note that $\varphi_\alpha(X) = X - \alpha$. By the definition of $\varphi_{\alpha,s}(X)$, we get $\varphi_{\alpha,0}(X) = 1$, $\varphi_{\alpha,1}(X) = \cdots = \varphi_{\alpha,n-1}(X) = 0$. Hence $\beta_0 = -\alpha$, $\beta_1 = \sqrt[n]{a}$, $\beta_2 = \cdots = \beta_{n-1} = 0$. By expanding the first column of the determinant $D(\sqrt[n]{a})$, we have $D(\sqrt[n]{a}) = (-1)^n (\alpha^n - a)$.

**Lemma 1.8.** For every term $\pm \beta_{i_1} \beta_{i_2} \cdots \beta_{i_n}$ of the cyclic determinant (1), the following equality holds:

$$i_1 + i_2 + \cdots + i_n \equiv 0 \pmod{n}.$$

**Proof.** Let $|a_{i,j}|$ be the cyclic determinant defined by (1). Then there exists a permutation $\sigma$ such that

$$\beta_{i_1} \beta_{i_2} \cdots \beta_{i_n} = a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

Then

$$i_1 \equiv \sigma(1) - 1 \pmod{n}, \ldots, i_n \equiv \sigma(n) - n \pmod{n}.$$
Therefore

\[ i_1 + i_2 + \cdots + i_n \equiv \sigma(1) + \sigma(2) + \cdots + \sigma(n) - (1 + 2 + \cdots + n) \mod n \]

\[ \equiv 0 \mod n. \]

**Lemma 1.9.** \( I_{\alpha}^n D(\sqrt[n]{a}) \) is an ideal of \( R \) and \( I_{\alpha}^n D(\sqrt[n]{a}) \subseteq J_{\alpha}^n. \)

**Proof.** First we shall show that \( I_{\alpha}^n D(\sqrt[n]{a}) \subseteq R. \) Let \( \pm \beta_1, \beta_2, \ldots, \beta_n \) be a term of the cyclic determinant \( D(\sqrt[n]{a}). \) By Lemma 1.8 there exists a non-negative integer \( q \) such that \( i_1 + i_2 + \cdots + i_n = nq. \) Then

\[
\beta_1 \beta_2 \cdots \beta_n = (\sqrt[n]{a})^q \varphi_{\alpha,i_1}(\sqrt[n]{a}) \cdots \varphi_{\alpha,i_n}(\sqrt[n]{a})
\]

By the definitions of \( I_{\alpha} \) and \( \varphi_{\alpha,i}(X), \) we have

\[ I_{\alpha}^n \beta_1 \beta_2 \cdots \beta_n \subseteq R. \]

Therefore \( I_{\alpha}^n D(\sqrt[n]{a}) \subseteq R. \)

It is easily verified that \( I_{\alpha}^n D(\sqrt[n]{a}) \) is an ideal of \( R. \) Furthermore,

\[ I_{\alpha}^n \beta_1 \beta_2 \cdots \beta_n \subseteq J_{\alpha}^n. \]

Hence \( I_{\alpha}^n D(\sqrt[n]{a}) \subseteq J_{\alpha}^n. \)

**Proposition 1.10.** Let \( R \) be an integral domain with quotient field \( K. \) Let \( n \) be a positive integer and \( a \) an element of \( R. \) Let \( \omega \) be a primitive \( n \)-th root of unity. Assume that the following six conditions hold:

1. \( \alpha \) is an anti-integral element of degree \( d \) over \( R. \)
2. \( R[\omega] \) is a flat extension over \( R. \)
3. \( [K(\sqrt[n]{a}) : K] = n. \)
4. \( [K(\sqrt[n]{a})(\alpha) : K(\sqrt[n]{a})] = d. \)
5. \( [K(\omega)(\sqrt[n]{a}) : K(\omega)] = n. \)
6. \( [K(\omega)(\sqrt[n]{a})(\alpha) : K(\omega)(\sqrt[n]{a})] = d. \)

Set \( A = R[\alpha]. \) If \( I_{\alpha}^n D(\sqrt[n]{a}) = R, \) then \( \alpha^n - a \) is a unit of \( A. \)

**Proof.** Set \( B = R[\sqrt[n]{a}]. \) First we will prove the case that \( \omega \) is in \( R. \) The condition \( I_{\alpha}^n D(\sqrt[n]{a}) = R \) implies that

\[ I_{\alpha}^n D(\sqrt[n]{a})B = B. \]

Hence

\[ I_{\alpha}^n \prod_{k=0}^{n-1} \left( \sum_{s=0}^{n-1} (\sqrt[n]{a} \omega^k)^s \varphi_{\alpha,i}(\sqrt[n]{a}) \right) B = B. \]
Since \( I_{[\alpha]}(\sum_{s=0}^{n-1} (\sqrt[n]{\alpha} \omega^k)^s \varphi_{\alpha,s}(\sqrt[n]{\alpha}))B \) is an ideal of \( B \), we get

\[
I_{[\alpha]} \left( \sum_{s=0}^{n-1} (\sqrt[n]{\alpha} \omega^k)^s \varphi_{\alpha,s}(\sqrt[n]{\alpha}) \right) B = B.
\]

Hence \( I_{[\alpha]}(\sqrt[n]{\alpha} \omega^k)B = B \) for \( k = 0, 1, \ldots, n-1 \). Theorem 1.6 asserts that \( \alpha^n - \alpha \) is a unit of \( A \).

Next we prove the general case. Set \( R' = R[\omega] \), \( A' = R'[\alpha] \). We have \( I_{[\alpha]}(\sqrt[n]{\alpha})D(\sqrt[n]{\alpha}) = R' \) because \( I_{[\alpha]} R' \subset I_{R'[\alpha]} \) and \( I_{[\alpha]} D(\sqrt[n]{\alpha}) = R \). Besides, \( \alpha \) is an anti-integral element of degree \( d \) over \( R' \) by Lemma 1.1 (2). Then the argument above implies that \( \alpha^n - \alpha \) is a unit of \( A' \). But \( \alpha^n - \alpha \in A \) and \( A' \) is integral over \( A \). Hence \( \alpha^n - \alpha \in A \).

Let \( n \) be a positive integer and \( \omega \) a primitive \( n \)-th root of unity. Set \( R' = R[\omega] \). The case that \( R' \) is not flat over \( R \) happens as the following example shows. Let \( Z \) be the ring of integers and \( Q \) the field of rational numbers.

**Example 1.11.** Set \( i = \sqrt{-1} \), \( R = Z[i, \sqrt{2}] \) and \( \omega = (1 + i)/\sqrt{2} \). Then \( \omega \) is a primitive 8-th root of unity. Let \( K \) be the quotient field of \( R \). Then \( K = Q(\sqrt{2}, i) \) and \( \omega \) is in \( K \). Hence \( \varphi_\omega(X) = X - \omega \). We have the following:

1. \( \omega \) is not in \( R \).
2. \( I_{[\omega]} = \{ p + q\sqrt{2} + ri + s\sqrt{2}i ; p, q, r, s \in Z \text{ and } p + r \in 2Z \} \).
3. \( R[\omega] \) is neither anti-integral nor flat over \( R \).

**Proof.** Note that \( R \) is a free \( Z \)-module with a basis \( \{ 1, \sqrt{2}, i, \sqrt{2}i \} \) and \( \{ 1, \sqrt{2}, i, \sqrt{2}i \} \) is linearly independent over \( Q \).

1. If \( \omega = \sqrt{2}/2 + \sqrt{2}i/2 \) is in \( R \), there exist elements \( p, q, r, s \) of \( Z \) such that \( \omega = p + q\sqrt{2} + ri + s\sqrt{2}i \). Then \( p = 0, q = 1/2, r = 0 \) and \( s = 1/2 \). This is a contradiction.

2. Set \( I_0 = \{ p + q\sqrt{2} + ri + s\sqrt{2}i ; p, q, r, s \in Z \text{ and } p + r \in 2Z \} \). Then it is easily proved that \( I_0 \) is an ideal of \( R \). Let \( x \) be an element of \( R \). It is easily seen that \( x\omega \) is in \( R \) if and only if \( x \) is in \( I_0 \). Hence \( I_{[\omega]} = I_0 \).

3. Assume that \( \omega \) is an anti-integral element over \( R \). Then \( I_{[\omega]} = R \) by Lemma 1.4 because \( \omega \) is integral over \( R \). The assertion (2) claims that \( 1 \) is not in \( I_{[\omega]} \). This is a contradiction. Hence \( \omega \) is not an anti-integral element over \( R \).

Next we will assume that \( R[\omega] \) is flat over \( R \). Since \( R[\omega] \) is integral over \( R \) and \( 1 \) is not in \( I_{[\omega]} \), we have \( I_{[\omega]} R[\omega] \neq R[\omega] \). On the other hand, by Lemma 1.1 (1), we see that \( I_{[\omega]} R[\omega] = I_{R[\omega],[\omega]} \). Since \( \omega \) is in \( R[\omega] \), we get \( I_{R[\omega],[\omega]} = R[\omega] \). This is absurd. Therefore \( R[\omega] \) is not flat over \( R \).

We can’t delete the assumption (5) of Proposition 1.10. To show an example for it, we need the following lemmas:

**Lemma 1.12.** ([4, Theorem 49]) Let \( K \) be a field and \( a \) an element of \( K \). Let \( m \) and \( n \) be relatively prime positive integers. Then \( X^{mn} - a \) is irreducible over \( K \) if and only if both \( X^m - a \) and \( X^n - a \) are irreducible over \( K \).
Lemma 1.13. ([4, Theorem 51]) Let $K$ be a field and $a$ an element of $K$. Let $p$ be a prime number and $n$ a positive integer. Assume that no $p$-th root of $a$ is in $K$. Then:

1. If $p$ is odd, then $X^p - a$ is irreducible over $K$.
2. If $p = 2$ and the characteristic of $K$ is 2, then $X^{2^n} - a$ is irreducible over $K$.
3. If $p = 2$, $n \geq 2$ and the characteristic of $K$ is not 2, then $X^{2^n} - a$ is irreducible over $K$ if and only if $-4a$ is not a fourth power in $K$.

Let $R$ be an integral domain with quotient field $K$. Let $n$ be a positive integer and $\omega$ a primitive $n$-th root of unity. Then $[K(\sqrt[n]{a}) : K] = n$ does not imply $[K(\omega)(\sqrt[n]{a}) : K(\omega)] = n$ as the following example shows:

Example 1.14. Set $n = 12$, $K = \mathbb{Q}$ and $a = -\frac{9}{2}$. Let $\omega$ be a primitive 12-th root of unity. Then the following two assertions hold:

1. $X^{12} + 9/4$ is irreducible over $\mathbb{Q}$ and $[K(\sqrt[12]{a}) : K] = n$
2. $X^{12} + 9/4$ is reducible over $\mathbb{Q}(\omega)$ and $[K(\omega)(\sqrt[12]{a}) : K(\omega)] < n$.

Proof. (1) By Lemma 1.12, we have only to prove that both $X^3 + 9/4$ and $X^4 + 9/4$ are irreducible over $\mathbb{Q}$. Lemma 1.13 (1) implies that $X^3 + 9/4$ is irreducible over $\mathbb{Q}$. Since $-4 \times \frac{9}{4} = -9$ is not a fourth power in $\mathbb{Q}$, we see that $X^4 + 9/4$ is irreducible over $\mathbb{Q}$ by Lemma 1.13 (3).

2. Set $f = 1 - 2\omega^2$. Then we have

$$X^{12} + \frac{9}{4} = \left(X^6 + fX^3 - \frac{3}{2}\right)\left(X^6 - fX^3 - \frac{3}{2}\right)$$

and $f$ is in $\mathbb{Q}(\omega)$. Hence $X^{12} + 9/4$ is reducible over $\mathbb{Q}(\omega)$ and $[K(\omega)(\sqrt[12]{a}) : K(\omega)] < n$.

Theorem 1.15. Let $R$ be an integral domain with quotient field $K$. Let $n$ be a positive integer and $a$ an element of $R$. Let $\omega$ be a primitive $n$-th root of unity. Assume that the following six conditions hold:

1. $\alpha$ is an anti-integral element of degree $d$ over $R$.
2. $R[\omega]$ is a flat extension over $R$.
3. $[K(\sqrt[n]{a}) : K] = n$.
4. $[K(\sqrt[n]{a})(\alpha) : K(\sqrt[n]{a})] = d$.
5. $[K(\omega)(\sqrt[n]{a}) : K(\omega)] = n$.
6. $[K(\omega)(\sqrt[n]{a})(\alpha) : K(\omega)(\sqrt[n]{a})] = d$.

Set $A = R[\alpha]$. Then the following conditions are equivalent to each other:

(i) $\alpha^n - a$ is a unit of $A$.
(ii) $I_{[\alpha]} D(\sqrt[n]{a}) = R$.

Proof. (ii) $\Rightarrow$ (i). It is clear from Proposition 1.10.

(i) $\Rightarrow$ (ii). (In this part of the proof, we don’t use the assumptions (5) and (6).) Set $B = R[\sqrt[n]{a}]$. First we will prove the case that $\omega$ is in $R$. By Theorem 1.6 we obtain

$I_{[\alpha]}(\sqrt[n]{a} \omega^k)B = B$
for $k = 0, 1, \ldots, n - 1$. Therefore

$$I_{[a]}^{n} \prod_{k=0}^{n-1} \varphi_{a}(\sqrt[n]{a} \cdot \omega^{k})B = B.$$ 

Note that

$$\varphi_{a}(\sqrt[n]{a} \cdot \omega^{k}) = \sum_{s=0}^{n-1} (\sqrt[n]{a} \cdot \omega^{k})^{s} \varphi_{a,s}(\sqrt[n]{a} \cdot \omega^{k})$$

$$= \sum_{s=0}^{n-1} (\sqrt[n]{a} \cdot \omega^{k})^{s} \varphi_{a,s}(\sqrt[n]{a}).$$

Hence

$$I_{[a]}^{n} \prod_{k=0}^{n-1} \left( \sum_{s=0}^{n-1} (\sqrt[n]{a} \cdot \omega^{k})^{s} \varphi_{a,s}(\sqrt[n]{a}) \right)B = B,$$

that is, $I_{[a]}^{n}D(\sqrt[n]{a})B = B$.

We will prove that $I_{[a]}^{n}D(\sqrt[n]{a}) = R$. By Lemma 1.9 we know that $I_{[a]}^{n}D(\sqrt[n]{a}) \subset R$, and $I_{[a]}^{n}D(\sqrt[n]{a})$ is an ideal of $R$. Suppose the contrary, i.e., $I_{[a]}^{n}D(\sqrt[n]{a}) \neq R$. Then there exists a prime ideal $p$ of Spec $R$ such that $I_{[a]}^{n}D(\sqrt[n]{a}) \subset p$. Since $B$ is integral over $R$, we can take a prime ideal $P$ of Spec $B$ such that $P \cap R = p$. Then $I_{[a]}^{n}D(\sqrt[n]{a})B \subset P$. This is a contradiction.

Next we prove the general case. Set $R' = R[\omega]$ and $A' = R'[\alpha]$. Then $\alpha^{n} - a$ is a unit of $A'$ because $\alpha^{n} - a$ is a unit of $A$. By the assumption (2), $R'$ is a flat extension over $R$. Hence $I_{[a]}^{n}D(\sqrt[n]{a})R' = I_{R'[\alpha]}^{n}D(\sqrt[n]{a}) = R'$. Since $R'$ is an integral extension of $R$ and $I_{[a]}^{n}D(\sqrt[n]{a}) \subset R$, we see that $I_{[a]}^{n}D(\sqrt[n]{a}) = R$.

**Theorem 1.16.** Let $R$ be an integral domain with quotient field $K$. Let $a$ be an element of $R$ and $n$ a positive integer. Let $\omega$ be a primitive $n$-th root of unity. Assume that the following four conditions hold:

1. $\alpha$ is an anti-integral element of degree $d$ over $R$.
2. $R[\omega]$ is a flat extension over $R$.
3. $[K(\sqrt[n]{a}) : K] = n$.
4. $[K(\sqrt[n]{a})(\alpha) : K(\sqrt[n]{a})] = d$.

If $\alpha^{n} - a$ is a unit of $R[\alpha]$, then $R[\alpha]/R$ is a flat extension.

**Proof.** Since $\alpha$ is an anti-integral element of degree $d$ over $R$, we have only to prove that $J_{[\alpha]} = R$ by [6, Proposition 2.6]. By the assumption that $\alpha^{n} - a$ is a unit of $R[\alpha]$, Theorem 1.15 implies that $I_{[\alpha]}^{n}D(\sqrt[n]{a}) = R$. Hence we have $J_{[\alpha]} = R$ by Lemma 1.9. So we get $J_{[\alpha]} = R$.

**Remark 1.17.** Under the assumptions in Theorem 1.16 (including the condition that $\alpha^{n} - a$ is a unit of $R[\alpha]$), we know that $I_{[\alpha]}$ is an invertible ideal of $R$ because $R = J_{[\alpha]} = I_{[\alpha]}(1, \eta_{1}, \ldots, \eta_{d})$. 
2. Ideals $I^n_\alpha D(\sqrt[n]\alpha)$ and $J^n_\alpha$

We investigate the relation between $I^n_\alpha D(\sqrt[n]\alpha)$ and $J^n_\alpha$. We know that $I^n_\alpha D(\sqrt[n]\alpha) \subseteq J^n_\alpha$ by Lemma 1.9. We will study what can occur under the condition that $I^n_\alpha D(\sqrt[n]\alpha) = J^n_\alpha$. We need the following lemma:

Lemma 2.1. ([1, Theorem 4]) Let $R$ be an integral domain and $\alpha$ an anti-integral element over $R$. Let $\gamma$ be a linear fractional transform of $\alpha$. Then $\gamma$ is also an anti-integral element over $R$ and $J_{(\gamma)} = J_\alpha$. In particular, $J_{(\alpha - \alpha^{-1})} = J_\alpha$ for every element $a \in R$.

Remark 2.2. Though in [1, Theorem 4] we assume that $R$ is Noetherian, we can delete the assumption that $R$ is Noetherian because we don’t assume it except [1, Lemmas 2 and 3] by [2, Theorem 6], [11, Fact 2].

Note that $I_\alpha \neq (0)$ by the definition of $I_\alpha$.

Proposition 2.3. Let $R$ be an integral domain and $\alpha_i$ anti-integral elements of degree $d_i$ over $R$ for $i = 1, 2, \ldots, n$. Assume that $I_{(\alpha_i)}$ is a finitely generated ideal of $R$ for $i = 1, 2, \ldots, n$. If $\prod_{i=1}^n I_{(\alpha_i)} = \prod_{i=1}^n J_{(\alpha_i)}$, then $I_{(\alpha_i)} = J_{(\alpha_i)} = R$ for $i = 1, 2, \ldots, n$.

Proof. Set 
\[ \varphi_{\alpha_i}(X) = X^{d_i} + \eta_1^{(i)} X^{d_i-1} + \cdots + \eta_{d_i}^{(i)}. \]
Then $J_{(\alpha_i)} = I_{(\alpha_i)}(1, \eta_1^{(i)}, \ldots, \eta_{d_i}^{(i)})$. Since 1 is in $(1, \eta_1^{(i)}, \ldots, \eta_{d_i}^{(i)})$ for each $i$, we have $\eta_j^{(i)} \in \prod_{i=1}^n (1, \eta_1^{(i)}, \ldots, \eta_{d_i}^{(i)})$ for $j = 1, 2, \ldots, d_i$. Let $c_1, \ldots, c_r$ be a system of generators of $\prod_{i=1}^n I_{(\alpha_i)}$. Since
\[ \prod_{i=1}^n I_{(\alpha_i)} = \prod_{i=1}^n J_{(\alpha_i)}(1, \eta_1^{(i)}, \ldots, \eta_{d_i}^{(i)}), \]
there exist elements $a_{11}, a_{12}, \ldots, a_{1r}, \ldots, a_{rr}$ of $R$ such that
\[ c_1 \eta_j^{(i)} = a_{11} c_1 + \cdots + a_{1r} c_r, \]
\[ \cdots \]
\[ a_{r1} c_1 + \cdots + a_{rr} c_r. \]
Hence
\[ \begin{vmatrix} a_{11} - \eta_1^{(i)} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} - \eta_j^{(i)} \end{vmatrix} = 0. \]
This implies that $\eta_j^{(i)}$ is integral over $R$ for $j = 1, 2, \ldots, d_i$. Therefore $R[\eta_1^{(i)}, \ldots, \eta_{d_i}^{(i)}]$ is integral over $R$. Since $\alpha_i$ is integral over $R[\eta_1^{(i)}, \ldots, \eta_{d_i}^{(i)}]$, we see that $\alpha_i$ is integral over $R$. By Lemma 1.4 we get $I_{(\alpha_i)} = R$ for $i = 1, 2, \ldots, n$. This shows that $\prod_{i=1}^n J_{(\alpha_i)} = \prod_{i=1}^n I_{(\alpha_i)} = R$. Then we see that $I_{(\alpha_i)} = J_{(\alpha_i)} = R$ for $i = 1, 2, \ldots, n$. 
Proposition 2.4. Let $R$ be an integral domain and $\alpha$ an anti-integral element over $R$. Set $A = R[\alpha]$ and assume that $I_{[\alpha]}$ is a finitely generated ideal of $R$. Then the following conditions are equivalent:

(i) There exists a positive integer $n$ such that $I_{[\alpha]}^n = J_{[\alpha]}^n$.

(ii) $A/R$ is both integral and flat extension.

Proof. (i) $\Rightarrow$ (ii). Proposition 2.3 implies that $I_{[\alpha]} = J_{[\alpha]} = R$. By Lemma 4, [6, Proposition 2.6], $A/R$ is integral and flat extension.

(ii) $\Rightarrow$ (i). It is clear from $I_{[\alpha]} = R = J_{[\alpha]}$.

Remark 2.5. Let $R$ be an integral domain and $n$ a positive integer. Let $a$ be an element of $R$. Let $\omega$ be a primitive $n$-th root of unity and assume that $\omega \in R$. If $I_{[\alpha]}^n D(\sqrt[n]{\alpha}) = J_{[\alpha]}^n$, then $\alpha^n a \neq 0$.

Proof. Suppose that $\alpha^n a = 0$. Then there exists an integer $k$ such that $0 \leq k \leq n - 1$ and $\alpha - \sqrt[n]{\alpha} \omega^k = 0$. Set $B = R[\sqrt[n]{\alpha}]$, then $I_{[\alpha]} B \varphi_{\alpha}(\sqrt[n]{\alpha} \omega^k) B = (0)$ and $J_{[\alpha]} B = B$ by Lemma 1.1 and Remark 1.3. Hence

\[
I_{[\alpha]}^n D(\sqrt[n]{\alpha}) = \prod_{k=0}^{n-1} I_{[\alpha]} \varphi_{\alpha}(\sqrt[n]{\alpha} \omega^k) = (0)
\]

and $J_{[\alpha]} = R$ because $B$ is an integral extension of $R$. Therefore $I_{[\alpha]}^n D(\sqrt[n]{\alpha}) = (0) \subseteq R = J_{[\alpha]}$. This is a contradiction.

Proposition 2.6. Let $R$ be an integral domain with quotient field $K$. Let $n$ be a positive integer and $\omega$ a primitive $n$-th root of unity. Let $a$ be an element of $R$. Assume that the following five conditions hold:

1. $\omega \in R$.
2. $\alpha$ is an anti-integral element of degree $d$ over $R$.
3. $[K(\sqrt[n]{\alpha}) : K] = n$.
4. $[K(\sqrt[n]{\alpha}) \alpha : K(\sqrt[n]{\alpha})] = d$.
5. $I_{[\alpha]}$ is a finitely generated ideal of $R$.

Set $B = R[\sqrt[n]{\alpha}]$. If $I_{[\alpha]}^n D(\sqrt[n]{\alpha}) = J_{[\alpha]}^n$, then $J_{[\alpha]} B = B$.

Proof. Note that $\alpha^n a \neq 0$ by Remark 2.5. Set $\gamma_k = (\alpha - \sqrt[n]{\alpha} \omega^k)^{-1}$ for $k = 0, 1, \ldots, n - 1$. Then we obtain $I_{[\alpha]} B \varphi_{\alpha}(\sqrt[n]{\alpha} \omega^k) = I_{B, [\gamma_k]}$ for $k = 0, 1, \ldots, n - 1$ by Lemmas 1.1 and 1.2. Since

\[
I_{[\alpha]}^n D(\sqrt[n]{\alpha}) = \prod_{k=0}^{n-1} I_{[\alpha]} \varphi_{\alpha}(\sqrt[n]{\alpha} \omega^k),
\]

we see that $I_{[\alpha]}^n D(\sqrt[n]{\alpha}) B = \prod_{k=0}^{n-1} I_{B, [\gamma_k]}$. It follows from Lemmas 1.1 and 2.1 that $J_{[\alpha]} B = J_{B, [\gamma_k]}$ for $k = 0, 1, \ldots, n - 1$. Hence $I_{[\alpha]}^n D(\sqrt[n]{\alpha}) = J_{[\alpha]}^n$ means that $\prod_{k=0}^{n-1} I_{B, [\gamma_k]} = \prod_{k=0}^{n-1} J_{B, [\gamma_k]}$. By the assumption, $I_{[\alpha]}$ is a finitely generated ideal of $R$. Let $b_1, \ldots, b_s$
be a system of generators of $I_{[\alpha]}$. Then \( \{ b_1\varphi_\alpha(\sqrt[n]{\alpha}\omega^k), \ldots, b_n\varphi_\alpha(\sqrt[n]{\alpha}\omega^k) \} \) is a subset of $B$ and it is a system of generators of $I_{[\alpha]}B\varphi_\alpha(\sqrt[n]{\alpha}\omega^k)$. Hence $I_{[\alpha]}B\varphi_\alpha(\sqrt[n]{\alpha}\omega^k) = I_{B,[\gamma_k]}$ is a finitely generated ideal of $B$ for $k = 0, 1, \ldots, n - 1$. Then by Proposition 2.3, $I_{B,[\gamma_k]} = J_{B,[\gamma_k]} = B$ for $k = 0, 1, \ldots, n - 1$. Hence $J_{[\alpha]}B = J_{B,[\gamma_k]} = B$.

**Theorem 2.7.** Let $R$ be an integral domain with quotient field $K$. Let $\alpha$ be an element of $R$ and $n$ a positive integer. Let $\omega$ be a primitive $n$-th root of unity. Assume that the following five conditions hold:

1. $\alpha$ is an anti-integral element of degree $d$ over $R$.
2. $R[\omega]$ is a flat extension over $R$.
3. $[K(\sqrt[n]{\alpha}) : K] = n$.
4. $[K(\sqrt[n]{\alpha})(\alpha) : K(\sqrt[n]{\alpha})] = d$.
5. $I_{[\alpha]}$ is a finitely generated ideal of $R$.

If $I_{[\alpha]}D(\sqrt[n]{\alpha}) = J^n_{[\alpha]}$, then $J_{[\alpha]} = R$.

**Proof.** Set $B = R[\sqrt[n]{\alpha}]$. First we will prove the case that $\omega$ is in $R$. Then $J_{[\alpha]}B = B$ by Proposition 2.6. This shows that $J_{[\alpha]} = R$ because $B$ is an integral extension of $R$.

Next we will prove the general case. Set $R' = R[\omega]$ and $A' = R'[\alpha]$. Then the former case shows that $J_{[\alpha]}R' = J_{R'[\alpha]} = R'$. Since $R'$ is an integral extension of $R$, we get $J_{[\alpha]} = R$.

**Theorem 2.8.** Let $R$ be an integral domain with quotient field $K$. Let $\alpha$ be an element of $R$ and $n$ a positive integer. Let $\omega$ be a primitive $n$-th root of unity. Assume that the following seven conditions hold:

1. $\alpha$ is an anti-integral element of degree $d$ over $R$.
2. $R[\omega]$ is a flat extension over $R$.
3. $[K(\sqrt[n]{\alpha}) : K] = n$.
4. $[K(\sqrt[n]{\alpha})(\alpha) : K(\sqrt[n]{\alpha})] = d$.
5. $[K(\omega)(\sqrt[n]{\alpha}) : K(\omega)] = n$.
6. $[K(\omega)(\sqrt[n]{\alpha})\omega : K(\omega)(\sqrt[n]{\alpha})] = d$.
7. $I_{[\alpha]}$ is a finitely generated ideal of $R$.

Then the following conditions are equivalent:

1. $I_{[\alpha]}D(\sqrt[n]{\alpha}) = J^n_{[\alpha]}$.
2. $I_{[\alpha]}D(\sqrt[n]{\alpha}) = R$.
3. $\alpha^n - \alpha$ is a unit of $R[\alpha]$.

**Proof.** Equivalence (ii) ⇔ (iii) is proved by Theorem 1.15. We will prove the implication (i) ⇒ (ii). By Theorem 2.7 we have $J^n_{[\alpha]} = R$. Hence $I_{[\alpha]}D(\sqrt[n]{\alpha}) = R$. Next we will prove the implication (ii) ⇒ (i). Lemma 1.9 shows that $I^n_{[\alpha]}D(\sqrt[n]{\alpha}) \subset J^n_{[\alpha]}$. Therefore by the condition (ii), we get $I^n_{[\alpha]}D(\sqrt[n]{\alpha}) = R = J^n_{[\alpha]}$.

The converse of Theorem 2.7 is not true. We have an example that $J_{[\alpha]} = R$ but $I_{[\alpha]}\varphi_\alpha(\alpha) \not\subset J_{[\alpha]}$ as the following shows.
Example 2.9. Let $R = F[u,v]$ be a polynomial ring over a field $F$ in two variables $u$ and $v$. Let $a$ be a root of $\varphi_a(X) = X^2 + (v/u)X + 1/u$. It is easily verified that $\varphi_a(X)$ is irreducible over the quotient field of $R$. Since $R$ is a unique factorization domain, we can get $I[\alpha] = uR$. Furthermore, $J[\alpha] = I[\alpha](1,v/u,1/u) = uR(1,v/u,1/u) = (1,v) = R$. We will show that $I[\alpha] \varphi_a(a) \subseteq J[\alpha]$ for every non-zero element $a$ of $R$. Assume that there exists a non-zero element $a$ of $R$ such that $I[\alpha] \varphi_a(a) = J[\alpha]$. Then there exists an element $b$ of $R$ such that $(ua^2 + va + 1)b = 1$ because $J[\alpha] = R$ and $I[\alpha] = uR$. This implies that $ua^2 + va + 1$ is a unit of $R$, hence, a unit of $F$. Since $a$ is not zero, we can draw a contradiction by comparing the degrees of the both sides of the equation above.

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References