TAMKANG JOURNAL OF MATHEMATICS
Volume 35, Number 1, Spring 2004

STABILITY OF LINEAR FUNCTIONAL EQUATIONS IN BANACH MODULES

CHUN-GIL PARK

Abstract. We prove the Hyers-Ulam-Rassias stability of the linear functional equation in Banach modules over a unital Banach algebra.

1. Introduction

In 1940, S. M. Ulam [12] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

Let E_1 and E_2 be Banach spaces. Hyers [5] showed that if $\epsilon > 0$ and $f : E_1 \to E_2$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E_1$, then there exists a unique additive mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \epsilon$$

for all $x \in E_1$.

Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th. M. Rassias [9] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E_1$. Gajda [3] generalized the Rassias' result.

Throughout this paper, let B be a unital Banach algebra with norm $|\cdot|$, $B_1 = \{a \in B \mid \ |a| = 1\}$, $\mathbb{R}^+$ the set of nonnegative real numbers, and let ${}_BM_1$ and ${}_BM_2$ be left Banach B-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively. Let d and s be positive integers.

In this paper, we are going to prove the Hyers-Ulam-Rassias stability of the linear functional equation in Banach modules over a unital Banach algebra.

Received March 5, 2002.
2000 *Mathematics Subject Classification*. Primary 39B72, 39B52, Secondary 46Bxx.
Key words and phrases. Stability, B-linear, Banach algebra.
This work was supported by grant No. R05-2003-000-10006-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

2. Stability of the Linear Functional Equation in Banach Modules

In this section, we prove the Hyers-Ulam-Rassias stability of the linear functional equation in Banach modules over a unital Banach algebra.

Theorem 2.1. *Let* $f : {}_B M_1 \to {}_B M_2$ *be a mapping for which there exists a function* $\varphi : {}_B M_1 \times {}_B M_1 \to [0, \infty)$ *such that*

$$\widetilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < \infty, \tag{i}$$

$$\|f(a^d x + a^d y) - a^s f(x) - a^s f(y)\| \le \varphi(x, y),$$

$$\|f(tx + ty) - tf(x) - tf(y) \le \varphi(x, y)$$

for all $a \in B_1$, *all* $t \in \mathbb{R}^+$, *and all* $x, y \in {}_B M_1$. *Then there exists a unique* $\mathbb{R}$*-linear mapping* $T : {}_B M_1 \to {}_B M_2$ *such that*

$$T(a^d x) = a^s T(x),$$

$$\|f(x) - T(x)\| \le \frac{1}{2} \widetilde{\varphi}(x, x) \tag{ii}$$

for all $a \in B_1$ *and all* $x \in {}_B M_1$.

Proof. Let $a = 1 \in B_1$. By the Găvruta result [4], it follows from the second inequality of the statement that there exists a unique additive mapping $T : {}_B M_1 \to {}_B M_2$ satisfying (ii). The mapping $T : {}_B M_1 \to {}_B M_2$ was given by $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ for all $x \in {}_B M_1$.

By the assumption, for each $a \in B_1$,

$$\|f(2^n a^d x) - 2a^s f(2^{n-1} x)\| \le \varphi(2^{n-1} x, 2^{n-1} x)$$

for all $x \in {}_B M_1$. Using the fact that for each $a \in B$ and each $z \in {}_B M_2$ $\|az\| \le K|a| \cdot \|z\|$ for some $K > 0$, one can show that

$$\|a^s f(2^n x) - 2a^s f(2^{n-1} x)\| \le K|a^s| \cdot \|f(2^n x) - 2f(2^{n-1} x)\| \le K\varphi(2^{n-1} x, 2^{n-1} x)$$

for all $a \in B_1$ and all $x \in {}_B M_1$. So

$$\begin{aligned} \|f(2^n a^d x) - a^s f(2^n x)\| &\le \|f(2^n a^d x) - 2a^s f(2^{n-1} x)\| + \|2a^s f(2^{n-1} x) - a^s f(2^n x)\| \\ &\le \varphi(2^{n-1} x, 2^{n-1} x) + K\varphi(2^{n-1} x, 2^{n-1} x) \end{aligned}$$

for all $a \in B_1$ and all $x \in {}_B M_1$. Thus $2^{-n}\|f(2^n a^d x) - a^s f(2^n x)\| \to 0$ as $n \to \infty$ for all $a \in B_1$ and all $x \in {}_B M_1$. Hence

$$T(a^d x) = \lim_{n\to\infty} \frac{f(2^n a^d x)}{2^n} = \lim_{n\to\infty} \frac{a^s f(2^n x)}{2^n} = a^s T(x)$$

for each $a \in B_1$.

Similarly, one can obtain that

$$T(tx) = \lim_{n\to\infty} \frac{f(2^n tx)}{2^n} = \lim_{n\to\infty} \frac{tf(2^n x)}{2^n} = tT(x)$$

for each $t \in \mathbb{R}^+$. Since $t = |t| \cdot \frac{t}{|t|}$ for each $t \in \mathbb{R}$ $(t \neq 0)$,

$$\begin{aligned} T(t_1x + t_2y) &= T(t_1x) + T(t_2y) = T(|t_1|\frac{t_1}{|t_2|}x) + T(|t_2|\frac{t_2}{|t_2|}y) \\ &= |t_1|T(\frac{t_1}{|t_1|}x) + |t_2|T(\frac{t_2}{|t_2|}y) = |t_1|\frac{t_1}{|t_1|}T(x) + |t_2|\frac{t_2}{|t_2|}T(y) \\ &= t_1T(x) + t_2T(y) \end{aligned}$$

for all $t_1, t_2 \in \mathbb{R}^+ (t_1, t_2 \neq 0)$ and all $x,\ y \in {}_BM_1$. And $T(x) = T(x - y + y) = T(x - y) + T(y)$ for all $x \in {}_BM_1$. Hence $T(x - y) = T(x) - T(y)$ for all $x \in {}_BM_1$. So the unique additive mapping $T : {}_BM_1 \to {}_BM_2$ is an $\mathbb{R}$-linear mapping satisfying the conditions given in the statement.

Let $d = s = 1$ in Theorem 2.1. Since $T(ax) = aT(x)$ for all $a \in B_1$ and all $x \in {}_BM_1$,

$$\begin{aligned} T(ax + by) &= T(ax) + T(by) = T(|a|\frac{a}{|a|}x) + T(|b|\frac{b}{|b|}y) \\ &= |a|T(\frac{a}{|a|}x) + |b|T(\frac{b}{|b|}y) = |a|\frac{a}{|a|}T(x) + |b|\frac{b}{|b|}T(y) \\ &= aT(x) + bT(y) \end{aligned}$$

for all $a,\ b \in B(a, b \neq 0)$ and all $x,\ y \in {}_BM_1$. And $T(0x) = 0T(x)$ for all $x \in {}_BM_1$. So the $\mathbb{R}$-linear mapping $T : {}_BM_1 \to {}_BM_2$ is B-linear.

Remark 2.1. If the second inequality in the statement of Theorem 2.1 is replaced by

$$\|f(a^dx + y) - a^sf(x) - f(y)\| \le \varphi(x, y),$$

then

$$\begin{aligned} &\|f(a^dx + a^dy) - a^sf(x) - f(a^dy)\| \le \varphi(x, a^dy), \\ &\|f(a^dx + a^dy) - f(a^dx) - a^sf(y)\| \le \varphi(y, a^dx), \\ &\|f(a^dx + a^dy) - f(a^dx) - f(a^dy)\| \le \varphi(a^dx, a^dy). \end{aligned}$$

So

$$\|f(a^dx + a^dy) - a^sf(x) - a^sf(y)\| \le \varphi(x, a^dy) + \varphi(y, a^dx) + \varphi(a^dx, a^dy),$$

hence the result does also hold.

Corollary 2.2. *Let $f : {}_BM_1 \to {}_BM_2$ be a mapping for which there exist constants $\epsilon \ge 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} \|f(a^dx + a^dy) - a^sf(x) - a^sf(y)\| &\le \epsilon(||x||^p + ||y||^p), \\ \|f(tx + ty) - tf(x) - tf(y)\| &\le \epsilon(||x||^p + ||y||^p) \end{aligned}$$

for all $a \in B_1$, all $t \in \mathbb{R}^+$, and all $x, y \in {}_BM_1$. Then there exists a unique $\mathbb{R}$-linear mapping $T : {}_BM_1 \to {}_BM_2$ such that

$$T(a^d x) = a^s T(x),$$
$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2 - 2^p} ||x||^p$$

for all $a \in B_1$ and all $x \in {}_BM_1$.

Proof. Define $\varphi(x, y) = \epsilon(||x||^p + ||y||^p)$, and apply Theorem 2.1.

Corollary 2.3. *Let E_1 and E_2 be complex Banach spaces and $f : E_1 \to E_2$ a mapping for which there exists a function $\varphi : E_1 \times E_1 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < \infty,$$
$$\|f(\mu^d x + \mu^d y) - \mu^s f(x) - \mu^s f(y)\| \le \varphi(x, y),$$
$$\|f(\lambda x + \lambda y) - \lambda f(x) - \lambda f(y)\| \le \varphi(x, y)$$

for all $\mu \in \mathbb{C}_1 = \mathbb{T}^1$, all $\lambda \in \mathbb{R}^+$, and all x, $y \in E_1$. Then there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$T(\mu^d x) = \mu^s T(x),$$
$$\|f(x) - T(x)\| \le \frac{1}{2} \widetilde{\varphi}(x, x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in E_1$.

Proof. Since $\mathbb{C}$ is a unital Banach algebra, the Banach spaces E_1 and E_2 are considered as Banach modules over $\mathbb{C}$. By Theorem 2.1, there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ satisfying the conditions given in the statement.

Now we prove the Hyers-Ulam-Rassias stability of another linear functional equation in Banach modules over a unital Banach algebra.

Theorem 2.4. *Let $f : {}_BM_1 \to {}_BM_2$ be a mapping for which there exists a function $\varphi : {}_BM_1 \times {}_BM_1 \to [0, \infty)$ satisfying (i) such that*

$$\|a^s f(x + y) - f(a^d x) - f(a^d y)\| \le \varphi(x, y),$$
$$\|t f(x + y) - f(tx) - f(tx)\| \le \varphi(x, y)$$

for all $a \in B_1$, all $t \in \mathbb{R}^+$, and all x, $y \in {}_BM_1$. Then there exists a unique $\mathbb{R}$-linear mapping $T : {}_BM_1 \to {}_BM_2$ satisfying (ii) *such that*

$$a^s T(x) = T(a^d x)$$

for all $a \in B_1$ and all $x \in {}_BM_1$.

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique additive mapping $T : {}_BM_1 \to {}_BM_2$ satisfying (ii).

By the assumption, for each $a \in B_1$,

$$\|a^s f(2^n x) - 2f(2^{n-1}a^d x)\| \le \varphi(2^{n-1}x, 2^{n-1}x)$$

for all $x \in {}_BM_1$. So

$$\begin{aligned}\|a^s f(2^n x) - f(2^n a^d x)\| &\le \|a^s f(2^n x) - 2f(2^{n-1}a^d x)\| + \|2f(2^{n-1}a^d x) - f(2^n a^d x)\| \\ &\le \varphi(2^{n-1}x, 2^{n-1}x) + \varphi(2^{n-1}a^d x, 2^{n-1}a^d x)\end{aligned}$$

for all $a \in B_1$ and all $x \in {}_BM_1$. So $2^{-n}\|a^s f(2^n x) - f(2^n a^d x)\| \to 0$ as $n \to \infty$ for all $a \in B_1$ and all $x \in {}_BM_1$. Hence

$$a^s T(x) = \lim_{n\to\infty} \frac{a^s f(2^n x)}{2^n} = \lim_{n\to\infty} \frac{f(2^n a^d x)}{2^n} = T(a^d x)$$

for all $a \in B_1$.

Similarly, one can obtain that

$$tT(x) = \lim_{n\to\infty} \frac{tf(2^n x)}{2^n} = \lim_{n\to\infty} \frac{f(2^n tx)}{2^n} = T(tx)$$

for each $t \in \mathbb{R}^+$.

The rest of the proof is the same as the proof of Theorem 2.1. So the unique additive mapping $T : {}_BM_1 \to {}_BM_2$ is an $\mathbb{R}$-linear mapping satisfying (ii) such that

$$a^s T(x) = T(a^d x)$$

for all $a \in B_1$ and all $x \in {}_BM_1$, as desired.

Theorem 2.5. *Let $f : {}_BM_1 \to {}_BM_2$ be a mapping for which there exists a function $\varphi : {}_BM_1 \times {}_BM_1 \to [0, \infty)$ satisfying* (i) *such that*

$$\begin{aligned}\|f(tx + ty) - tf(x) - tf(y)\| &\le \varphi(x, y), \\ \|f(a^d x) - a^s f(x)\| &\le \varphi(x, x)\end{aligned}$$

for all $a \in B_1$, all $t \in \mathbb{R}^+$, and all x, $y \in {}_BM_1$. Then there exists a unique $\mathbb{R}$-linear mapping $T : {}_BM_1 \to {}_BM_2$ satisfying (ii) *such that*

$$T(a^d x) = a^s T(x)$$

for all $a \in B_1$ and all $x \in {}_BM_1$.

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique additive mapping $T : {}_BM_1 \to {}_BM_2$ satisfying (ii). Combining the definition of the mapping T and the second inequality given in the statement yields that

$$T(a^d x) = \lim_{n\to\infty} \frac{f(2^n a^d x)}{2^n} = \lim_{n\to\infty} \frac{a^s f(2^n x)}{2^n} = a^s T(x)$$

for all $a \in B_1$ and all $x \in {}_BM_1$.

The rest of the proof is the same as the proof of Theorem 2.1. So the unique additive mapping $T : {}_BM_1 \to {}_BM_2$ is an $\mathbb{R}$-linear mapping satisfying the conditions given in the statement, as desired.

Remark 2.2. If the second inequality in the statement of Theorem 2.1 is replaced by

$$\|f(a^d x + y) - a^s f(x) - f(y)\| \le \varphi(x, y),$$

then

$$\|f(a^d x + x) - a^s f(x) - f(x)\| \le \varphi(x, x),$$
$$\|f(a^d x + x) - f(a^d x) - f(x)\| \le \varphi(a^d x, x).$$

So

$$\|f(a^d x) - a^s f(x)\| \le \varphi(x, x) + \varphi(a^d x, x),$$

hence the result does also hold.

3. Stability of the Pexider Functional Equation in Banach Modules

In this section, we prove the Hyers-Ulam-Rassias stability of the Pexider functional equation in Banach modules over a unital Banach algebra.

Theorem 3.1. *Let $f,\ g,\ h : {}_BM_1 \to {}_BM_2$ be mappings for which there exists a function $\varphi : {}_BM_1 \setminus \{0\} \times {}_BM_1 \setminus \{0\} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{k=0}^{\infty} 3^{-k} \varphi(3^k x, 3^k y) < \infty, \tag{iii}$$
$$\|f(a^d x + a^d y) - a^s g(x) - a^s h(y)\| \le \varphi(x, y),$$
$$\|f(tx + ty) - tg(x) - th(y)\| \le \varphi(x, y)$$

for all $a \in B_1$, all $t \in \mathbb{R}^+$, and all $x,\ y \in {}_BM_1 \setminus \{0\}$. Then there exists a unique $\mathbb{R}$-linear mapping $T : {}_BM_1 \to {}_BM_2$ such that

$$T(a^d x) = a^s T(x),$$
$$\|f(x) - f(0) - T(x)\| \le \frac{1}{3}\widetilde{\varphi}(\frac{x}{2}, \frac{-x}{2}) + \frac{1}{3}\widetilde{\varphi}(\frac{-x}{2}, \frac{x}{2}) + \frac{1}{3}\widetilde{\varphi}(\frac{x}{2}, \frac{x}{2}) + \frac{2}{3}\widetilde{\varphi}(\frac{-x}{2}, \frac{x}{2})$$
$$+ \frac{1}{3}\widetilde{\varphi}(\frac{-x}{2}, \frac{3x}{2}) + \frac{1}{3}\widetilde{\varphi}(\frac{3x}{2}, \frac{-x}{2}) + \frac{1}{3}\widetilde{\varphi}(\frac{3x}{2}, \frac{3x}{2}) \tag{iv}$$

for all $a \in B_1$ and all $x \in {}_BM_1 \setminus \{0\}$.

Proof. Put $a = 1$. By [7, Theorem 2.2], there exists a unique additive mapping $T : {}_BM_1 \to {}_BM_2$ satisfying (iv). The mapping $T : {}_BM_1 \to {}_BM_2$ was given by

$$T(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^n} = \lim_{n\to\infty} \frac{g(3^n x)}{3^n} = \lim_{n\to\infty} \frac{h(3^n x)}{3^n}$$

for all $x \in {}_B M_1$. For each fixed $a \in B_1$, it follows from the second inequality of the statement and the definition of the mapping T that

$$T(a^d x) = \lim_{n\to\infty} \frac{f(3^n a^d x)}{3^n} = \lim_{n\to\infty} \frac{a^s g(3^n x)}{3^n} = a^s T(x)$$

for all $a \in B_1$ and all $x \in {}_B M_1$.

Similarly, one can obtain that

$$T(tx) = \lim_{n\to\infty} \frac{f(3^n tx)}{3^n} = \lim_{n\to\infty} \frac{t f(3^n x)}{3^n} = tT(x)$$

for each $t \in \mathbb{R}^+$.

The rest of the proof is the same as the proof of Theorem 2.1. So the unique additive mapping $T : {}_B M_1 \to {}_B M_2$ is an $\mathbb{R}$-linear mapping satisfying the conditions given in the statement, as desired.

Corollary 3.2. *Let $p < 1$, and f, g, $h : {}_B M_1 \to {}_B M_2$ mappings such that*

$$\|f(a^d x + a^d y) - a^s g(x) - a^s h(y)\| \le ||x||^p + ||y||^p,$$
$$\|f(tx + ty) - tg(x) - th(y)\| \le ||x||^p + ||y||^p$$

for all $a \in B_1$, all $t \in \mathbb{R}^+$, and all x, $y \in {}_B M_1 \setminus \{0\}$. Then there exists a unique $\mathbb{R}$-linear mapping $T : {}_B M_1 \to {}_B M_2$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{4(3+3^p)}{2^p(3-3^p)}||x||^p, \qquad T(a^d x) = a^s T(x)$$

for all $a \in B_1$ and all $x \in {}_B M_1 \setminus \{0\}$.

Proof. Define $\varphi(x, y) = ||x||^p + ||y||^p$, and apply Theorem 3.1.

Now we prove the Hyers-Ulam-Rassias stability of another Pexider functional equation in Banach modules over a unital Banach algebra.

Theorem 3.3. *Let f, g, h : ${}_B M_1 \to {}_B M_2$ be mappings for which there exists a function $\varphi : {}_B M_1 \setminus \{0\} \times {}_B M_1 \setminus \{0\} \to [0, \infty)$ satisfying (iii) such that*

$$\|a^s f(x + y) - g(a^d x) - h(a^d y)\| \le \varphi(x, y),$$
$$\|tf(x + y) - g(tx) - h(ty)\| \le \varphi(x, y)$$

for all $a \in B_1$, all $t \in \mathbb{R}^+$, and all x, $y \in {}_B M_1 \setminus \{0\}$. Then there exists a unique $\mathbb{R}$-linear mapping $T : {}_B M_1 \to {}_B M_2$ satisfying (iv) such that

$$a^s T(x) = T(a^d x)$$

for all $a \in B_1$ all $x \in {}_B M_1 \setminus \{0\}$.

Proof. By the same reasoning as the proof of Theorem 3.1, there exists a unique additive mapping $T : {}_BM_1 \to {}_BM_2$ satisfying (iv). For each fixed $a \in B_1$, it follows from the first inequality of the statement and the definition of the mapping T that

$$a^sT(x) = \lim_{n\to\infty} \frac{a^s f(3^n x)}{3^n} = \lim_{n\to\infty} \frac{g(3^n a^d x)}{3^n} = T(a^d x)$$

for all $a \in B_1$ and all $x \in {}_BM_1$.

The rest of the proof is similar to the proof of Theorem 2.1. So the unique additive mapping $T : {}_BM_1 \to {}_BM_2$ is an $\mathbb{R}$-linear mapping satisfying the conditions given in the statement, as desired.

Similarly, one can prove the stability of the other linear functional equations in Banach modules over a unital Banach algebra.

References

[1] J. Aczél and J. K. Chung, *Integrable solutions of functional equations of a general type*, Studia Sci. Math. Hungar. **17**(1982), 51-67.
[2] F. Bonsall and J. Duncan, Complete Normed Algebras, Springer-Verlag, New York, Heidelberg and Berlin, 1973.
[3] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. **14**(1991), 431-434.
[4] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184**(1994), 431-436.
[5] D. H. Hyers, *On the stability of the linear functional equation*, Pro. Nat'l. Acad. Sci. U.S.A. **27**(1941), 222-224.
[6] D. H. Hyers, G. Isac and Th. M. Rassias, Stability of Functional Equations in Several Variables, Birkhäuser, Berlin, Basel and Boston, 1998.
[7] K. Jun and Y. Lee, *A generalization of the Hyers-Ulam-Rassias stability of Pexider equation*, J. Math. Anal. Appl. **246**(2000), 627-638.
[8] K. Kovács, *On the characterization of additive and multiplicative functions*, Studia Sci. Math. Hungar. **18**(1983), 1-11.
[9] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72**(1978), 297-300.
[10] H. Schröder, *K-theory for real C^*-algebras and applications*, Pitman Research Notes in Math. Ser., Longman Sci. Tech. Essex, **290**(1993).
[11] F. Skof, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53**(1983), 113-129.
[12] S. M. Ulam, Problems in Modern Mathematics, Wiley, New York, 1960.

Department of Mathematics, Chungnam National University, DaeJeon 305-764, South Korea.
E-mail: cgpark@math.cnu.ac.kr