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ANTI-INTEGRAL EXTENSIONS $R[\alpha]/R$ AND INVERTIBILITY OF $\alpha^n - a$

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Abstract. Let R be an integral domain and α an anti-integral element of degree d over R. In the paper [3] we give a condition for $\alpha^2 - a$ to be a unit of $R[\alpha]$. In this paper we will generalize the result to an arbitrary positive integer n and give a condition, in terms of the ideal $I^n_{[\alpha]}D(\sqrt[n]{a})$ of R, for $\alpha^n - a$ to be a unit of $R[\alpha]$.

1. Conditions of Invertibility of $\alpha^n - a$

Let R be an integral domain with quotient field K and $R[X]$ a polynomial ring over R in an indeterminate X. Let α be an element of an algebraic field extension of K and $\pi : R[X] \longrightarrow R[\alpha]$ the R-algebra homomorphism defined by $\pi(X) = \alpha$. Let $\varphi_\alpha(X)$ be the minimal polynomial of α over K with deg $\varphi_\alpha(X) = d$ and write $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d, (\eta_1, \ldots, \eta_d \in K)$. We will define $I_{[\alpha]} := \bigcap_{i=1}^d (R :_R \eta_i)$ and $J_{[\alpha]} := I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)$ where $(R :_R \eta_i) = \{c \in R; c\eta_i \in R\}$ and $(1, \eta_1, \ldots, \eta_d)$ is an R-module generated by $1, \eta_1, \ldots, \eta_d$. An element α is called an anti-integral element of degree d over R if

$$\operatorname{Ker} \pi = I_{[\alpha]}\varphi_\alpha(X)R[X].$$

We say that the extension $R[\alpha]/R$ is an anti-integral extension if α is an anti-integral element of degree d over R.

Our notation is standard and our general reference for unexplained terms is [5].

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We will frequently use the following lemma throughout this paper:

Lemma 1.1. *Let $R \subset S$ be a flat extension of integral domains. Let α be an element of an algebraic field extension of the quotient field of S.*

(1) *Set $I_{S,[\alpha]} = \bigcap_{i=1}^d (S :_S \eta_i)$ and $J_{S,[\alpha]} = I_{S,[\alpha]}(1, \eta_1, \ldots, \eta_d)$ where $(S :_S \eta_i) = \{b \in S; b\eta_i \in S\}$ and $(1, \eta_1, \ldots, \eta_d)$ is an S-module generated by 1, $\eta_1, \ldots, \eta_d$. Then $I_{S,[\alpha]} = I_{[\alpha]}S$ and $J_{S,[\alpha]} = J_{[\alpha]}S$.*

(2) *If α is an anti-integral element of degree d over R, then α is also an anti-integral element of degree d over S.*

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Proof. (1) Since S is a flat extension over R, we have

$$I_{S,[\alpha]} = \bigcap_{i=1}^{d} (S :_S \eta_i) = \bigcap_{i=1}^{d} (R :_R \eta_i)S = (\bigcap_{i=1}^{d} (R :_R \eta_i))S = I_{[\alpha]}S$$

and

$$J_{S,[\alpha]} = I_{S,[\alpha]}(1, \eta_1, \ldots, \eta_d) = I_{[\alpha]}(1, \eta_1, \ldots, \eta_d)S = J_{[\alpha]}S.$$

(2) By tensoring S to the following exact sequence:

$$0 \longrightarrow I_{[\alpha]}\varphi_\alpha(X)R[X] \longrightarrow R[X] \longrightarrow R[\alpha] \longrightarrow 0,$$

we have an exact sequence:

$$0 \longrightarrow I_{[\alpha]}\varphi_\alpha(X)S[X] \longrightarrow S[X] \longrightarrow S[\alpha] \longrightarrow 0,$$

By (1), we know that $I_{[\alpha]}S = I_{S,[\alpha]}$. Hence α is also an anti-integral element of degree d over S.

We list some facts which will be used later for reference sake:

Lemma 1.2. ([10, Theorem 1] and [1, Theorem 4]) *Let R be an integral domain and α an algebraic element over the quotient field of R. Let a be an element of R such that $\alpha - a$ is not zero. Then the following hold:*
(1) $I_{[(\alpha-a)^{-1}]} = I_{[\alpha]}\varphi_\alpha(a)$.
(2) *If α is an anti-integral element over R, then so is $(\alpha - a)^{-1}$.*

Remark 1.3. If $\alpha - a = 0$, then $\varphi_\alpha(X) = X - a$, and so $I_{[\alpha]}\varphi_\alpha(a) = (0)$ and $J_{[\alpha]} = R$. Hence $I_{[\alpha]}\varphi_\alpha(a) \subsetneq J_{[\alpha]}$. Especially, $I_{[\alpha]}\varphi_\alpha(a) \neq R$.

Lemma 1.4. ([6, Theorem 2.2] and [8, Lemma 8]) *Let R be an integral domain and γ an anti-integral element over R. Then the following conditions are equivalent:*
(i) *γ is integral over R.*
(ii) $I_{[\gamma]} = R$.

By making use of Lemmas 1.3 and 1.4 we have the following:

Lemma 1.5. (cf. [7, Theorem 6]) *Let R be an integral domain and α an anti-integral element over R. Let a be an element of R. Then $\alpha - a$ is a unit of $R[\alpha]$ if and only if $I_{[\alpha]}\varphi_\alpha(a) = R$.*

Proof. First we shall prove the 'only if' part. Since $\alpha - a$ is a unit of $R[\alpha]$, then there exist elements $f(X)$ of $R[X]$ and $g(X)$ of $I_{[\alpha]}R[X]$ such that $f(X)(X - a) - 1 = g(X)\varphi_\alpha(X)$. Hence $g(a)\varphi_\alpha(a) = -1$. This means that $I_{[\alpha]}\varphi_\alpha(a) = R$.

Next we shall prove the 'if' part. By Remark 1.3, $\alpha - a$ is not zero. Therefore we get $I_{[(\alpha-a)^{-1}]} = R$ by Lemma 1.2 (1). Then Lemma 1.4 asserts that $(\alpha - a)^{-1}$ is integral over R. Therefore there exist elements $c_1, \ldots, c_n$ of R such that

$$((\alpha - a)^{-1})^n + c_1((\alpha - a)^{-1})^{n-1} + \cdots + c_n = 0.$$

Hence $1 = -(c_1 + \cdots + c_n(\alpha - a)^{n-1})(\alpha - a)$. This shows that $\alpha - a$ is a unit of $R[\alpha]$.

Let n be a positive integer and a an element of R. Assume that the following three conditions hold:

(1) α is an anti-integral element of degree d over R.

(2) $[K(\sqrt[n]{a}) : K] = n$.

(3) $[K(\sqrt[n]{a})(\alpha) : K(\sqrt[n]{a})] = d$.

Set $B = R[\sqrt[n]{a}]$. By the condition (3), the minimal polynomial of α over $K(\sqrt[n]{a})$ coincides with $\varphi_\alpha(X)$. The condition (2) implies that B is a free R-module of rank n. Hence B is a flat extension over R. Therefore, by Lemma 1.1, $I_{B,[\alpha]} = I_{[\alpha]}B$, $J_{B,[\alpha]} = J_{[\alpha]}B$ and α is also an anti-integral element of degree d over B.

We give a condition for the element $\alpha^n - a$ to be a unit of $R[\alpha]$.

Theorem 1.6. *Let R be an integral domain with quotient field K. Let n be a positive integer and ω a primitive n-th root of unity. Let a be an element of R. Assume that the following four conditioins hold:*

(1) $\omega \in R$.

(2) *α is an anti-integral element of degree d over R.*

(3) $[K(\sqrt[n]{a}) : K] = n$.

(4) $[K(\sqrt[n]{a})(\alpha) : K(\sqrt[n]{a})] = d$.

Set $A = R[\alpha]$ and $B = R[\sqrt[n]{a}]$. Then the following conditions are equivalent to each other:

(i) *$\alpha^n - a$ is a unit of A.*

(ii) *$I_{[\alpha]}\varphi_\alpha(\sqrt[n]{a}\,\omega^k)B = B$ for $k = 0, \ldots, n-1$.*

Proof. (i) $\Rightarrow$ (ii). Since

$$\alpha^n - a = (\alpha - \sqrt[n]{a})(\alpha - \sqrt[n]{a}\,\omega)\cdots(\alpha - \sqrt[n]{a}\,\omega^{n-1}),$$

we see that $\alpha - \sqrt[n]{a}\,\omega^k$ is a unit of $B[\alpha]$ for $k = 0, 1, \ldots, n-1$. Hence by Lemma 1.5 we get

$$I_{[\alpha]}\varphi_\alpha(\sqrt[n]{a}\,\omega^k)B = B$$

for $k = 0, 1, \ldots, n-1$.

(ii) $\Rightarrow$ (i). By Lemma 1.5 we see that

$$\alpha - \sqrt[n]{a}\,\omega^k$$

is a unit of $B[\alpha]$ for $k = 0, 1, \ldots, n-1$. Hence $\alpha^n - a$ is also a unit of $B[\alpha]$. Since $\alpha^n - a$ is an element of A and $B[\alpha]$ is an integral extension of A, we know that $\alpha^n - a$ is a unit of A.

Recall that $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Let s be an integer such that $0 \leq s \leq n-1$. Then we define

$$\varphi_{\alpha,s}(X) = (\sum \eta_i X^{d-i})X^{-s}$$

where $\eta_0 = 1$ and the sum is taken over i such that $0 \le i \le d$ and that the remainder of $d - i$ divided by n is s. Then it is easily verified:

$$\varphi_\alpha(X) = \sum_{s=0}^{n-1} X^s \varphi_{\alpha,s}(X)$$

and

$$\varphi_{\alpha,s}(X) \in R[\eta_1, \ldots, \eta_d][X^n].$$

Set $\beta_s = (\sqrt[n]{a})^s \varphi_{\alpha,s}(\sqrt[n]{a})$ for $s = 0, 1, \ldots, n-1$. The cyclic determinant

$$\begin{vmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \\ \beta_{n-1} & \beta_0 & \cdots & \beta_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1 & \beta_2 & \cdots & \beta_0 \end{vmatrix} \tag{1}$$

is denoted by $D(\sqrt[n]{a})$. Then

$$D(\sqrt[n]{a}) = \prod_{k=0}^{n-1} \left(\sum_{s=0}^{n-1} (\sqrt[n]{a}\,\omega^k)^s \varphi_{\alpha,s}(\sqrt[n]{a}) \right).$$

For cyclic determinants, see [9, p. 91].

Example 1.7. Let R be an integral domain with quotient field K. Let α be an element of K. Let a be an element of R and n a positive integer. Then $D(\sqrt[n]{a}) = (-1)^n(\alpha^n - a)$. Hence, if $I^n_{[\alpha]} D(\sqrt[n]{a}) = R$, then $\alpha^n - a$ is a unit of $R[\alpha]$.

Proof. Note that $\varphi_\alpha(X) = X - \alpha$. By the definition of $\varphi_{\alpha,s}(X)$, we get $\varphi_{\alpha,0}(X) = -\alpha$, $\varphi_{\alpha,1}(X) = 1$, $\varphi_{\alpha,2}(X) = \cdots = \varphi_{\alpha,n-1}(X) = 0$. Hence $\beta_0 = -\alpha$, $\beta_1 = \sqrt[n]{a}$, $\beta_2 = \cdots = \beta_{n-1} = 0$. By expanding the first column of the determinant $D(\sqrt[n]{a})$, we have $D(\sqrt[n]{a}) = (-1)^n(\alpha^n - a)$.

Lemma 1.8. *For every term $\pm\beta_{i_1}\beta_{i_2}\cdots\beta_{i_n}$ of the cyclic determinant* (1)*, the following equality holds:*

$$i_1 + i_2 + \cdots + i_n \equiv 0 \pmod{n}.$$

Proof. Let $|a_{ij}|$ be the cyclic determinant defined by (1). Then there exists a permutation σ such that

$$\beta_{i_1}\beta_{i_2}\cdots\beta_{i_n} = a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}.$$

Then

$$i_1 \equiv \sigma(1) - 1 \pmod{n}, \ldots, i_n \equiv \sigma(n) - n \pmod{n}.$$

Hence

$$\begin{aligned} i_1 + i_2 + \cdots + i_n &\equiv \sigma(1) + \sigma(2) + \cdots \sigma(n) - (1 + 2 + \cdots + n) \,(\mathrm{mod} n) \\ &\equiv 0 \,(\mathrm{mod} n). \end{aligned}$$

Lemma 1.9. *$I_{[\alpha]}^n D(\sqrt[n]{a})$ is an ideal of R and $I_{[\alpha]}^n D(\sqrt[n]{a}) \subset J_{[\alpha]}^n$.*

Proof. First we shall show that $I_{[\alpha]}^n D(\sqrt[n]{a}) \subset R$. Let $\pm\beta_{i_1}\beta_{i_2}\cdots\beta_{i_n}$ be a term of the cyclic determinant $D(\sqrt[n]{a})$. By Lemma 1.8 there exists a non-negative integer q such that $i_1 + i_2 + \cdots + i_n = nq$. Then

$$\begin{aligned} &\beta_{i_1}\beta_{i_2}\cdots\beta_{i_n} \\ &= (\sqrt[n]{a})^{nq}\varphi_{\alpha,i_1}(\sqrt[n]{a})\cdots\varphi_{\alpha,i_n}(\sqrt[n]{a}) \\ &= a^q\varphi_{\alpha,i_1}(\sqrt[n]{a})\cdots\varphi_{\alpha,i_n}(\sqrt[n]{a}). \end{aligned}$$

By the definitions of $I_{[\alpha]}$ and $\varphi_{\alpha,s}(X)$, we have

$$I_{[\alpha]}^n \beta_{i_1}\beta_{i_2}\cdots\beta_{i_n} \subset R.$$

Therefore $I_{[\alpha]}^n D(\sqrt[n]{a}) \subset R$.

It is easily verified that $I_{[\alpha]}^n D(\sqrt[n]{a})$ is an ideal of R. Furthermore,

$$I_{[\alpha]}^n \beta_{i_1}\beta_{i_2}\cdots\beta_{i_n} \subset J_{[\alpha]}^n.$$

Hence $I_{[\alpha]}^n D(\sqrt[n]{a}) \subset J_{[\alpha]}^n$.

Proposition 1.10. *Let R be an integral domain with quotient field K. Let n be a positive integer and a an element of R. Let ω be a primitive n-th root of unity. Assume that the following six conditioins hold:*

(1) *α is an anti-integral element of degree d over R.*
(2) *$R[\omega]$ is a flat extension over R.*
(3) *$[K(\sqrt[n]{a}) : K] = n$.*
(4) *$[K(\sqrt[n]{a})(\alpha) : K(\sqrt[n]{a})] = d$.*
(5) *$[K(\omega)(\sqrt[n]{a}) : K(\omega)] = n$.*
(6) *$[K(\omega)(\sqrt[n]{a})(\alpha) : K(\omega)(\sqrt[n]{a})] = d$.*

Set $A = R[\alpha]$. If $I_{[\alpha]}^n D(\sqrt[n]{a}) = R$, then $\alpha^n - a$ is a unit of A.

Proof. Set $B = R[\sqrt[n]{a}]$. First we will prove the case that ω is in R. The condition $I_{[\alpha]}^n D(\sqrt[n]{a}) = R$ implies that

$$I_{[\alpha]}^n D(\sqrt[n]{a})B = B.$$

Hence

$$I_{[\alpha]}^n \prod_{k=0}^{n-1}\left(\sum_{s=0}^{n-1}(\sqrt[n]{a}\,\omega^k)^s\varphi_{\alpha,s}(\sqrt[n]{a})\right)B = B.$$

Since $I_{[\alpha]}(\sum_{s=0}^{n-1}(\sqrt[n]{a}\,\omega^k)^s\varphi_{\alpha,s}(\sqrt[n]{a}))B$ is an ideal of B, we get

$$I_{[\alpha]}\left(\sum_{s=0}^{n-1}(\sqrt[n]{a}\,\omega^k)^s\varphi_{\alpha,s}(\sqrt[n]{a})\right)B = B.$$

Hence $I_{[\alpha]}\varphi_\alpha(\sqrt[n]{a}\,\omega^k)B = B$ for $k = 0, 1, \ldots, n-1$. Theorem 1.6 asserts that $\alpha^n - a$ is a unit of A.

Next we prove the general case. Set $R' = R[\omega]$, $A' = R'[\alpha]$. We have $I^n_{R',[\alpha]}D(\sqrt[n]{a}) = R'$ because $I_{[\alpha]}R' \subset I_{R',[\alpha]}$ and $I^n_{[\alpha]}D(\sqrt[n]{a}) = R$. Besides, α is an anti-integral element of degree d over $R^{'}$ by Lemma 1.1 (2). Then the argument above implies that $\alpha^n - a$ is a unit of A'. But $\alpha^n - a \in A$ and A' is integral over A. Hence $\alpha^n - a$ is a unit of A.

Let n be a positive integer and ω a primitive n-th root of unity. Set $R' = R[\omega]$. The case that R' is not flat over R happens as the following example shows. Let $\mathbf{Z}$ be the ring of integers and $\mathbf{Q}$ the field of rational numbers.

Example 1.11. Set $i = \sqrt{-1}$, $R = \mathbf{Z}[\sqrt{2}, i]$ and $\omega = (1+i)/\sqrt{2}$. Then ω is a primitive 8-th root of unity. Let K be the quotient field of R. Then $K = \mathbf{Q}(\sqrt{2}, i)$ and ω is in K. Hence $\varphi_\omega(X) = X - \omega$. We have the following:

(1) ω is not in R.

(2) $I_{[\omega]} = \{p + q\sqrt{2} + ri + s\sqrt{2}i; p, q, r, s \in \mathbf{Z}$ and $p + r \in 2\mathbf{Z}\}$.

(3) $R[\omega]$ is neither anti-integral nor flat over R.

Proof. Note that R is a free $\mathbf{Z}$-module with a basis $\{1, \sqrt{2}, i, \sqrt{2}i\}$ and $\{1, \sqrt{2}, i, \sqrt{2}i\}$ is linearly independent over $\mathbf{Q}$.

(1) If $\omega = \sqrt{2}/2 + \sqrt{2}i/2$ is in R, there exist elements p, q, r, s of $\mathbf{Z}$ such that $\omega = p + q\sqrt{2} + ri + s\sqrt{2}i$. Then $p = 0$, $q = 1/2$, $r = 0$ and $s = 1/2$. This is a contradiction.

(2) Set $I_0 = \{p + q\sqrt{2} + ri + s\sqrt{2}i; p, q, r, s \in \mathbf{Z}$ and $p + r \in 2\mathbf{Z}\ \}$. Then it is easily proved that I_0 is an ideal of R. Let x be an element of R. It is easily seen that $x\omega$ is in R if and only if x is in I_0. Hence $I_{[\omega]} = I_0$.

(3) Assume that ω is an anti-integral element over R. Then $I_{[\omega]} = R$ by Lemma 1.4 because ω is integral over R. The assertion (2) claims that 1 is not in $I_{[\omega]}$. This is a contradiction. Hence ω is not an anti-integral element over R.

Next we will assume that $R[\omega]$ is flat over R. Since $R[\omega]$ is integral over R and 1 is not in $I_{[\omega]}$, we have $I_{[\omega]}R[\omega] \neq R[\omega]$. On the other hand, by Lemma 1.1 (1), we see that $I_{[\omega]}R[\omega] = I_{R[\omega],[\omega]}$. Since ω is in $R[\omega]$, we get $I_{R[\omega],[\omega]} = R[\omega]$. This is absurd. Therefore $R[\omega]$ is not flat over R.

We can't delete the assumption (5) of Proposition 1.10. To show an example for it, we need the following lemmas:

Lemma 1.12. ([4, Theorem 49]) *Let K be a field and a an element of K. Let m and n be relatively prime positive integers. Then $X^{mn} - a$ is irreducible over K if and only if both $X^m - a$ and $X^n - a$ are irreducible over K.*

Lemma 1.13. ([4, Theorem 51]) *Let K be a field and a an element of K. Let p be a prime number and n a positive integer. Assume that no p-th root of a is in K. Then:*

(1) *If p is odd, then $X^{p^n} - a$ is irreducible over K.*

(2) *If $p = 2$ and the characteristic of K is 2, then $X^{2^n} - a$ is irreducible over K.*

(3) *If $p = 2$, $n \geqq 2$ and the characteristic of K is not 2, then $X^{2^n} - a$ is irreducible over K if and only if $-4a$ is not a fourth power in K.*

Let R be an integral domain with quotient field K. Let n be a positive integer and ω a primitive n-th root of unity. Then $[K(\sqrt[n]{a}) : K] = n$ does not imply $[K(\omega)(\sqrt[n]{a}) : K(\omega)] = n$ as the following example shows:

Example 1.14. Set $n = 12$, $K = \mathbf{Q}$ and $a = -\frac{9}{4}$. Let ω be a primitive 12-th root of unity. Then the following two assertions hold:

(1) $X^{12} + 9/4$ is irreducible over $\mathbf{Q}$ and $[K(\sqrt[n]{a}) : K] = n$

(2) $X^{12} + 9/4$ is reducible over $\mathbf{Q}(\omega)$ and $[K(\omega)(\sqrt[n]{a}) : K(\omega)] < n$.

Proof. (1) By Lemma 1.12, we have only to prove that both $X^3 + 9/4$ and $X^4 + 9/4$ are irreducible over $\mathbf{Q}$. Lemma 1.13 (1) implies that $X^3 + 9/4$ is irreducible over $\mathbf{Q}$. Since $-4 \times \frac{9}{4} = -9$ is not a fourth power in $\mathbf{Q}$, we see that $X^4 + 9/4$ is irreducible over $\mathbf{Q}$ by Lemma 1.13 (3).

(2) Set $f = 1 - 2\omega^2$. Then we have

$$X^{12} + \frac{9}{4} = \left(X^6 + fX^3 - \frac{3}{2}\right)\left(X^6 - fX^3 - \frac{3}{2}\right)$$

and f is in $\mathbf{Q}(\omega)$. Hence $X^{12} + 9/4$ is reducible over $\mathbf{Q}(\omega)$ and $[K(\omega)(\sqrt[n]{a}) : K(\omega)] < n$.

Theorem 1.15. *Let R be an integral domain with quotient field K. Let n be a positive integer and a an element of R. Let ω be a primitive n-th root of unity. Assume that the following six conditioins hold:*

(1) *α is an anti-integral element of degree d over R.*

(2) *$R[\omega]$ is a flat extension over R.*

(3) *$[K(\sqrt[n]{a}) : K] = n$.*

(4) *$[K(\sqrt[n]{a})(\alpha) : K(\sqrt[n]{a})] = d$.*

(5) *$[K(\omega)(\sqrt[n]{a}) : K(\omega)] = n$.*

(6) *$[K(\omega)(\sqrt[n]{a})(\alpha) : K(\omega)(\sqrt[n]{a})] = d$.*

Set $A = R[\alpha]$. Then the following conditions are equivalent to each other:

(i) *$\alpha^n - a$ is a unit of A.*

(ii) *$I_{[\alpha]}^n D(\sqrt[n]{a}) = R$.*

Proof. (ii) $\Rightarrow$ (i). It is clear from Proposition 1.10.

(i) $\Rightarrow$ (ii). (In this part of the proof, we don't use the assumptions (5) and (6).) Set $B = R[\sqrt[n]{a}]$. First we will prove the case that ω is in R. By Theorem 1.6 we obtain

$$I_{[\alpha]}\varphi_\alpha(\sqrt[n]{a}\,\omega^k)B = B$$

for $k = 0, 1, \dots, n-1$. Therefore

$$I^n_{[\alpha]} \prod_{k=0}^{n-1} \varphi_\alpha(\sqrt[n]{a}\,\omega^k)B = B.$$

Note that

$$\begin{aligned}\varphi_\alpha(\sqrt[n]{a}\,\omega^k) &= \sum_{s=0}^{n-1}(\sqrt[n]{a}\,\omega^k)^s\varphi_{\alpha,s}(\sqrt[n]{a}\,\omega^k)\\ &= \sum_{s=0}^{n-1}(\sqrt[n]{a}\,\omega^k)^s\varphi_{\alpha,s}(\sqrt[n]{a}).\end{aligned}$$

Hence

$$I^n_{[\alpha]} \prod_{k=0}^{n-1}\left(\sum_{s=0}^{n-1}(\sqrt[n]{a}\,\omega^k)^s\varphi_{\alpha,s}(\sqrt[n]{a})\right)B = B,$$

that is, $I^n_{[\alpha]}D(\sqrt[n]{a})B = B$.

We will prove that $I^n_{[\alpha]}D(\sqrt[n]{a}) = R$. By Lemma 1.9 we know that $I^n_{[\alpha]}D(\sqrt[n]{a}) \subset R$, and $I^n_{[\alpha]}D(\sqrt[n]{a})$ is an ideal of R. Suppose the contrary, i.e., $I^n_{[\alpha]}D(\sqrt[n]{a}) \neq R$. Then there exists a prime ideal p of $\operatorname{Spec} R$ such that $I^n_{[\alpha]}D(\sqrt[n]{a}) \subset p$. Since B is integral over R, we can take a prime ideal P of $\operatorname{Spec} B$ such that $P \cap R = p$. Then $I^n_{[\alpha]}D(\sqrt[n]{a})B \subset P$. This is a contradiction.

Next we prove the general case. Set $R' = R[\omega]$ and $A' = R'[\alpha]$. Then $\alpha^n - a$ is a unit of A' because $\alpha^n - a$ is a unit of A. By the assumption (2), R' is a flat extension over R. Hence $I^n_{[\alpha]}D(\sqrt[n]{a})R' = I^n_{R',[\alpha]}D(\sqrt[n]{a}) = R'$. Since R' is an integral extension of R and $I^n_{[\alpha]}D(\sqrt[n]{a}) \subset R$, we see that $I^n_{[\alpha]}D(\sqrt[n]{a}) = R$.

Theorem 1.16. *Let R be an integral domain with quotient field K. Let a be an element of R and n a positive integer. Let ω be a primitive n-th root of unity. Assume that the following four conditioins hold:*
(1) *α is an anti-integral element of degree d over R.*
(2) *$R[\omega]$ is a flat extension over R.*
(3) $[K(\sqrt[n]{a}) : K] = n$.
(4) $[K(\sqrt[n]{a})(\alpha) : K(\sqrt[n]{a})] = d$.
If $\alpha^n - a$ is a unit of $R[\alpha]$, then $R[\alpha]/R$ is a flat extension.

Proof. Since α is an anti-integral element of degree d over R, we have only to prove that $J_{[\alpha]} = R$ by [6, Proposition 2.6]. By the assumption that $\alpha^n - a$ is a unit of $R[\alpha]$. Theorem 1.15 implies that $I^n_{[\alpha]}D(\sqrt[n]{a}) = R$. Hence we have $J^n_{[\alpha]} = R$ by Lemma 1.9. So we get $J_{[\alpha]} = R$.

Remark 1.17. Under the assumptions in Theorem 1.16 (including the condition that $\alpha^n - a$ is a unit of $R[\alpha]$), we know that $I_{[\alpha]}$ is an invertible ideal of R because $R = J_{[\alpha]} = I_{[\alpha]}(1, \eta_1, \dots, \eta_d)$.

2. Ideals $I^n_{[\alpha]}D(\sqrt[n]{a})$ and $J^n_{[\alpha]}$

We investgate the relation between $I^n_{[\alpha]}D(\sqrt[n]{a})$ and $J^n_{[\alpha]}$. We know that $I^n_{[\alpha]}D(\sqrt[n]{a}) \subset J^n_{[\alpha]}$ by Lemma 1.9. We will study what can occur under the condition that $I^n_{[\alpha]}D(\sqrt[n]{a}) = J^n_{[\alpha]}$. We need the following lemma:

Lemma 2.1. ([1, Theorem 4]) *Let R be an integral domain and α an anti-integral element over R. Let γ be a linear fractional transform of α. Then γ is also an anti-integral element over R and $J_{[\gamma]} = J_{[\alpha]}$. In particular, $J_{[(\alpha-a)^{-1}]} = J_{[\alpha]}$ for every element $a \in R$.*

Remark 2.2. Though in [1, Theorem 4] we assume that R is Noetherian, we can delete the assumption that R is Noetherian because we don't assume it except [1, Lemmas 2 and 3] and we don't need it in [1, Lemmas 2 and 3] by [2, Theorem 6], [11, Fact 2].

Note that $I_{[\alpha]} \neq (0)$ by the definition of $I_{[\alpha]}$.

Proposition 2.3. *Let R be an integral domain and α_i anti-integral elements of degree d_i over R for $i = 1, 2, \ldots, n$. Assume that $I_{[\alpha_i]}$ is a finitely generated ideal of R for $i = 1, 2, \ldots, n$. If $\prod_{i=1}^n I_{[\alpha_i]} = \prod_{i=1}^n J_{[\alpha_i]}$, then $I_{[\alpha_i]} = J_{[\alpha_i]} = R$ for $i = 1, 2, \ldots, n$.*

Proof. Set
$$\varphi_{\alpha_i}(X) = X^{d_i} + \eta_1^{(i)} X^{d_i-1} + \cdots + \eta_{d_i}^{(i)}.$$
Then $J_{[\alpha_i]} = I_{[\alpha_i]}(1, \eta_1^{(i)}, \ldots, \eta_{d_i}^{(i)})$. Since 1 is in $(1, \eta_1^{(i)}, \ldots, \eta_{d_i}^{(i)})$ for each i, we have $\eta_j^{(i)} \in \prod_{i=1}^n (1, \eta_1^{(i)}, \ldots, \eta_{d_i}^{(i)})$ for $j = 1, 2, \ldots, d_i$. Let $c_1, \ldots, c_r$ be a system of generators of $\prod_{i=1}^n I_{[\alpha_i]}$. Since
$$\prod_{i=1}^n I_{[\alpha_i]} = \prod_{i=1}^n I_{[\alpha_i]}(1, \eta_1^{(i)}, \ldots, \eta_{d_i}^{(i)}),$$
there exist elements $a_{11}, \ldots, a_{1r}, \ldots, a_{r1}, \ldots, a_{rr}$ of R such that
$$\begin{aligned} c_1\eta_j^{(i)} &= a_{11}c_1 + \cdots + a_{1r}c_r, \\ &\cdots \\ c_r\eta_j^{(i)} &= a_{r1}c_1 + \cdots + a_{rr}c_r. \end{aligned}$$
Hence
$$\begin{vmatrix} a_{11} - \eta_j^{(i)} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} - \eta_j^{(i)} \end{vmatrix} = 0.$$
This implies that $\eta_j^{(i)}$ is integral over R for $j = 1, 2, \ldots, d_i$. Therefore $R[\eta_1^{(i)}, \ldots, \eta_{d_i}^{(i)}]$ is integral over R. Since α_i is integral over $R[\eta_1^{(i)}, \ldots, \eta_{d_i}^{(i)}]$, we see that α_i is integral over R. By Lemma 1.4 we get $I_{[\alpha_i]} = R$ for $i = 1, 2, \ldots, n$. This shows that $\prod_{i=1}^n J_{[\alpha_i]} = \prod_{i=1}^n I_{[\alpha_i]} = R$. Then we see that $I_{[\alpha_i]} = J_{[\alpha_i]} = R$ for $i = 1, 2, \ldots, n$.

Proposition 2.4. *Let R be an integral domain and α an anti-integral element over R. Set $A = R[\alpha]$ and assume that $I_{[\alpha]}$ is a finitely generated ideal of R. Then the following conditions are equivalent:*

(i) *There exists a positive integer n such that $I^n_{[\alpha]} = J^n_{[\alpha]}$.*

(ii) *A/R is both integral and flat extension.*

Proof. (i) $\Rightarrow$ (ii). Proposition 2.3 implies that $I_{[\alpha]} = J_{[\alpha]} = R$. By Lemma 4, [6, Proposition 2.6], A/R is integral and flat extension.

(ii) $\Rightarrow$ (i). It is clear from $I_{[\alpha]} = R = J_{[\alpha]}$.

Remark 2.5. Let R be an integral domain and n a positive integer. Let a be an element of R. Let ω be a primitive n-th root of unity and assume that $\omega \in R$. If $I^n_{[\alpha]}D(\sqrt[n]{a}) = J^n_{[\alpha]}$, then $\alpha^n - a \neq 0$.

Proof. Suppose that $\alpha^n - a = 0$. Then there exists an integer k such that $0 \leq k \leq n-1$ and $\alpha - \sqrt[n]{a}\,\omega^k = 0$. Set $B = R[\sqrt[n]{a}]$, then $I_{[\alpha]}B\varphi_\alpha(\sqrt[n]{a}\,\omega^k)B = (0)$ and $J_{[\alpha]}B = B$ by Lemma1.1 and Remark 1.3. Hence

$$I^n_{[\alpha]}D(\sqrt[n]{a}) = \prod_{k=0}^{n-1} I_{[\alpha]}\varphi_\alpha(\sqrt[n]{a}\,\omega^k) = (0)$$

and $J_{[\alpha]} = R$ because B is an integral extension of R. Therefore $I^n_{[\alpha]}D(\sqrt[n]{a}) = (0) \subsetneq R = J^n_{[\alpha]}$. This is a contradiction.

Proposition 2.6. *Let R be an integral domain with quotient field K. Let n be a positive integer and ω a primitive n-th root of unity. Let a be an element of R. Assume that the following five conditioins hold:*

(1) *$\omega \in R$.*

(2) *α is an anti-integral element of degree d over R.*

(3) *$[K(\sqrt[n]{a}) : K] = n$.*

(4) *$[K(\sqrt[n]{a})(\alpha) : K(\sqrt[n]{a})] = d$.*

(5) *$I_{[\alpha]}$ is a finitely generated ideal of R.*

Set $B = R[\sqrt[n]{a}]$. If $I^n_{[\alpha]}D(\sqrt[n]{a}) = J^n_{[\alpha]}$, then $J_{[\alpha]}B = B$.

Proof. Note that $\alpha^n - a \neq 0$ by Remark 2.5. Set $\gamma_k = (\alpha - \sqrt[n]{a}\,\omega^k)^{-1}$ for $k = 0, 1, \ldots, n-1$. Then we obtain $I_{[\alpha]}B\varphi_\alpha(\sqrt[n]{a}\,\omega^k) = I_{B,[\gamma_k]}$ for $k = 0, 1, \ldots, n-1$ by Lemmas 1.1 and 1.2. Since

$$I^n_{[\alpha]}D(\sqrt[n]{a}) = \prod_{k=0}^{n-1} I_{[\alpha]}\varphi_\alpha(\sqrt[n]{a}\,\omega^k),$$

we see that $I^n_{[\alpha]}D(\sqrt[n]{a})B = \prod_{k=0}^{n-1} I_{B,[\gamma_k]}$. It follows from Lemmas 1.1 and 2.1 that $J_{[\alpha]}B = J_{B,[\gamma_k]}$ for $k = 0, 1, \ldots, n-1$. Hence $I^n_{[\alpha]}D(\sqrt[n]{a}) = J^n_{[\alpha]}$ means that $\prod_{k=0}^{n-1} I_{B,[\gamma_k]} = \prod_{k=0}^{n-1} J_{B,[\gamma_k]}$. By the assumption, $I_{[\alpha]}$ is a finitely generated ideal of R. Let $b_1, \ldots, b_s$

be a system of generators of $I_{[\alpha]}$. Then $\{b_1\varphi_\alpha(\sqrt[n]{a}\,\omega^k), \ldots, b_s\varphi_\alpha(\sqrt[n]{a}\,\omega^k)\}$ is a subset of B and it is a system of generators of $I_{[\alpha]}B\varphi_\alpha(\sqrt[n]{a}\,\omega^k)$. Hence $I_{[\alpha]}B\varphi_\alpha(\sqrt[n]{a}\,\omega^k) = I_{B,[\gamma_k]}$ is a finitely generated ideal of B for $k = 0, 1, \ldots, n-1$. Then by Proposition 2.3, $I_{B,[\gamma_k]} = J_{B,[\gamma_k]} = B$ for $k = 0, 1, \ldots, n-1$. Hence $J_{[\alpha]}B = J_{B,[\gamma_k]} = B$.

Theorem 2.7. *Let R be an integral domain with quotient field K. Let a be an element of R and n a positive integer. Let ω be a primitive n-th root of unity. Assume that the following five conditioins hold:*
(1) *α is an anti-integral element of degree d over R.*
(2) *$R[\omega]$ is a flat extension over R.*
(3) *$[K(\sqrt[n]{a}) : K] = n$.*
(4) *$[K(\sqrt[n]{a})(\alpha) : K(\sqrt[n]{a})] = d$.*
(5) *$I_{[\alpha]}$ is a finitely generated ideal of R.*
If $I^n_{[\alpha]}D(\sqrt[n]{a}) = J^n_{[\alpha]}$, then $J_{[\alpha]} = R$.

Proof. Set $B = R[\sqrt[n]{a}]$. First we will prove the case that ω is in R. Then $J_{[\alpha]}B = B$ by Proposition 2.6. This shows that $J_{[\alpha]} = R$ because B is an integral extension of R.

Next we will prove the general case. Set $R' = R[\omega]$ and $A' = R'[\alpha]$. Then the former case shows that $J_{[\alpha]}R' = J_{R',[\alpha]} = R'$. Since R' is an integral extension of R, we get $J_{[\alpha]} = R$.

Theorem 2.8. *Let R be an integral domain with quotient field K. Let a be an element of R and n a positive integer. Let ω be a primitive n-th root of unity. Assume that the following seven conditioins hold:*
(1) *α is an anti-integral element of degree d over R.*
(2) *$R[\omega]$ is a flat extension of R.*
(3) *$[K(\sqrt[n]{a}) : K] = n$.*
(4) *$[K(\sqrt[n]{a})(\alpha) : K(\sqrt[n]{a})] = d$.*
(5) *$[K(\omega)(\sqrt[n]{a}) : K(\omega)] = n$.*
(6) *$[K(\omega)(\sqrt[n]{a})(\alpha) : K(\omega)(\sqrt[n]{a})] = d$.*
(7) *$I_{[\alpha]}$ is a finitely generated ideal of R.*
Then the following conditions are equivalent:
(i) *$I^n_{[\alpha]}D(\sqrt[n]{a}) = J^n_{[\alpha]}$.*
(ii) *$I^n_{[\alpha]}D(\sqrt[n]{a}) = R$.*
(iii) *$\alpha^n - a$ is a unit of $R[\alpha]$.*

Proof. Equivalence (ii) $\Leftrightarrow$ (iii) is proved by Theorem 1.15. We will prove the implication (i) $\Rightarrow$ (ii). By Theorem 2.7 we have $J^n_{[\alpha]} = R$. Hence $I^n_{[\alpha]}D(\sqrt[n]{a}) = R$. Next we will prove the implication (ii) $\Rightarrow$ (i). Lemma 1.9 shows that $I^n_{[\alpha]}D(\sqrt[n]{a}) \subset J^n_{[\alpha]}$. Therefore by the condition (ii), we get $I^n_{[\alpha]}D(\sqrt[n]{a}) = R = J^n_{[\alpha]}$.

The converse of Theorem 2.7 is not true. We have an example that $J_{[\alpha]} = R$ but $I_{[\alpha]}\varphi_\alpha(a) \subsetneq J_{[\alpha]}$ as the following shows.

Example 2.9. Let $R = F[u, v]$ be a polynomial ring over a field F in two variables u and v. Let α be a root of $\varphi_\alpha(X) = X^2 + (v/u)X + 1/u$. It is easily verified that $\varphi_\alpha(X)$ is irreducible over the quotient field of R. Since R is a unique factorization domain, we can get $I_{[\alpha]} = uR$. Furthermore, $J_{[\alpha]} = I_{[\alpha]}(1, v/u, 1/u) = uR(1, v/u, 1/u) = (1.v) = R$. We will show that $I_{[\alpha]}\varphi_\alpha(a) \subsetneq J_{[\alpha]}$ for every non-zero element a of R. Assume that there exists a non-zero element a of R such that $I_{[\alpha]}\varphi_\alpha(a) = J_{[\alpha]}$. Then there exists an element b of R such that $(ua^2 + va + 1)b = 1$ because $J_{[\alpha]} = R$ and $I_{[\alpha]} = uR$. This implies that $ua^2 + va + 1$ is a unit of R, hence, a unit of F. Since a is not zero, we can draw a contradiction by comparing the degrees of the both sides of the equation above.

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