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FRACTIONAL INTEGRATION OF CERTAIN SPECIAL FUNCTIONS

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Abstract. We derive an Eulerian integral and a main theorem based upon the fractional integral operator associated with generalized polynomials given by Srivastava [8. 185, Eq. (7)] and H-function of several complex variables given by Srivastava and Panda [11, p.271, Eq. (4.1)] which provide unification and extension of numerous results in the theory of fractional calculus of special functions in one and more variables. Certain interesting sepcial cases (known and new) have also been discussed.

1. Introduction

In recent years certain fractional integrals deduced from Eulerian integrals have been established by several authors namely Saxena and Saigo [4], Saigo and Saxena [6], Srivastava and Hussain [10], Nishimoto and Saxena [5].

we start by giving the following definitions:

The Riemann-Liouville operator of fractional integration $R^m f$ of order m is defined by:

$$ {}_xD_y^{-m}[f(y)] = \frac{1}{\Gamma(m)} \int_x^y (y-t)^{m-1} f(t)dt. \tag{1.1}$$

for Re $(m) > 0$, and a constant x.

An equivalent form of beta function is [2, p.10, Eq. (1.3)]:

$$\int_m^n (t-m)^{a-1}(n-t)^{b-1}dt = (n-m)^{a+b-1}.B(a,b) \tag{1.2}$$

where m, $n \in \mathcal{R}$ $(m < n)$, Re $(a) > 0$, Re $(b) > 0$.

Using [2, p.62, Eq. (15)], we have:

$$\begin{aligned}(pt+q)^\alpha &= (xp+q)^\alpha \left[1 + \frac{p(t-x)}{xp+q}\right]^\alpha \\ &= \frac{(xp+q)^\alpha}{\Gamma(-\alpha)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(-\beta).\Gamma(\beta-\alpha) \left[\frac{p(t-x)}{xp+q}\right]^\beta d\beta. \end{aligned} \tag{1.3}$$

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where $i = \sqrt{-1}$; $p,\ q,\ \alpha \in C$; $x,\ t \in \mathcal{R}$; $|\arg\ (\frac{p}{xp+q})| < \pi$ and the path of integration is indented, if necessary in such a manner so as to separate the poles of $\Gamma(-\beta)$ from those of $\Gamma(\beta - \alpha)$.

The H-function of several complex variables [11] is defined in the following manner:

$$H[z_1, \ldots, z_r]$$
$$\equiv H^{0,\lambda:(u',v'):\cdots:(u^{(r)},v^{(r)})}_{A,C:[B',D']:\cdots:[B^{(r)},D^{(r)}]}\left[\begin{matrix}[(a)\colon \theta', \ldots, \theta^{(r)}]\colon [(b')\colon \Phi']; \cdots; [(b^{(r)})\colon \Phi^{(r)}];\\ [(c)\colon \psi', \ldots, \psi^{(r)}]\colon [(d')\colon \delta']; \cdots; [(d^{(r)})\colon \delta^{(r)}];\end{matrix} z_1, \ldots, z_r\right] \tag{1.4}$$
$$= \frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r}\psi(\xi_1, \ldots, \xi_r)\Phi_1(\xi_1)\cdots\Phi_r(\xi_r)z_1^{\xi_1}\cdots z_r^{\xi_r}d\xi_1\cdots d\xi_r \tag{1.5}$$

where $i = \sqrt{(-1)}$

The convergence conditions and other details of the H-function of several complex variables $H[z_1 \cdots z_r]$ are given by Srivastava, Gupta and Goyal [9, p.251].

The Lauricella function $F_D^{(h)}$ is defined in the integral form as

$$\frac{\Gamma(a)\Gamma(b_1)\cdots\Gamma(b_h)}{\Gamma(c)}F_D^{(h)}[a, b_1, \ldots, b_h; c; x_1, \ldots, x_h]$$
$$= \frac{1}{(2\pi i)^h}\int_{-i\infty}^{i\infty}\overset{(h)}{\cdots}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+\xi_1+\cdots+\xi_h)\Gamma(b_1+\xi_1)\cdots\Gamma(b_h+\xi_h)}{\Gamma(c+\xi_1+\cdots+\xi_h)}\Gamma(-\xi_1)\cdots$$
$$\Gamma(-\xi_h)(-x_1)^{\xi_1}\cdots(-x_h)^{\xi_h}d\xi_1\cdots d\xi_h, \tag{1.6}$$

where $\max[|\arg(-x_1)|, \ldots, |\arg(-x_h)|] < \pi$; $c \neq 0, -1, -2, \ldots$.

The following result will be used in establishing the Eulerian integral:

$$\int_x^y (t-x)^{a-1}(y-t)^{b-1}(p_1t+q_1)^{\rho_1}\cdots(p_ht+q_h)^{\rho_h}dt$$
$$= (y-x)^{a+b-1}B(a,b)(p_1x+q_1)^{\rho_1}\cdots(p_hx+q_h)^{\rho_h}$$
$$.F_D^{(h)}\left[a, -\rho_1, \ldots, -\rho_h; a+b; -\frac{(y-x)p_1}{p_1x+q_1}, \ldots, -\frac{(y-x)p_h}{p_hx+q_h}\right], \tag{1.7}$$

where $x,\ y \in \mathcal{R}$ $(x < y)$; $p_j,\ q_j,\ \rho_j \in C$ $(j = 1, \ldots, h)$;

$$\min[\mathrm{Re}(a), \mathrm{Re}(b)] > 0 \quad \text{and} \quad \max\left[\left|\frac{(y-x)p_1}{(xp_1+q_1)}\right|, \ldots, \left|\frac{(y-x)p_h}{xp_h+q_h}\right|\right] < 1.$$

The formula (1.7) can be developed by making use of (1.2), (1.3) and (1.6).

The known results [3, p.301, entry (2.2.6.1)] and [10, p.81, Eq. (3.6)] are deducible for $h = 1$ and $h = 2$ respectively.

In what follows h is a positive integer and $0, \ldots, 0$ would mean h zeros.

The generalized polynomials defined by Srivastava [8], is as follows:

$$S^{M_1,\ldots,M_k}_{N_1,\ldots,N_k}[x_1, \ldots, x_k]$$
$$= \sum_{\alpha_1=0}^{[N_1/M_1]}\cdots\sum_{a_k=0}^{[N_k/M_k]}\frac{(-N_1)_{M_1\alpha_1}}{\alpha_1!}\cdots\frac{(-N_k)_{M_k\alpha_k}}{\alpha_k!}.B[N_1, \alpha_1; \cdots; N_k, \alpha_k]x_1^{\alpha_1}\cdots x_k^{\alpha_k}, \tag{1.8}$$

where $N_{i'} = 0, 1, 2, \dots \ \forall i' = (1, \dots, k)$, $M_1, \dots, M_k$ arbitrary positive integers and the coefficient $B[N_1, \alpha_1; \cdots; N_k, \alpha_k]$ are arbitrary constants, real or complex.

2. Eulerian Integral

The main integral to be established here is

$$\int_m^n (t-m)^{a-1}(n-t)^{b-1}\left\{\prod_{j=1}^h (p_j t + q_j)^{\rho_j}\right\}$$

$$.S_{N_1,\dots,N_k}^{M_1,\dots,M_k}\begin{bmatrix} x_1(t-m)^{\lambda_1}(n-t)^{\mu_1}\prod_{j=1}^h (p_j t+q_j)^{a'_j} \\ \vdots \\ x_k(t-m)^{\lambda_k}(n-t)^{\mu_k}\prod_{j=1}^h (p_j t+q_j)^{a_j^{(k)}} \end{bmatrix}$$

$$.H\begin{bmatrix} w_1(t-m)^{\gamma_1}(n-t)^{\tau_1}\prod_{j=1}^h (p_j t+q_j)^{-c'_j} \\ \vdots \\ w_r(t-m)^{\gamma_r}(n-t)^{\tau_r}\prod_{j=1}^h (p_j t+q_j)^{-c_j^{(r)}} \end{bmatrix} dt$$

$$= G_1 \sum_{\alpha_1}^{[N_1/M_1]} \cdots \sum_{\alpha_k}^{[N_k/M_k]} \frac{(-N_1)_{M_1\alpha_1}}{\alpha_1!} \cdots \frac{(-N_k)_{M_k\alpha_k}}{\alpha_k!} B[N_1, \alpha_1; \cdots; N_k, \alpha_k] x_1^{\alpha_1} \cdots x_k^{\alpha_k}$$

$$.G_2 H^{0,\lambda+h+2:(u',v');\cdots;(u^{(r)},v^{(r)});(1,0);\cdots;(1,0)}_{A+h+2,C+h+1:[B',D'];\cdots;[B^{(r)},D^{(r)}];[0,1];\cdots;[0,1]}$$

$$\left[\begin{matrix} A_1, A_2, A_3, [(a)\colon \theta', \dots, \theta^{(r)}, 0, \dots, 0] \colon [(b')\colon \Phi'];\cdots;[(b^{(r)})\colon \Phi^{(r)}]; -;\cdots;-;R_1 \\ [(c)\colon \psi', \dots, \psi^{(r)}, 0, \dots, 0], A_4, A_5 \colon [(d')\colon \delta'];\cdots;[(d^{(r)})\colon \delta^{(r)}];[0,1];\cdots;[0,1];R_2 \end{matrix}\right] \quad (2.1)$$

The followings are the conditions of the validity of (2.1):

(1) $m, n \in \mathcal{R}$ $(m < n)$; γ_i, τ_i, $c_j^{(i)}$, $\lambda_{i'}$, $\mu_{i'}$, $a_j^{(i')} \in \mathcal{R}^+$, $\rho_j \in \mathcal{R}$,
p_j, $q_j \in C$, $w_i \in C$ $(i = 1, \dots, r; i' = 1, \dots, k; j = 1, \dots, h)$;

(2) $\max_{1\le j\le h}\left[\left|\dfrac{(n-m)p_j}{mp_j+q_j}\right|\right] < 1$;

(3) $\mathrm{Re}\left[a + \sum_{i=1}^r \dfrac{\gamma_i d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$ $(j = 1, \dots, u^{(i)})$,

$\mathrm{Re}\left[b + \sum_{i=1}^r \dfrac{\tau_i d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$ $(j = 1, \dots, u^{(i)})$;

(4) $R'_i = \sum_{j=1}^A \theta_j^{(i)} - \sum_{j=1}^C \psi_j^{(i)} + \sum_{j=1}^{B^{(i)}} \Phi_j^{(i)} - \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} \le 0$,

$$T_i = -\sum_{j=\lambda+1}^{A} \theta_j^{(i)} - \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{v^{(i)}} \Phi_j^{(i)} - \sum_{j=v^{(i)}+1}^{B^{(i)}} \Phi_j^{(i)} + \sum_{j=1}^{u^{(i)}} \delta_j^{(i)} - \sum_{j=u^{(i)}+1}^{D^{(i)}} \delta_j^{(i)} - \gamma_i$$
$$-\tau_i - \textstyle\sum_{j=1}^{h} c_j^{(i)} > 0 \ (i = 1, \ldots, r);$$

(5) $$\left| \arg(w_i) \prod_{j=1}^{h} (p_j t + q_j)^{-c_j^{(i)}} \right| < \frac{T_i \pi}{2} \ (m \le t \le n; i = 1, \ldots, r).$$

Here

$$G_1 = (n-m)^{(a+b-1)} \left(\prod_{j=1}^{h} (p_j m + q_j)^{\rho_j} \right),$$

$$G_2 = (n-m)^{\sum_{i'=1}^{k} (\lambda_{i'} + \mu_{i'}) \alpha_{i'}} \left(\prod_{j=1}^{h} (p_j m + q_j)^{\sum_{i'=1}^{k} a_j^{(i')} \alpha_{i'}} \right),$$

$$A_1 = \left[1 - a - \sum_{i'=1}^{k} \lambda_{i'} \alpha_{i'} : \gamma_1, \ldots, \gamma_r, 1, \ldots, 1 \right],$$

$$A_2 = \left[1 - b - \sum_{i'=1}^{k} \mu_{i'} \alpha_{i'} : \tau_1, \ldots, \tau_r, 0, \ldots, 0 \right],$$

$$A_3 = \left[1 + \rho_j + \sum_{i'=1}^{k} a_j^{(i')} \alpha_{i'} : c'_j, \ldots, c_j^{(r)}, 0, \ldots, \overset{j}{1}, \ldots, 0 \right]_{1.h},$$

$$A_4 = \left[1 + \rho_j + \sum_{i'=1}^{k} a_j^{(i')} \alpha_{i'} : c'_j, \ldots, c_j^{(r)}, 0, \ldots, 0 \right]_{1.h},$$

$$A_5 = \left[1 - a - b - \sum_{i'=1}^{k} (\lambda_{i'} + \mu_{i'}) \alpha_{i'} : (\gamma_1 + \tau_1), \ldots, (\gamma_r + \tau_r), 1, \ldots, 1 \right];$$

$$R_1 = \begin{cases} w_1 (n-m)^{\gamma_1 + \tau_1} \Big/ \prod_{j=1}^{h} (m p_j + q_j)^{c'_j} \\ \vdots \\ w_r (n-m)^{\gamma_r + \tau_r} \Big/ \prod_{j=1}^{h} (m p_j + q_j)^{c_j^{(r)}}, \end{cases}$$

$$R_2 = \begin{cases} (n-m) p_1 / (m p_1 + q_1) \\ \vdots \\ (n-m) p_h / (m p_h + q_h). \end{cases}$$

Proof. In order to prove (2.1), express the multivariable H-function in terms of Mellin-Barnes type of contour integrals by (1.5) and generalized polynomials given by (1.8) and interchange the order of summation and integration (which is permissible under the conditions of validity stated above). Appealing to the results in (1.3), (1.6) and (1.7), we get the desired result.

3. Interesting Special Cases

I. For $\gamma_1 = 0 = \cdots = \gamma_r$ and $\lambda_1 = 0 = \cdots = \lambda_k$, the integral (2.1) reduces to

$$\int_m^n (t-m)^{a-1}(n-t)^{b-1}\left\{\prod_{j=1}^{h}(p_jt+q_j)^{\rho_j}\right\} S_{N_1,\ldots,N_k}^{M_1,\ldots,M_k}\begin{bmatrix} x_1(n-t)^{\mu_1}\prod_{j=1}^{h}(p_jt+q_j)^{a'_j} \\ \vdots \\ x_k(n-t)^{\mu_k}\prod_{j=1}^{h}(p_jt+q_j)^{a_j^{(k)}} \end{bmatrix}$$

$$.H\begin{bmatrix} w_1(n-t)^{\tau_1}\prod_{j=1}^{h}(p_jt+q_j)^{-c'_j} \\ \vdots \\ w_r(n-t)^{\tau_r}\prod_{j=1}^{h}(p_jt+q_j)^{-c_j^{(r)}} \end{bmatrix} dt$$

$$= E_1\sum_{\alpha_1=0}^{[N_1/M_1]}\cdots\sum_{\alpha_k=0}^{[N_k/M_k]}\frac{(-N_1)_{M_1\alpha_1}}{\alpha_1!}\cdots\frac{(-N_k)_{M_k\alpha_k}}{\alpha_k!}B[N_1,\alpha_1;\cdots;N_k,\alpha_k]x_1^{\alpha_1}\cdots x_k^{\alpha_k}$$

$$.E_2 H^{0,\lambda+h+2:(u',v');\cdots;(u^{(r)},v^{(r)});(1,0);\cdots;(1,0)}_{A+h+2,C+h+1:[B',D'];\cdots;[B^{(r)},D^{(r)}];[0,1];\cdots;[0,1]}$$

$$\left[\begin{matrix} F_1,F_2,F_3,[(a):\theta',\ldots,\theta^{(r)},0,\ldots,0]:[(b'):\Phi'];\cdots;[(b^{(r)}):\Phi^{(r)}];-;\cdots;-;P_1 \\ [(c):\psi',\ldots,\psi^{(r)},0,\ldots,0],F_4,F_5:[(d'):\delta'];\cdots;[d^{(r)}:\delta^{(r)}];[0,1];\cdots;[0,1];P_2 \end{matrix}\right], \quad (3.1)$$

which holds under the conditions surrounding equation (2.1).

Here

$$E_1 = (n-m)^{(a+b-1)}\left(\prod_{j=1}^{h}(p_jm+q_j)^{\rho_j}\right)$$

$$E_2 = (n-m)^{\sum_{i'=1}^{k}\mu_{i'}\alpha_{i'}}\left(\prod_{j=1}^{h}(p_jm+q_j)^{\sum_{i'=1}^{k}a_j^{(i')}\alpha_{i'}}\right);$$

$$F_1 = [1-a:\overbrace{0,\ldots,0}^{r},1,\ldots,1],$$

$$F_2 = \left[1-b-\sum_{i'=1}^{k}\mu_{i'}\alpha_{i'}:\tau_1,\ldots,\tau_r,0,\ldots,0\right],$$

$$F_3 = \left[1+\rho_j+\sum_{i'=1}^{k}a_j^{(i')}\alpha_{i'}:c'_j,\ldots,c_j^{(r)},0,\ldots,\overset{j}{1},\ldots,0\right]_{1.h},$$

$$F_4 = \left[1+\rho_j+\sum_{i'=1}^{k}a_j^{(i')}\alpha_{i'}:c'_j,\ldots,c_j^{(r)},0,\ldots,0\right]_{1.h},$$

$$F_5 = \left[1-a-b-\sum_{i'=1}^{k}\mu_{i'}\alpha_{i'} : \tau_1,\ldots,\tau_r,1,\ldots,1\right];$$

$$P_1 = \begin{cases} w_1(n-m)^{\tau_1}\Big/\prod_{j=1}^{h}(p_jm+q_j)^{c'_j} \\ \vdots \\ w_r(n-m)^{\tau_r}\Big/\prod_{j=1}^{h}(p_jm+q_j)^{c_j^{(r)}}, \end{cases}$$

$$P_2 = \begin{cases} (n-m)p_1/(mp_1+q_1) \\ \vdots \\ (n-m)p_h/(mp_h+q_h). \end{cases}$$

II. For $\tau_1 = 0 = \cdots = \tau_r$ and $\mu_1 = 0 = \cdots = \mu_k$, integral (2.1) reduces to

$$\int_m^n (t-m)^{a-1}(n-t)^{b-1}\left\{\prod_{j=1}^{h}(p_jt+q_j)^{\rho_j}\right\} S_{N_1,\ldots,N_k}^{M_1,\ldots,M_k}\begin{bmatrix} x_1(t-m)^{\lambda_1}\prod_{j=1}^{h}(p_jt+q_j)^{a'_j} \\ \vdots \\ x_k(t-m)^{\lambda_k}\prod_{j=1}^{h}(p_jt+q_j)^{a_j^{(k)}} \end{bmatrix}$$

$$.H\begin{bmatrix} w_1(t-m)^{\gamma_1}\prod_{j=1}^{h}(p_jt+q_j)^{-c'_j} \\ \vdots \\ w_r(t-m)^{\gamma_r}\prod_{j=1}^{h}(p_jt+q_j)^{-c_j^{(r)}} \end{bmatrix} dt$$

$$= \Gamma(b)T_1\sum_{\alpha_1=0}^{[N_1/M_1]}\cdots\sum_{\alpha_k=0}^{[N_k/M_k]}\frac{(-N_1)_{M_1\alpha_1}}{\alpha_1!}\cdots\frac{(-N_k)_{M_k\alpha_k}}{\alpha_k!}B[N_1,\alpha_1;\cdots;N_k,\alpha_k]x_1^{\alpha_1}\cdots x_k^{\alpha_k}$$

$$.T_2H^{0,\lambda+h+1:(u',v');\cdots;(u^{(r)},v^{(r)});(1,0);\cdots;(1,0)}_{A+h+1,C+h+1:[B',D'];\cdots;[B^{(r)},D^{(r)}];[0,1];\cdots;[0,1]}$$

$$\left[\begin{matrix} Q_1,Q_2,[(a):\theta',\ldots,\theta^{(r)},0,\ldots,0]:[b':\Phi'];\cdots;[b^{(r)}:\Phi^{(r)}];-;\cdots;-;X_1 \\ [(c):\psi',\ldots,\psi^{(r)},0,\ldots,0],Q_3,Q_4:[d':\delta'];\cdots;[d^{(r)}:\delta^{(r)}];[0,1];\cdots;[0,1];X_2 \end{matrix}\right], \quad (3.2)$$

which holds true under the same conditions as given in (2.1).

Where

$$T_1 = (n-m)^{(a+b-1)}\left(\prod_{j=1}^{h}(p_jm+q_j)^{\rho_j}\right),$$

$$T_2 = (n-m)^{\sum_{i'=1}^{k}\lambda_{i'}\alpha_{i'}}\left(\prod_{j=1}^{h}(p_jm+q_j)^{\sum_{i'=1}^{k}a_j^{(i')}\alpha_{i'}}\right);$$

$$Q_1 = \left[1-a-\sum_{i'=1}^{k}\lambda_{i'}\alpha_{i'} : \gamma_1,\ldots,\gamma_r,1,\ldots,1\right],$$

$$Q_2 = \left[1+\rho_j+\sum_{i'=1}^{k}a_j^{i'}\alpha_{i'} : c'_j,\ldots,c_j^{(r)},0,\ldots,\overset{j}{1},\ldots,0\right]_{1.h},$$

$$Q_3 = \left[1 + \rho_j + \sum_{i'=1}^{k} a_j^{i'} \alpha_{i'} : c'_j, \ldots, c_j^{(r)}, 0, \ldots, 0\right]_{1.h},$$

$$Q_4 = \left[1 - a - b - \sum_{i'=1}^{k} \lambda_{i'} \alpha_{i'} : \gamma_1, \ldots, \gamma_r, 1, \ldots, 1\right];$$

$$X_1 = \begin{cases} w_1 (n-m)^{\gamma_1} \Big/ \prod_{j=1}^{h} (p_j m + q_j)^{c'_j} \\ \vdots \\ w_r (n-m)^{\gamma_r} \Big/ \prod_{j=1}^{h} (p_j m + q_j)^{c_j^{(r)}}, \end{cases}$$

$$X_2 = \begin{cases} (n-m) p_1 / (m p_1 + q_1) \\ \vdots \\ (n-m) p_h / (m p_h + q_h). \end{cases}$$

III. When $\tau_1 = \cdots = \tau_r = 0 = \gamma_1 = \cdots = \gamma_r$ and $\lambda_1 = \cdots = \lambda_k = 0 = \mu_1 = \cdots = \mu_k$, equation (2.1) reduces to

$$\int_m^n (t-m)^{a-1} (n-t)^{b-1} \left\{ \prod_{j=1}^{h} (p_j t + q_j)^{\rho_j} \right\} S_{N_1,\ldots,N_k}^{M_1,\ldots,M_k} \begin{bmatrix} x_1 \prod_{j=1}^{h} (p_j t + q_j)^{a'_j} \\ \vdots \\ x_k \prod_{j=1}^{h} (p_j t + q_j)^{a_j^{(k)}} \end{bmatrix}$$

$$.H \begin{bmatrix} w_1 \prod_{j=1}^{h} (p_j t + q_j)^{-c'_j} \\ \vdots \\ w_r \prod_{j=1}^{h} (p_j t + q_j)^{-c_j^{(r)}} \end{bmatrix} dt$$

$$= \Gamma(b) L_1 \sum_{\alpha_1=0}^{[N_1/M_1]} \cdots \sum_{\alpha_k=0}^{[N_k/M_k]} \frac{(-N_1)_{M_1\alpha_1}}{\alpha_1!} \cdots \frac{(-N_k)_{M_k\alpha_k}}{\alpha_k!} B[N_1, \alpha_1; \cdots; N_k, \alpha_k] x_1^{\alpha_1} \cdots x_k^{\alpha_k}$$

$$.L_2 H^{0,\lambda+h+1:(u',v');\cdots;(u^{(r)},v^{(r)});(1,0);\cdots;(1,0)}_{A+h+1,C+h+1:[B',D'];\cdots;[B^{(r)},D^{(r)}];[0,1];\cdots;[0,1]}$$

$$\left[\begin{matrix} B_1, B_2, [(a) \colon \theta', \ldots, \theta^{(r)}, 0, \ldots, 0] \colon [b' \colon \Phi']; \cdots; [b^{(r)} \colon \Phi^{(r)}]; -; \cdots; -; Y_1 \\ [(c) \colon \psi', \ldots, \psi^{(r)}, 0, \ldots, 0], B_3, B_4 \colon [d' \colon \delta']; \cdots; [d^{(r)} \colon \delta^{(r)}]; [0,1]; \cdots; [0,1]; Y_2 \end{matrix}\right], \quad (3.3)$$

valid under the same conditions as required in (2.1).

Where

$$L_1 = (n-m)^{(a+b-1)} \left(\prod_{j=1}^{h} (p_j m + q_j)^{\rho_j} \right),$$

$$L_2 = \left\{ \prod_{j=1}^{h} (p_j m + q_j)^{\sum_{i'=1}^{k} a_j^{(i')} \alpha_{i'}} \right\};$$

$$B_1 = [(1-a) : \overbrace{0,\ldots,0}^{r}, 1,\ldots,1],$$

$$B_2 = \left[1+\rho_j+\sum_{i'=1}^{k} a_j^{i'}\alpha_{i'} : c_j',\ldots,c_j^{(r)},0,\ldots,\overset{j}{1},\ldots,0\right]_{1.h},$$

$$B_3 = \left[1+\rho_j+\sum_{i'=1}^{k} a_j^{i'}\alpha_{i'} : c_j',\ldots,c_j^{(r)},0,\ldots,0\right]_{1.h},$$

$$B_4 = [(1-a-b) : \overbrace{0,\ldots,0}^{r}, 1,\ldots,1];$$

$$Y_1 = \begin{cases} \prod_{j=1}^{h}(p_j m+q_j)^{-c_j'} \\ \vdots \\ \prod_{j=1}^{h}(p_j m+q_j)^{-c_j^{(r)}}, \end{cases}$$

$$Y_2 = \begin{cases} (n-m)p_1/(mp_1+q_1) \\ \vdots \\ (n-m)p_h/(mp_h+q_h). \end{cases}$$

4. Main Theorem

Let

$$f(t) = (t-m)^{a-1}\left\{\prod_{j=1}^{h}(p_j t+q_j)^{\rho_j}\right\} S_{N_1,\ldots,N_k}^{M_1,\ldots,M_k}\begin{bmatrix} X_1(t-m)^{\lambda_1}\prod_{j=1}^{h}(p_j t+q_j)^{a_j'} \\ \vdots \\ X_k(t-m)^{\lambda_k}\prod_{j=1}^{h}(p_j t+q_j)^{a_j^{(k)}} \end{bmatrix}$$

$$.H\begin{bmatrix} z_1(t-m)^{\gamma_1}\prod_{j=1}^{h}(p_j t+q_j)^{-c_j'} \\ \vdots \\ z_r(t-m)^{\gamma_r}\prod_{j=1}^{h}(p_j t+q_j)^{-c_j^{(r)}} \end{bmatrix}.$$

then

$$\begin{aligned} &{}_mD_y^{-b}[f(y)] \\ &= I_1 \sum_{\alpha_1=0}^{[N_1/M_1]}\cdots\sum_{\alpha_k=0}^{[N_k/M_k]} \frac{(-N_1)_{M_1\alpha_1}}{\alpha_1!}\cdots\frac{(-N_k)_{M_k\alpha_k}}{\alpha_k!} B[N_1,\alpha_1;\cdots;N_k,\alpha_k] X_1^{\alpha_1}\cdots X_k^{\alpha_k} \\ &.I_2 H^{0,\lambda+h+1:(u',v');\cdots;(u^{(r)},v^{(r)});(1,0);\cdots;(1,0)}_{A+h+1,C+h+1:[B',D'];\cdots;[B^{(r)},D^{(r)}];[0,1];\cdots;[0,1]} \\ &\left[\begin{matrix} K_1,K_2,[(a):\theta',\ldots,\theta^{(r)},0,\ldots,0]:[b':\Phi'];\cdots;[b^{(r)}:\Phi^{(r)}];-;\cdots;-;W_1 \\ [(c):\psi',\ldots,\psi^{(r)},0,\ldots,0],K_3,K_4:[d':\delta'];\cdots;[d^{(r)}:\delta^{(r)}];[0,1];\cdots;[0,1];W_2 \end{matrix}\right], \quad (4.1) \end{aligned}$$

holds true with the conditions associated with (2.1).

Where

$$I_1 = (y-m)^{a+b-1}\left(\prod_{j=1}^{h}(p_jm+q_j)^{\rho_j}\right),$$

$$I_2 = (y-m)^{\sum_{i'=1}^{k}\lambda_{i'}\alpha_{i'}}\left(\prod_{j=1}^{h}(p_jm+q_j)^{\sum_{i'=1}^{k}a_j^{(i')}\alpha_{i'}}\right);$$

$$K_1 = \left[1-a-\sum_{i'=1}^{k}\lambda_{i'}\alpha_{i'} : \gamma_1,\ldots,\gamma_r,1,\ldots,1\right],$$

$$K_2 = \left[1+\rho_j+\sum_{i'=1}^{k}a_j^{i'}\alpha_{i'} : c_j',\ldots,c_j^{(r)},0,\ldots,\overset{j}{1},\ldots,0\right]_{1.h},$$

$$K_3 = \left[1+\rho_j+\sum_{i'=1}^{k}a_j^{i'}\alpha_{i'} : c_j',\ldots,c_j^{(r)},0,\ldots,0\right]_{1.h},$$

$$K_4 = \left[1-a-b-\sum_{i'=1}^{k}\lambda_{i'}\alpha_{i'} : \gamma_1,\ldots,\gamma_r,1,\ldots,1\right];$$

$$W_1 = \begin{cases} z_1(y-m)^{\gamma_1}\Big/\prod_{j=1}^{h}(p_jm+q_j)^{c_j'} \\ \vdots \\ z_r(y-m)^{\gamma_r}\Big/\prod_{j=1}^{h}(p_jm+q_j)^{c_j^{(r)}}, \end{cases}$$

$$W_2 = \begin{cases} (y-m)p_1/(mp_1+q_1) \\ \vdots \\ (y-m)p_h/(mp_h+q_h). \end{cases}$$

5. Special Cases

1. If we set $\gamma_1=\cdots=\gamma_r=0$ and $\lambda_1=\cdots=\lambda_k=0$, then (4.1) reduces to

$$\begin{aligned} &{}_mD_y^{-b}[f(y)] \\ &= I_1\sum_{\alpha_1=0}^{[N_1/M_1]}\cdots\sum_{\alpha_k=0}^{[N_k/M_k]}\frac{(-N_1)_{M_1\alpha_1}}{\alpha_1!}\cdots\frac{(-N_k)_{M_k\alpha_k}}{\alpha_k!}B[N_1,\alpha_1;\cdots;N_k,\alpha_k]X_1^{\alpha_1}\cdots X_k^{\alpha_k} \\ &.I_2H^{0,\lambda+h+1:(u',v');\cdots;(u^{(r)},v^{(r)});(1,0);\cdots;(1,0)}_{A+h+1,C+h+1:[B',D'];\cdots;[B^{(r)},D^{(r)}];[0,1];\cdots;[0,1]} \\ &\left[\begin{matrix} K_1,K_2,[(a):\theta',\ldots,\theta^{(r)},0,\ldots,0]:[b':\Phi'];\cdots;[b^{(r)}:\Phi^{(r)}];-;\cdots;-;W_1 \\ [(c):\psi',\ldots,\psi^{(r)},0,\ldots,0],K_3,K_4:[d':\delta'];\cdots;[d^{(r)}:\delta^{(r)}];[0,1];\cdots;[0,1];W_2 \end{matrix}\right], \quad (5.1) \end{aligned}$$

valid under the same conditions as required for integral (2.1) and where I_1, I_2, K_1,

K_2, K_3, K_4, W_1 and W_2 are the same as in integral (4.1) after eliminating $\lambda_{i'}$ and $\gamma_{i'}$, $(i' = 1, \ldots, k; i = 1, \ldots, r)$.

2. Setting $M_i \to 0$, the results in (2.1) and (4.1) reduce to known results given by Saigo and Saxena [4, Eq. 4.1, Eq. 6.1].

3. If we set $M_i \to 0$, $\gamma_1 = \cdots = \gamma_r = 0 = \tau_1 = \cdots = \tau_r$, $h = 2$ then our results (2.1) and (4.1) reduce to the results derived by Srivastava and Hussain [10, Eq. 2.5, Eq. 3.14].

4. For $k = 1$, the results in (2.1) and (4.1) can be reduced to the results recently obtained by Chaurasia and Godika [1, Eq. 3.1, Eq. 4.1] with $s = 1$.

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