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SOME UNIFORMLY STARLIKE FUNCTIONS WITH VARYING ARGUMENTS

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Abstract. In this paper two new subclasses of starlike functions that are analytic and normalized in the open unit disc with varying arguments is introduced. For functions in these classes we obtained coefficient bound, distortion results and the extreme points.

1. Introduction

Denote by S the family of functions $f(z)$ of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{1.1}$$

that are analytic, univalent and normalized in the unit disc $U = \{z : |z| < 1\}$ and by CV and ST the subfamilies of S that are respectively convex and starlike. Goodman [2, 3] defined the following subclass of CV and ST.

Definition 1.1. [2, 3] A function f is uniformly convex UCV (uniformly starlike UST) in U if f is in CV (ST) and has the property that for every circular arc, γ contained in U, with centre ξ also in U, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\xi)$.

Rønning [6] and Ma and Minda [5] indepentently characterised the class UCV analytically by

$$UCV = \left\{ f \in S : \left| \frac{zf''(z)}{f'(z)} \right| \leq \mathrm{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\}, \quad z \in U \right\} \tag{1.2}$$

and Rønning [6] defined a new subclass of starlike functions related to UST by

$$S_p = \left\{ f \in S : \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\}, \quad z \in U \right\}. \tag{1.3}$$

Note that $f(z) \in UCV \iff zf'(z) \in S_p$. Further Rønning [7, 8] generalized the class UCV and S_p by introducing a parameter α in the following way

$$S_p(\alpha) = \left\{ f \in S : \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} - \alpha, \quad z \in U \right\}$$

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and $f(z)$ is in $UCV(\alpha)$ if and only if $zf'(z) \in S_p(\alpha)$.

Recently Kanas and Srivastava [4] and Srivastava and Mishra [11] defined the new subclasses of the families UCV and UST making use of linear operators and fractional calculus respectively and obtained various interesting properties. In light of this in this paper we study the classes UCV and S_p defining by Ruscheweyh derivative operator.

For functions $f(z)$ of the form (1.1) Ruscheweyh [9] defined the derivative operator $D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z)$ where $(*)$ stands for Hadamard product (or convolution product) of two power series, equivalently $D^n f(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!} n > -1$. Al-Amiri [1] called this $D^n f(z)$ as the n^{th} order Ruscheweyh derivative of $f(z)$. It is easy to see that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$ and

$$D^n f(z) = z + \sum_{m=2}^{\infty} c(n,m) a_m z^m, \quad \text{where } c(n,m) = \binom{n+m-1}{n} \tag{1.4}$$

For $0 \le \alpha < 1$, we let $S_p(n,\alpha)$ denote the new subclass of family of starlike functions corresponding to the families UCV and UST and for functions $f(z)$ of the form (1.1) such that

$$\operatorname{Re}\left\{\frac{z(D^n f(z))'}{D^n f(z)} - \alpha\right\} \ge \left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| \tag{1.5}$$

where $D^n f(z)$ is defined by (1.4).

Definition 1.2. [10] A function $f(z)$ of the form (1.1) is said to be in the class $V(\theta_m)$ if $f \in S$ and arg $(a_m) = \theta_m$ for all $m \ge 2$. If furthermore there exist a real number β such that $\theta_m + (m-1)\beta \equiv \pi (\text{mod} 2\pi)$, then $f(z)$ is said to be in the class $V(\theta_m, \beta)$. The union of $V(\theta_m, \beta)$ taken over all possible sequences $\{\theta_m\}$ and all possible real numbers β is denote by V.

In this paper we introduce two new subclasses $VS_p(n,\alpha)$ and $VS_p(n,\alpha,\beta)$ of starlike functions $S_p(\alpha)$ with varying arguments. First we obtain a sufficient coefficient bound for functions in $S_p(n,\alpha)$. We prove that these coefficient conditions are also necessary for functions in the classes $VS_p(n,\alpha)$ and $VS_p(n,\alpha,\beta)$ further we obtained distortion bounds and the extreme points for functions in these classes.

2. The Class $VS_p(n,\alpha)$

Definition 2.1. For $0 \le \alpha < 1$ we define $VS_p(n,\alpha) = S_p(n,\alpha) \cap V$.

In our first theorem, we obtain a sufficient coefficient bound for functions in $S_p(n,\alpha)$.

Theorem 2.1. *Let $f(z)$ be given by* (1.1). *If*

$$\sum_{m=2}^{\infty} (2m-1-\alpha) c(n,m) |a_m| \le 1-\alpha, \quad 0 \le \alpha < 1, \tag{2.1}$$

then $f(z) \in S_p(n,\alpha)$.

Proof. By definition of the class $S_p(n,\alpha)$ it suffices to show that

$$\left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| \leq \operatorname{Re}\left\{\frac{z(D^n f(z))'}{D^n f(z)} - \alpha\right\}$$

That is

$$\left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| - \operatorname{Re}\left\{\frac{z(D^n f(z))'}{D^n f(z)} - 1\right\} \leq 2\left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right|$$
$$\leq 2\frac{\sum_{m=2}^{\infty}(m-1)c(n,m)|a_m|\,|z|^{m-1}}{1-\sum_{m=2}^{\infty}c(n,m)|a_m|\,|z|^{m-1}}.$$

Now the last expression is bounded above by $(1-\alpha)$ and only if $\sum_{m=2}^{\infty}(2m-1-\alpha)$ $c(n,m)|a_m| \leq 1-\alpha$. In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f \in V$.

Theorem 2.2. *Let $f(z)$ of the form* (1.1)*, f is in $VS_p(n,\alpha)$ if and only if*

$$\sum_{m=2}^{\infty}(2m-1-\alpha)c(n,m)|a_m| \leq 1-\alpha.$$

Proof. In view of theorem (2.1) we need only to show that f is in $VS_p(n,\alpha)$ satisfies the coefficient inequality. If $f \in VS_p(n,\alpha)$ then by definition.

$$\left|\frac{z+\sum_{m=2}^{\infty}mc(n,m)a_m z^m}{z+\sum_{m=2}^{\infty}c(n,m)a_m z^m} - 1\right| \leq \operatorname{Re}\left\{\frac{z+\sum_{m=2}^{\infty}mc(n,m)a_m z^m}{z+\sum_{m=2}^{\infty}c(n,m)a_m z^m} - \alpha\right\}$$

That is

$$\left|\frac{\sum_{m=2}^{\infty}(m-1)c(n,m)a_m z^{m-1}}{1+\sum_{m=2}^{\infty}c(n,m)a_m z^{m-1}}\right| \leq \operatorname{Re}\left\{\frac{(1-\alpha)+\sum_{m=2}^{\infty}(m-\alpha)c(n,m)a_m z^{m-1}}{1+\sum_{m=2}^{\infty}c(n,m)a_m z^{m-1}}\right\}$$

since $f(z) \in V$ and $f(z)$ lies in $V(\theta_m,\beta)$ for some sequence $\{\theta_m\}$ and a real number β such that $\theta_m + (m-1)\beta \equiv \pi(\mathrm{mod} 2\pi)$ set $z = re^{i\beta}$ in the above inequality

$$\frac{\sum_{m=2}^{\infty}(m-1)c(n,m)a_m r^{m-1}}{1-\sum_{m=2}^{\infty}c(n,m)a_m r^{m-1}} \leq \frac{(1-\alpha)-\sum_{m=2}^{\infty}(m-\alpha)c(n,m)a_m r^{m-1}}{1-\sum_{m=2}^{\infty}c(n,m)a_m r^{m-1}}.$$

Letting $r \to 1$, leads the desired inequality $\sum_{m=2}^{\infty}(2m-1-\alpha)c(n,m)|a_m| \leq 1-\alpha$.

Corollary 2.1. *If $f \in VS_p(n,\alpha)$ then $|a_m| \leq \frac{1-\alpha}{(2m-1-\alpha)c(n,m)}$ for $m \geq 2$. The inequality holds for $f(z) = z + \sum_{m=2}^{\infty}\frac{(1-\alpha)e^{i\theta_m}z^m}{(2m-1-\alpha)c(n,m)}$ for $m \geq 2$, $z \in U$.*

Theorem 2.3. (Distortion theorem) *Let $f(z)$ of the form* (1.1) *be in the class $VS_p(n,\alpha)$. Then*

$$r - \frac{1-\alpha}{(3-\alpha)c(n,2)}r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{(3-\alpha)c(n,2)}r^2$$

and

$$1-\frac{2(1-\alpha)}{(3-\alpha)c(n,2)}r \le |f'(z)| \le 1+\frac{2(1-\alpha)}{(3-\alpha)c(n,2)}r.\quad \textit{The result is sharp.}$$

Proof. Let $f(z)$ of the form (1.1) be in the class $VS_p(n,\alpha)$. By taking absolute value of $f(z)$

$$|f(z)| = \left| z + \sum_{m=2}^{\infty} a_m z^m \right| \le |z| + |z|^2 \sum_{m=2}^{\infty} |a_m|,$$

since $f(z) \in VS_p(n,\alpha)$ and by Theorem (2.1), we have

$$(3-\alpha)c(n,2)\sum_{m=2}^{\infty}|a_m| \le \sum_{m=2}^{\infty}(2m-1-\alpha)c(n,m)|a_m| \le 1-\alpha.$$

Thus $\quad |f(z)| \le |z| + \frac{1-\alpha}{(3-\alpha)c(n,2)}|z|^2.$

That is $\quad |f(z)| \le r + \frac{1-\alpha}{(3-\alpha)c(n,2)}r^2,$

similarly we get $\quad |f(z)| \ge r - \frac{1-\alpha}{(3-\alpha)c(n,2)}r^2.$

On other hand $\quad f'(z) = 1 + \sum_{m=2}^{\infty} m a_m z^{m-1},$

and $\quad |f'(z)| = 1 + \sum_{m=2}^{\infty} m|a_m|\,|z|^{m-1} \le 1 + |z|\sum_{m=2}^{\infty} m|a_m|,$ since $f(z) \in VS_p(n,\alpha)$.

Then by Theorem (2.1) we have $\quad \sum_{m=2}^{\infty} m|a_m| \le \frac{2(1-\alpha)}{(3-\alpha)c(n,2)}.$

Thus $\quad |f'(z)| \le 1 + \frac{2(1-\alpha)}{(3-\alpha)c(n,2)}r.$

Similarly we get $\quad |f'(z)| \ge 1 - \frac{2(1-\alpha)}{(3-\alpha)c(n,2)}r.$ This completes the result.

Theorem 2.4. *Let the function $f(z)$ defined by (1.1) be in the class $VS_p(n,\alpha)$, with arg $a_m = \theta_m$ where $[\theta_m + (m-1)\beta] \equiv \pi(\text{mod}2\pi)$. Define $f_l(z) = z$ and $f_m(z) = z + \frac{1-\alpha}{(2m-1-\alpha)c(n,m)} e^{i\theta_m} z^m$, $m \ge 2$, $z \in U$.*

Then $f(z) \in VS_p(n,\alpha)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{m=2}^{\infty} \mu_m f_m(z)$ where $\mu_m \ge 0$ and $\sum_{m=2}^{\infty} \mu_m = 1$.

Proof. If $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ with $\sum_{m=1}^{\infty} \mu_m = 1$ and $\mu_m \ge 0$ then

$$\sum_{m=2}^{\infty}(2m-1-\alpha)c(n,m)\frac{(1-\alpha)}{(2m-1-\alpha)c(n,m)}\mu_m = \sum_{m=2}^{\infty}\mu_m(1-\alpha) = (1-\mu_1)(1-\alpha) \le 1-\alpha,$$

Hence $f(z) \in VS_p(n,\alpha)$.

Conversely, let the function $f(z)$ defined by (1.1) be in the class $VS_p(n,\alpha)$, since $|a_m| \le \frac{1-\alpha}{(2m-1-\alpha)c(n,m)}$, $m = 2, 3, \ldots$. We may set $\mu_m = \frac{(2m-1-\alpha)c(n,m)|a_m|}{1-\alpha}$, $m \ge 2$ and $\mu_1 = 1 - \sum_{m=2}^{\infty} \mu_m$. Then $f(z) = \sum_{m=1}^{\infty} \lambda_m f_m(z)$, this completes the proof.

3. The Class $VS_p(n,\alpha,\beta)$

In this section we introduce a new subclass $VS_p(n,\alpha,\beta)$ and state the coefficient estimates, distortion theorems and extreme points analogous to the class $VS_p(n,\alpha)$.

Definition 3.1. A function $f(z) \in V$ of the form (1.1) is in the class $VS_p(n,\alpha,\beta)$ if $f(z)$ satisfies the analytic criteria

$$\operatorname{Re}\left\{\frac{z(D^n f(z))'}{D^n f(z)}\right\} \ge \alpha \left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right| + \beta \quad \text{where } \alpha \ge 0,\ \beta \ge 0 \text{ and } z \in U.$$

The proof of the following is similar to that of Theorem (2.2) and will be omitted.

Theorem 3.1. *For f of the form* (1.1), *f is in $VS_p(n,\alpha,\beta)$ if and only if*

$$\sum_{m=2}^{\infty} E_m c(n,m)|a_m| \le 1 - \beta, \quad \text{where } E_m = m(\alpha+1) - (\alpha+\beta).$$

Corollary 3.1. *If $f \in V$ is in $VS_p(n,\alpha,\beta)$ then $a_m \le \frac{(1-\beta)}{E_m c(n,m)}$ for $m \ge 2$.*

The inequality holds for the function f given by $f(z) = z + \sum_{m=2}^{\infty}[\frac{(1-\beta)}{E_m c(n,m)}]e^{i\theta_m} z^m$, $z \in U$ $(m \ge 2)$.

On lines similar to Theorem (2.3) and Theorem (2.4) we get the distortion bounds and extreme points for function $f(z) \in V$ in $VS_p(n,\alpha,\beta)$.

Theorem 3.2. (Distortion Theorem) *Let the function $f(z)$ of the form* (1.1) *be in the class $VS_p(n,\alpha,\beta)$. Then,*

$$r - \frac{1-\beta}{E_2 c(n,2)} r^2 \le |f(z)| \le r + \frac{1-\beta}{E_2 c(n,2)} r^2$$

and

$$1 - \frac{2(1-\beta)}{E_2 c(n,2)} r^2 \le |f'(z)| \le 1 + \frac{2(1-\beta)}{E_2 c(n,2)} r^2.$$

Theorem 3.3. *Let the function $f(z)$ defined by* (1.1) *be in the class $VS_p(n,\alpha,\beta)$, with arg $a_m = \theta_m$ where $[\theta_m + (m-1)\beta] \equiv \pi(\text{mod} 2\pi)$. Define $f_1(z) = z$ and $f_m(z) = z + \frac{1-\beta}{E_m c(n,m)} e^{i\theta_m} z^m$, $z \in U$; $m \ge 2$, $z \in U$. Then $f(z) \in VS_p(n,\alpha,\beta)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ where $\mu_m \ge 0$ and $\sum_{m=1}^{\infty} \mu_m = 1$.*

Remark 1. By taking $n = 0$, $\beta = 0$, these results reduces to the result obtained for the functions $f(z)$ of the form $f(z) = z - \sum_{m=2}^{\infty} |a_m| z^m$, in the $TS_p(\alpha)$ [12].

Remark 2. If $\alpha = 0$, $n = 0$. The above results coincide with the resutls obtained in [10].

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References

[1] H. S. Al-Amiri, *On Ruscheweyh derivatives*, Ann. Polon. Math. **38**(1980), 87-94.
[2] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. **56**(1991), 87-92.
[3] A. W. Goodman, *On uniformly starlike functions*, J. Math. Anal. and Appl. **155**(1991), 364-370.
[4] S. Kanas and H. M. Srivastava, *Linear operators involving k-uniformly convex functions*, Integral Transforms. Spec. Funct. **9**(2000), 121-132.
[5] W. Ma and D. Minda, *On uniformly covnex functions*, Ann. Polon. Math. **57**(1992), 165-175.
[6] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. **18**(1993), 189-196.
[7] F. Rønning, *Integral representations for bounded starlike functions*, Ann. Polon. Math. **60**(1995), 289-297.
[8] F. Rønning, *On starlike functions associated with parabolic region*, Ann. Univ. Marie-Curie. Sklodowska Sect. A. Vol.XIV **14**(1991), 117-122.
[9] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49**(1975), 109-115.
[10] H. Silverman, *Univalent functions with varying arguments*, Houston. J. Math. **17**(1981), 283-287.
[11] H. M. Srivastava and A. K. Mishra, *Applications of fractional calculus to parabolic starlike and uniformly convex functions*, Comput. Math. Appl. **39**(2000), 57-69.
[12] K. G. Subramanian, G. Murugusundaramoorthy, P. Balasubramanyam and H. Silverman, *Subclasses of uniformly convex and uniformly starlike functions*, Math. Japonica **42**(1995), 517-522.

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