TAMKANG JOURNAL OF MATHEMATICS
Volume 48, Number 4, 301-320, December 2017
doi:10.5556/j.tkjm.48.2017.2237
This paper is available online at http://journals.math.tku.edu.tw/index.php/TKJM/pages/view/onlinefirst

REGULAR CLIQUE ASSEMBLIES, CONFIGURATIONS, AND FRIENDSHIP IN EDGE-REGULAR GRAPHS

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Abstract. An edge-regular graph is a regular graph in which, for some λ, any two adjacent vertices have exactly λ common neighbors. This paper is about the existence and structure of edge-regular graphs with $\lambda = 1$ and about edge-regular graphs with $\lambda > 1$ which have local neighborhood structure analogous to that of the edge-regular graphs with $\lambda = 1$.

1. Introduction

Graphs will be finite and simple with no isolated vertices. Notation will be fairly standard. For instance, if G is a graph and $v \in V(G)$, the vertex set of G, then $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $d_G(v) = |N_G(v)|$. When G is the only graph in the discussion, the subscript G may be dropped from the notation for the open neighbor set and for the degree of a vertex. The closed neighborhood in G of $v \in V(G)$ is $N[v] = N(v) \cup \{v\}$. If $S \subseteq V(G)$, $S \neq \emptyset$, the subgraph of G induced by S will be denoted $G[S]$.

If G and H are graphs, the *join* of G and H, formed by taking disjoint copies of G and H and putting in all edges with one end in $V(G)$ and the other in $V(H)$, will be denoted $G \vee H$. The *disjoint union*, or *sum*, of G and H, formed by taking disjoint copies of G and H and putting in no edges at all will be denoted $G + H$. [These notations are not entirely standard. Many authors prefer to denote the join by $+$.] If m is a positive integer, $mG = G + \cdots + G$, with G appearing m times in the sum. The *Cartesian product* of G and H, denoted $G \square H$, is the graph with the vertex set $V(G) \times V(H)$ with $(w, x), (y, z) \in V(G) \times V(H)$ adjacent in $G \square H$ if and only if either $w = y$ and $xz \in E(H)$ or $x = z$ and $wy \in E(G)$

A *friendship graph* is a graph of the form $K_1 \vee mK_2$ for some positive integer m. The single vertex in K_1 will be called a ringmaster of the graph. If $m > 1$, the ringmaster is unique. The friendship graphs are, famously, the only finite simple graphs in which each pair of distinct vertices have exactly one common neighbor [2].

Received September 25, 2016, accepted August 7, 2017.
2010 *Mathematics Subject Classification*. 05C75.
Key words and phrases. Edge-regular, friendship graph, clique.
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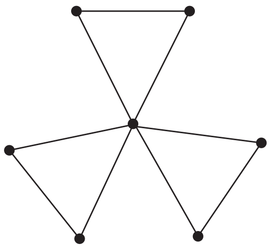


Figure 1: $K_1 \vee 3K_2$, a friendship graph.

An *edge-regular graph* is a regular graph G for which there exists an integer λ such that if $uv \in E(G)$ then $|N(u) \cap N(v)| = \lambda$. That is, every pair of adjacent vertices in G have exactly λ common neighbors. If G is an edge-regular graph with parameter λ, regular of degree d on n vertices, we write $G \in ER(n,d,\lambda)$.

A *strongly regular graph* is an edge-regular graph $G \in ER(n,d,\lambda)$ for some n, d, and λ, $0 < d < n$, for which there exists an integer μ such that for $u, v \in V(G)$, $u \neq v$, if $uv \notin E(G)$ then $|N(u) \cap N(v)| = \mu$. That is, each pair of distinct non-adjacent vertices in G have exactly μ common neighbors. If G is such a graph, we write $G \in SR(n,d,\lambda,\mu)$.

Those who are acquainted with developments in graph theory over the past 40–50 years will know that the quest for strongly regular graphs has become a small but important industry ([1, 2]). These graphs are just rare enough that the discovery of new ones is always of interest, and just numerous and varied enough that one despairs of easy classifications for them.

Therefore the richer class of edge-regular graphs is unlikely to collapse into tidy subclasses. However, there have been interesting characterization/classification results on edge regular graphs satisfying additional requirements of an extremal or structural nature. For instances:

1. In [5] all $G \in ER(n,d,\lambda)$ satisfying $d - \lambda \leq 3$ are described.
2. If $G \in ER(n,d,\lambda)$ and $\lambda > 0$ then $n \geq 3(d-\lambda)$ ([6, 8]). In [8] the edge-regular graphs with $\lambda = 2$ and $n = 3(d-\lambda)$ are completely characterized, and in [12] the main result in [8] is extended to a characterization of all edge-regular graphs satisfying $n = 3(d-\lambda)$ with $\lambda > 0$ even and d sufficiently large (depending on λ).
3. Edge-regular graphs with $n = 3(d-\lambda)+1$, $\lambda > 0$, satisfying certain local structural requirements are considered in [3] and [7]. The main result of [7] is of interest here: For every d, $ER(3d-2, d, 1) = \emptyset$.

Here we are mainly interested in edge-regular graphs with $\lambda = 1$, and a generalization: edge-regular graphs G such that for each $v \in V(G)$, $G[N_G(v)] \simeq mK_p$ for some m and p that do not vary with v. Before getting down to $\lambda = 1$ it will be useful to examine the more general class of graphs.

2. Regular clique assemblies

The clique number of a graph G, denoted $\omega(G)$, is the maximum order of a clique — complete subgraph — in G. The *clique graph* of G, denoted $CL(G)$, is the graph whose vertices are the maximal cliques of G, in which any two distinct maximal cliques of G are adjacent if and only if they have at least one vertex in common. If G has no isolated vertices and $\omega(G) = 2$, then $CL(G) = L(G)$, the line graph of G.

G is a *regular clique assembly* if G is regular, $\omega(G) \geq 2$, and

(1) every maximal clique of G is maximum;

(2) each edge of G is in exactly one maximum clique of G.

If G is a regular clique assembly on n vertices, regular of degree d, with $k = \omega(G)$, we write $G \in RCA(n, d, k)$. In all that follows, n, d, and k will be integers satisfying $n > d \geq k-1 \geq 1$. If $G \in RCA(n, k-1, k)$ then $G \simeq mK_k$, $m = \frac{n}{k}$. This is easy to see; also, it follows from Proposition 1, below. Notice that we do not require a regular clique assembly to be connected; a disjoint union of regular clique assemblies with the same degree and same clique number is a regular clique assembly. (Perhaps "assemblage" would have been a better choice than "assembly," but we are sticking with the latter.)

Lemma 1. *If G is a regular clique assembly, then any two different maximum cliques in G have at most one vertex in common. Further, if H_1, H_2, and H_3 are maximum cliques in G, $V(H_1) \cap V(H_2) = \{u\}$, $V(H_1) \cap V(H_3) = \{v\}$ and $u \neq v$, then $V(H_2) \cap V(H_3) = \emptyset$.*

Proof. If distinct maximum cliques in G had two vertices in common, then condition (2) in the RCA definition would be violated. Suppose H_1, H_2, H_3, u, and v are as described above. Then H_1, H_2, and H_3 are distinct maximum cliques. Suppose $w \in V(H_2) \cap V(H_3)$. If $w \in \{u, v\}$ then H_1 and one of H_2, H_3 have two vertices in common. Therefore $w \notin \{u, v\}$. Then u, v, w induce a K_3 in G, which is contained in a maximal, and therefore maximum, clique H_4 in G which is none of H_1, H_2, and H_3. Then uv is in both H_1 and H_4, violating (2). □

Proposition 1. *If $G \in RCA(n, d, k)$ then $k-1$ divides d, and for each $v \in V(G)$, $G[N_G(v)] \simeq \frac{d}{k-1} K_{k-1}$. Conversely, if G is a graph on n vertices such that, for some m, $p \geq 1$, $G[N_G(v)] \simeq mK_p$ for all $v \in V(G)$, then $G \in RCA(n, mp, p+1)$*

Proof. Suppose $G \in RCA(n, d, k)$. Suppose that $v \in V(G)$. A neighbor u of v is in the unique maximum clique $\simeq K_k$ containing the edge uv. Any two of the maximum cliques of G containing v have only v in common, by Lemma 1; thus $G[N_G(v)] \simeq mK_{k-1}$ for some m. Since G is d-regular, $d = m(k-1)$.

Now suppose that G is a finite simple graph such that for every $v \in V(G)$, $G[N_G(v)] \simeq mK_p$ for some positive integers m, p. Then G is regular of degree mp. G can contain no K_{p+2}, and any K_r in G, $r \le p+1$, must be contained in one of the K_{p+1}'s comprising the closed neighbor set of any one of its vertices. (For all $v \in V(G)$, $G[N_G[v]] \simeq K_1 \vee mK_p$.) Thus $\omega(G) = p+1$, and (1) in the RCA definition holds; (2) is obvious. □

Corollary 1. *$RCA(n,d,k) \subseteq ER(n,d,k-2)$, with equality when $k \in \{2,3\}$.*

Proof. If $G \in RCA(n,d,k)$, then, since every edge uv of G is contained in one K_k in G, vertices outside of which cannot be adjacent to both u and v, it follows that G is edge-regular with $\lambda = k-2$. If $G \in ER(n,d,0)$, then G is triangle-free and d-regular; clearly $G \in RCA(n,d,2)$. Suppose that $G \in ER(n,d,1)$. Since, for any $uv \in V(G)$, $|N_G(u) \cap N_G(v)| = 1$, G can contain no K_4, and no K_1 nor K_2 in G is a maximal clique ($d > 0$). Thus $\omega(G) = 3$ and (1) and (2) in the definition of RCAs hold. Therefore, $G \in RCA(n,d,3)$. □

Theorem 1. *If $G \in RCA(n,d,k)$, ($d > k-1$), then $CL(G) \in RCA\left(\frac{nd}{k(k-1)}, \frac{k(d-k+1)}{k-1}, \frac{d}{k-1}\right)$. Further, $CL(CL(G)) \simeq G$.*

Proof. The vertices of $CL(G)$ are the maximum cliques of G; counting the ordered pairs (v,K), $v \in V(K)$, $K \simeq K_k$ a maximum clique of G, in two different ways, using Proposition 1, we obtain $|V(CL(G))| = \frac{nd}{k(k-1)}$. By Lemma 1, two maximum cliques in G are adjacent as vertices in $CL(G)$ if and only if they have exactly one vertex in common. Let K be a maximum clique in G and $v \in V(K)$. In view of Proposition 1, K is adjacent in $CL(G)$ to each of $\frac{d}{k-1} - 1 = \frac{d-k+1}{k-1}$ other maximum cliques containing v — indeed, in $CL(G)$ these cliques induce, with K, a clique of order $\frac{d}{k-1}$. By Lemma 1, the maximum cliques "adjacent to K at v" are distinct from the maximum cliques adjacent to K at any other vertex of K — and not only distinct from, but also not adjacent to, since a clique H_2 sharing v with K shares no vertex with any clique H_3 sharing a vertex $u \neq v$ with K.

It follows that $CL(G)\left[N_{CL(G)}(K)\right] \simeq kK_{\frac{d}{k-1}-1}$. Since this holds for every vertex K of $CL(G)$, by Proposition 1 we conclude that $CL(G) \in RCA(\frac{nd}{k(k-1)}, k\frac{d-k+1}{k-1}, \frac{d}{k-1})$. Applying this result with $CL(G)$ replacing G, we find that $CL(CL(G)) \in RCA(n,d,k)$. From this we take that $CL(CL(G))$ has the same number of vertices as G and is d-regular.

For $v \in V(G)$, let $S(v)$ denote the $\frac{d}{k-1}$-clique induced in $CL(G)$ by the k-cliques in G that contain v; $S : V(G) \to V(CL(CL(G)))$ is clearly injective, and is therefore surjective. If u and v are adjacent in G then $S(u)$ and $S(v)$ have a vertex in common in $CL(G)$, namely, the unique maximum clique in G containing the edge uv; therefore, $S(u)$ and $S(v)$ are adjacent in $CL(CL(G))$. Since G and $CL(CL(G))$ are both d-regular, and S preserves adjacency, it must also preserve non-adjacency. Therefore S is a graph isomorphism; so G and $CL(CL(G))$ are isomorphic. □

Corollary 2. *$G \in RCA(n, 2k-2, k)$ for some n and $k > 2$ if and only if G is the line graph of a triangle-free k-regular graph.*

Proof. If $G \in RCA(n, 2k-2, k)$ and $k > 2$, then by Theorem 1, $CL(G) \in RCA(\frac{2n}{k}, k, 2)$, so $CL(G)$ is triangle-free and k-regular and $G \simeq CL(CL(G)) = L(CL(G))$. On the other hand, if $G = L(H)$, H triangle-free and k-regular, then $H \in RCA(t, k, 2)$, for $t = |V(H)|$, so $G = L(H) = CL(H) \in RCA(\frac{tk}{2}, 2(k-1), k)$, again by Theorem 1. □

By Corollary 1, regular clique assemblies with clique number $k = 2$ are plentiful: they are the triangle-free regular graphs. For $k = 3$ they are the edge regular graphs with $\lambda = 1$; we shall see that there are quite a few of these, although they are not so easy to find as the triangle-free regular graphs. For $k > 3$ we will see that there is a good supply of regular clique assemblies with clique number k by applying the following.

Proposition 2. *If $G \in RCA(n_1, d_1, k)$, and $H \in RCA(n_2, d_2, k)$ then $G \square H \in RCA(n_1 n_2, d_1 + d_2, k)$.*

Proof. The proof is elementary, from the definition of $G \square H$ and Proposition 1. □

Corollary 3. *For all integers $k, t \geq 2$ there are connected graphs in $RCA(k^t, t(k-1), k)$ and in $RCA(tk^{t-1}, k(t-1), t)$.*

Proof. Taking powers with use of the Cartesian product, by Proposition 2 $(K_k)^t \in RCA(k^t, t(k-1), k)$. Then by Theorem 1, $CL((K_k)^t) \in RCA(tk^{k-1}, k(t-1), t)$. □

The supply of regular clique assemblies with specified clique number k can be enlarged by swordplay with Theorem 1 and Proposition 2. If $G \in RCA(n, d, 2) = ER(n, d, 0)$ and $d > 1$ then $CL(G) = L(G) \in RCA(\frac{nd}{2}, 2(d-1), d)$, by Theorem 1. If $G \in RCA(n, d, 3) = ER(n, d, 1)$ and $d > 2$ then $CL(G) \in RCA(\frac{nd}{6}, \frac{3(d-2)}{2}, \frac{d}{2})$, by Theorem 1.

Since edge-regular graphs with $\lambda \in \{0, 1\}$ are relatively easy to obtain (see Section 4, in the case $\lambda = 1$), these observations give us starter supplies of regular clique assemblies with clique number k, for different k, and then taking Cartesian products enlarges the supply indefinitely. We already have, by Corollary 3, that there are infinitely many such graphs, for each $k \geq 2$; these latter observations bear on the orders, degrees, and isomorphism types to be found among the RCAs with clique number k. But we leave the examination of this bounty for now.

3. Configurations, another incarnation of regular clique assemblies

An *incidence structure* is a triple $(\mathscr{P}, \mathscr{B}, \mathscr{I})$ of sets such that $\mathscr{I} \subseteq \mathscr{P} \times \mathscr{B}$. The elements of $\mathscr{P}$ are called *points*, and the elements of $\mathscr{B}$ are called *lines* or *blocks*. If $(p, B) \in \mathscr{I}$ we say that

p and B are *incident*—but we will often resort to colloquial usages such as: p is *on* B, or p is an element of B, or even $p \in B$, treating each block as a set of points, which may as well be the case, however $\mathscr{B}$ is given initially.

The *dual* of an incidence structure $\mathscr{S} = (\mathscr{P}, \mathscr{B}, \mathscr{I})$ is the incidence structure $\mathscr{S}^T = (\mathscr{B}, \mathscr{P}, \mathscr{I}^T)$, where $\mathscr{I}^T = \{(B, p) \mid (p, B) \in \mathscr{I}\}$. Obviously $(\mathscr{S}^T)^T = \mathscr{S}$. A (v_r, b_k) *configuration* is an incidence structure $(\mathscr{P}, \mathscr{B}, \mathscr{I})$ in which $|\mathscr{P}| = v$, $|\mathscr{B}| = b$, each block contains (is incident to) exactly k points, each point lies in exactly r different blocks, and any two different points are together in at most one block. (Because $\mathscr{P}$ and $\mathscr{B}$ and the blocks are sets, there are no repeated points, nor repeated blocks.) In any such configuration, clearly $|\mathscr{I}| = vr = bk$. Also, no two different blocks can have more than one point in common (because two different points cannot be together in two different blocks), so the dual of a (v_r, b_k) configuration is a (b_k, v_r) configuration.

A *triangle* or *trilateral* in a configuration is a set of 3 points which are pairwise collinear but which do not lie together on the same line (block). In other words, each pair of the 3 points determines a line, and the 3 lines thus determined are distinct. A configuration is *trilateral-free* if its point set contains no trilateral. It is easy to see that every trilateral in a configuration corresponds to a trilateral in the dual; therefore, if $\mathscr{S}$ is trilateral-free, then so is $\mathscr{S}^T$.

Proposition 3. *Suppose $k \geq 2$. If $G \in RCA(n, d, k)$, then the incidence structure $(V(G), \mathscr{B}, \mathscr{I})$, where $\mathscr{B}$ is the set of vertex sets of the maximum cliques in G, and $(w, B) \in \mathscr{I}$ if and only if $w \in B \in \mathscr{B}$, is a trilateral-free $(n_{\frac{d}{k-1}}, (\frac{nd}{k(k-1)})_k)$ configuration.*

Conversely, if $(\mathscr{P}, \mathscr{B}, \mathscr{I})$ is a trilateral-free (v_r, b_k) configuration then the graph G defined by $V(G) = \mathscr{P}$ and $p, q \in \mathscr{P}$, $p \neq q$, are adjacent in G if and only if $p, q \in B$ for some $B \in \mathscr{B}$, is in $RCA(v, r(k-1), k)$.

Proof. If $G \in RCA(n, d, k)$, that the incidence structure derived from G as in the statement of Proposition 3 is an $(n_{\frac{d}{k-1}}, (\frac{nd}{k(k-1)})_k)$ configuration is a straightforward exercise from the definitions and the statements and proofs of Lemma 1, Proposition 1, and Theorem 1. To see that $\mathscr{S}$ is trilateral-free, suppose that $u, v, w \in V(G) = P$ are 3 different vertices of G (points of S) and are pairwise collinear. This means that there are maximum cliques H_1, H_2, H_3 in G such that $u, v \in V(H_1)$, $v, w \in V(H_2)$, and $u, w \in V(H_3)$. Then uvw is a K_3 in G, a clique, which is therefore contained in a maximal clique H of G. Because G is an RCA, H is a maximum clique, and then because it contains each of the edges uv, vw, and uw, it must be that $H = H_1 = H_2 = H_3$. This proves that $\mathscr{S}$ contains no trilaterals.

Now suppose that $\mathscr{S} = (\mathscr{P}, \mathscr{B}, \mathscr{I})$ is a trilateral-free (v_r, b_k) configuration, with $\mathscr{B}$ understood to be a collection of k-subsets of $\mathscr{P}$, and suppose that G is derived from $\mathscr{S}$ as described in the statement of Proposition 3. Because two different blocks of $\mathscr{S}$ can intersect in at most one point, it is clear that for each $w \in \mathscr{P}$, $G[N_G(w)]$ contains rK_{k-1} as a spanning subgraph. We have equality, $G[N_G(w)] \simeq rK_{k-1}$, unless there is an edge of G with ends in $B_1\backslash\{w\}$ and $B_2\backslash\{w\}$ for some $B_1, B_2 \in \mathscr{B}$, $B_1 \neq B_2$ with $\{w\} = B_1 \cap B_2$. There can be no such edge because $\mathscr{S}$ is trilateral-free. Therefore $G \in RCA(v, r(k-1), k)$, by Proposition 1. □

Remarks:

1. It is a straightforward chore to see that the derivations of trilateral-free configurations from regular clique assemblies, and of regular clique assemblies from trilateral-free configurations, as described in Proposition 3, are inverses of each other.
2. Another straightforward chore: verify that if $\mathscr{S}$ and G are, respectively, a trilateral-free configuration and a regular clique assembly which correspond à la Proposition 3, then $\mathscr{S}^T$ corresponds to $CL(G)$. Since $(\mathscr{S}^T)^T = \mathscr{S}$, this observation provides an elegant alternative proof of the claim in Theorem 1 that $CL(CL(G)) \simeq G$. The thrashing around on this matter in the proof of Theorem 1 has, in the alternative proof, been absorbed into the proof of Proposition 3.
3. Suppose that $\mathscr{S}_i = (\mathscr{P}_i, \mathscr{B}_i, \mathscr{I}_i)$ is a $((v_i)_r, (b_i)_k)$ configuration with $\mathscr{B}_i \subseteq \binom{\mathscr{P}_i}{k} = \{k\text{−subsets of } \mathscr{P}_i\}$, $i = 1, 2$. If $\mathscr{P}_1 \cap \mathscr{P}_2 = \emptyset$ then $\mathscr{S} = (\mathscr{P}_1 \cup \mathscr{P}_2, \mathscr{B}_1 \cup \mathscr{B}_2, \mathscr{I}_1 \cup \mathscr{I}_2)$ is a $((v_1+v_2)_r, (b_1+b_2)_k)$ configuration, trilateral free if and only if both $\mathscr{S}_1$ and $\mathscr{S}_2$ are. If $\mathscr{S}$ is obtained in this way from two other configurations, we will say that $\mathscr{S}$ is *reducible*. Otherwise, $\mathscr{S}$ is *irreducible*.

Proposition 4. *Suppose that $\mathscr{S}$ is a trilateral-free configuration and G is the regular clique assembly corresponding to $\mathscr{S}$. Then $\mathscr{S}$ is irreducible if and only if G is connected.*

Proof. It is straightforward to see that $\mathscr{S}$ is reducible if and only if G is not connected. □

In some precincts, irreducibility is part of the definition of configurations; not here, however.

A (v_r, b_k) configuration is symmetric if $v = b$ (equivalently, $r = k$), and is said to be a (v_k) configuration. By Proposition 3, the graph G corresponding to a trilateral-free (v_k) configuration belongs to $RCA(v, k(k-1), k)$ as does $CL(G)$, by Theorem 1, which naturally generates the questions: for which n and k is every graph in $RCA(n, k(k-1), k)$ isomorphic to its clique graph, and for which n and k does there exist $G \in RCA(n, k(k-1), k)$ such that $G \simeq CL(G)$? We have few answers (see Theorem 3 in the next section, for one), but we are encouraged to find that this graph isomorphism problem is equivalent to a geometric isomorphism problem, the question of which trilateral-free symmetric configurations are self-dual.

We embarked on this topic in search of answers about $ER(n, d, 1) = RCA(n, d, 3)$: for which n and d is this collection non-empty, and for such (n, d), how many different isomorphism classes of connected graphs are represented in $ER(n, d, 1)$? By Proposition 1 or by Theorem 1, $RCA(n, d, 3) \neq \emptyset$ only if d is even, and by Proposition 3 the graphs in $RCA(n, d, 3)$ correspond to symmetric trilateral-free configurations only if $d = 3(3-1) = 6$.

As we shall see in the next section, it is relatively easy to describe all graphs in $\bigcup_n ER(n,2t,1)$ if $t \in \{1,2\}$. For $t = 3$, using only graphical methods and results on RCAs, we had found that $ER(n,2t,1)$ contains a connected graph for $n = 15$ and $n \geq 17$, except possibly, for $n \in \{17,19,\ldots,26,28,29,30,40,\ldots,44,46\}$. Then a referee of an earlier version of this paper pointed out the connection between RCAs and trilateral-free configurations, and now we have this.

Theorem 2 (Raney [11], 2013)**.** *For every $n \geq 15$ except $n = 16$ there is a trilateral-free irreducible symmetric (n_3) configuration.*

Corollary 4. *For every $n \geq 15$ except $n = 16$, $RCA(n,6,3) = ER(n,6,1)$ contains a connected graph.*

In the next section we shall give alternate proofs of some of Theorem 2, and then repay some of our debt to configuration theory by giving easy proofs of the existence of graphs in $RCA(n,2t,3)$, $t > 3$ for various n.

4. Edge-regular graphs with $\lambda = 1$

By Corollary 1, $ER(n,d,1) = RCA(n,d,3)$. We sum up the conclusions of Section 2 for $ER(n,d,1)$ in the following.

Proposition 5. *Suppose $ER(n,d,1) \neq \emptyset$. Then*

1. *d is even;*
2. $3|nd$;
3. *for each $G \in ER(n,d,1)$ and $v \in V(G)$, $N_G[v]$ induces in G a friendship graph, $\{v\} \vee \frac{d}{2}K_2$;*
4. *if $d > 2$, each $G \in ER(n,d,1)$ is the clique graph of its clique graph, $CL(G) \in RCA(\frac{nd}{6}, \frac{3}{2}(d-2), \frac{d}{2})$.*

Conversely,

3.′ *If G is a graph such that for some positive integer m, for each $v \in V(G)$, $G[N_G[v]] \simeq \{v\} \vee mK_2$, then $G \in ER(n,2m,1)$, where $n = |V(G)|$; and*

4.′ *if G is the clique graph of some $H \in RCA(\frac{nd}{6}, \frac{3}{2}(d-2), \frac{d}{2})$, for some integers n and $d > 2$, then $G \in ER(n,d,1)$.*

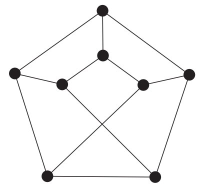
Clearly the only edge regular graphs with $\lambda = 1$ and $d = 2$ are the graphs mK_3. The edge-regular graphs with $d = 4$ and $\lambda = 1$ are completely characterized as follows.

Corollary 5. *$G \in ER(n,4,1)$ for some n if and only if G is the line graph of a triangle-free 3-regular graph.*

Proof. By Corollary 1, $ER(n,4,1) = RCA(n,4,3)$; the conclusion follows from Corollary 2. ☐

We note that Corollary 5 could also be extracted, almost entirely, from a passing remark appearing in Section 4 of [10].

Corollary 6. *There are exactly two graphs in* $ER(12,4,1)$, *the line graphs of* $K_{4,4} - M$, *where* M *is a perfect matching in* $K_{4,4}$, *and of*



Proof. Both $G_1 = K_{4,4} - M$ and the other graph, G_2, are 3-regular and triangle-free with 12 edges. Therefore, their line graphs are in $ER(12,4,1)$. Since, by Theorem 1, each G_i is the clique graph of its line graph, their line graphs are distinct.

Now suppose that $G \in ER(12,4,1)$. By Corollary 5, $G = L(H)$ for some $H \in ER(8,3,0)$. If H is bipartite, then, because H is bipartite and regular, H is 1-factorizable and so H must be $K_{4,4} - M$, for some perfect matching M.

If H is not bipartite, then, since H is K_3-free on 8 vertices, H must contain either a C_5 or a C_7 or both. If H contains a C_5, it must be induced in H, because H is triangle-free. Each vertex on the C_5 must therefore be adjacent to exactly one of the 3 vertices not on the C_5. If one of those vertices were adjacent to 3 vertices on the C_5, there would be a triangle in H. Therefore 2 of the 3 vertices off the C_5 are adjacent to 2 vertices each, on the C_5, and the third is adjacent to one vertex on the cycle and both of the other off-cycle vertices. From there it is easy to see that H must be G_2, the graph depicted above.

If H contains a C_7 then, because the one vertex off the cycle is adjacent to only 3 vertices on the cycle, H must contain two chords of the cycle. Any chord of a C_7 which does not create a K_3 must create a C_5, so H contains a C_5. Therefore $H \simeq G_2$. ☐

If, for some $n = |V(G)|$, $G \in ER(n,4,1) = RCA(n,4,3)$, then, by Theorem 1, $6|4n$, so $3|n$. (This conclusion also follows from Corollary 5.) Indeed, for any n and d, if $ER(n,d,1) \neq \emptyset$ then $6|nd$, and therefore, if $3 \nmid d$, then $3|n$. We mention this because it has been privately conjectured that whenever $ER(n,d,1) \neq \emptyset$, n must be divisible by 3. We shall see that this is not the case.

Corollary 5 shows that $ER(n,4,1)$ contains a connected graph for infinitely many n, and we shall soon see that $ER(n,6,1)$ contains a connected graph for infinitely many n. In passing, we note that these facts point to a powerful difference between the class of all edge regular graphs and the class of strongly regular graphs. An elementary necessary condition for

$SR(n,d,\lambda,\mu)$ to be non-empty is that $d(d-\lambda-1)=\mu(n-d-1)$ [2]. It follows that for given d, λ satisfying $d>\lambda+1$ there can be only finitely many pairs (n,μ) such that $SR(n,d,\lambda,\mu)\neq\emptyset$, and if $\mu>0$, any graph in $SR(n,d,\lambda,\mu)$ is connected.

In contrast, by Corollary 5 and either Proposition 2 or its analog, Proposition 6, to follow, it can be seen that for each even $d\geq 4$ there are infinitely many n such that $ER(n,d,1)$ contains a connected graph.

Proposition 6. *If $G\in ER(n_1,d_1,\lambda)$ and $H\in ER(n_2,d_2,\lambda)$, then $G\square H\in ER(n_1n_2,d_1+d_2,\lambda)$.*

The proof is straightforward.

Corollary 7. *For any integers $t\geq 3$, $q\geq 0$, $ER(3^{q+1}t,4+2q,1)$ contains a connected graph.*

Proof. It is easy to see that the smallest order of triangle-free 3-regular graph is 6 (and that the only such graph of that order is $K_{3,3}$). For any such graph H of order $2t$ one can make another such graph of order $2t+2$ by subdividing each of two independent edges of H and then making the two new vertices adjacent. Thus the possible orders of connected triangle-free 3-regular graphs are $2t,\ t\geq 3$. Therefore, by Corollary 5, $\{n \mid ER(n,4,1)$ contains a connected graph$\}=\{3t \mid t\geq 3$ and t is an integer$\}$. This establishes the claim of the corollary when $q=0$. Now suppose that $q>0,\ t\geq 3,\ G\in ER(3t,4,1)$, and G is connected. Then by Proposition 6, taking powers of K_3 with respect to the Cartesian product, $(K_3)^q\square G\in ER(3^q\cdot 3t,4+2q,1)$, and clearly $(K_3)^q\square G$ is connected. □

Fixing q and letting t vary, we see that for every even $d\geq 4$ there are infinitely many n such that $ER(n,d,1)$ contains a connected graph. Our ambition is to determine, for each even $d>4$, the spectrum $S_1^c(d)=\{n \mid ER(n,d,1)$ contains a connected graph$\}$. (We have $S_1^c(4)=\{9,12,15,\dots\}$ and $S_1^c(2)=\{3\}$.) Beyond that is the probably unachievable goal of determining, for each $n\in S_1^c(d)$, the different (isomorphism classes of) connected graphs in $ER(n,d,1)$. We make a start on these aims, after Corollary 8.

Corollary 8. *For any integers $t\geq 3,\ q\geq 0,$ there is an irreducible $((3^{q+1}t)_{2+q},(3^qt(2+q))_3)$ configuration.*

Proof. This is a consequence of Proposition 3, Proposition 4, and Corollary 7. □

By a remark in the Introduction, if $ER(n,6,1)\neq\emptyset$ then $n\geq 3(6-1)=15$. We shall see that $ER(15,6,1)$ contains exactly one graph and then use that graph to construct connected graphs in $ER(n,6,1)$ for infinitely many values of n.

Suppose m and k are positive integers. Let $[m]=\{1,\dots,m\}$ and let $\binom{[m]}{k}$ denote the set of all k-subsets of $[m]$. If $1\leq k\leq\frac{m}{2}$, the *Kneser graph* $K(m,k)$ has vertex set $\binom{[m]}{k}$, with $u,v\in\binom{[m]}{k}$ adjacent if and only if $u\cap v=\emptyset$

Lemma 2. *If m and k are integers satisfying $1 \le k \le \frac{m}{2}$, then $K(m,k) \in ER\left(\binom{m}{k}, \binom{m-k}{k}, \binom{m-2k}{k}\right)$. If $m \ge 4$, $K(m,2) \in SR\left(\binom{m}{2}, \binom{m-2}{2}, \binom{m-4}{2}, \binom{m-3}{2}\right)$.*

The verification is straightforward.

Corollary 9. *If $k \ge 1$, $K(3k,k) \in ER(\binom{3k}{k}, \binom{2k}{k}, 1)$.*

Theorem 3. *$K(6,2)$ is the unique graph in $ER(15,6,1)$.*

Proof. For any graph $G \in ER(n,6,1) = RCA(n,6,3)$, for any n, if $u,v,w \in V(G)$ induce a K_3 in G then, by Lemma 1 and its corollaries, the subgraph of G induced by $N[\{u,v,w\}] = N[u] \cup N[v] \cup N[w]$ has a spanning subgraph as depicted in Figure 2. By Corollary 9, $K(6,2) \in ER(15,6,1)$. For any $G \in ER(15,6,1)$, for any $u,v,w \in V(G)$ inducing K_3 in G, all 15 of G's vertices are on display in Figure 2. The edges of G not depicted are among the 12 vertices of $V(G)\backslash\{u,v,w\}$. Consider x_1. All 4 vertices to which x_1 is adjacent besides x_2 and v are among the z_j and the y_j. But x_1 cannot be adjacent to both z_1 and z_2, for instance, because the unique common neighbor of z_1 and z_2 is u. Therefore x_1 is adjacent to at most one of z_1, z_2, to at most one of z_3, z_4, to at most one of y_1, y_2, and to at most one of y_3, y_4. Therefore, x_1 is adjacent to *exactly* one of z_1, z_2, to exactly one of z_3, z_4, etc., because x_1 must have 4 neighbors among the 8 vertices.

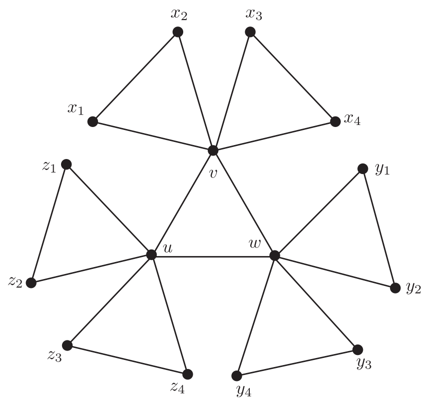


Figure 2: Spanning subgraph of $G[N[\{u,v,w\}]]$ for any $K_3 = G[\{u,v,w\}]$ in $G \in ER(n,6,1)$, for some n.

Therefore, u and x_1 have exactly 3 common neighbors, v and two among $z_1,\ldots,z_4$. But, because the diagram in Figure 2 will be the same (except for the vertex names), no matter which K_3 you start with, u and x_1 could be any two non-adjacent vertices in G. Therefore G is strongly regular: $G \in SR(15,6,1,3)$. According to [13], $K(6,2)$ is the only graph in $SR(15,6,1,3)$. □

For those who don't care for proof by appeal to websites, a more laborious proof can be given which provides an independent corroboration of the fact that $K(6,2)$ is the unique member of $SR(15,6,1,3)$. The full structure of the graph induced by the edges of G among the 12 vertices of $G-\{u,v,w\}$, excluding the edges shown in Figure 2 (x_1x_2, x_3x_4, etc.), can be deduced from the assumption that $G\in ER(15,6,1)$. For a somewhat shorter proof, note that that graph on 12 vertices must be in $ER(12,4,1)$; of the two possibilities given in Corollary 4, $L(G_2)$, where G_2 is the non-bipartite graph depicted, can be ruled out, and then it can be seen that $L(K_{4,4}-M)$ must be fitted onto the 12 vertices of degree 2 in Figure 2 so that the resulting graph is $K(6,2)$, if the resulting graph is to be edge regular with $d=6$, $\lambda=1$. But we omit the details.

By Proposition 3, the uniqueness of $K(6,2)$ in $ER(15,6,1)$ is equivalent to the uniqueness of the configuration associated with $K(6,2)$, as a symmetric (15_3) configuration. We are indebted to the previously mentioned referee of a previous version of this paper for pointing out that the uniqueness of this configuration has long been known, and is connected with the uniqueness of Tutte's 8-cage [4]. While we are on this subject, the well-known fact that there is no (16_3) configuration is a special case of the main result in [7], where it took the form of the claim that $ER(16,6,1)=\emptyset$. That main result, mentioned in the Introduction, is that $ER(3d-2,d,1)=\emptyset$ for all d.

4.1. Graphical construction of connected graphs in $ER(n,6,1)$ for infinitely many n

Start with the graph shown in Figure 2; we call this the *primary scaffold*. Each vertex in it has degree 2 or 6, any two vertices adjacent in the scaffold have a unique common neighbor in the scaffold, and non-adjacent vertices in the scaffold have at most one common neighbor. We can build new scaffolds with these properties from the primary scaffold in a number of ways. We shall describe the most straightforward construction method, leading to graphs in $ER(15+16k,6,1)$, $k=1,2,\ldots$, and then mention variations of the method that can produce graphs in $ER(n,6,1)$ for many other n, including all $n\geq 47$.

In a scaffold, each vertex of degree 6 is *finished*, and each vertex of degree 2 is *unfinished*. Produce a new scaffold by joining an unfinished vertex to the vertices of a $2K_2$ whose vertices are new to the scene. The 4 new vertices are unfinished in the new scaffold, and the formerly unfinished vertex to which they are joined is finished. The number of vertices has increased by 4 and the number of unfinished vertices has increased by 3.

The primary scaffold has 15 vertices, 12 of them unfinished. Therefore, after t iterations of the new-scaffold-generating process, the resulting scaffold will have $15+4t$ vertices, $12+3t$ of them unfinished. When $t=4k$ for some integer k, we have a scaffold on $15+16k$ vertices, with $12(k+1)$ of them unfinished.

At such a point we can stop building scaffolds and attempt to complete the scaffold we have to a graph in $ER(15+16k,6,1)$ by executing the following plan: partition the set of unfinished vertices in the scaffold into $k+1$ sets $P_1,\dots,P_{k+1}$ of 12 vertices each and then put edges among the vertices of P_j, for each j, so that the graph on those vertices, with those edges, is one of the two graphs in $ER(12,4,1)$ mentioned in Corollary 6.

For any choice of the P_j, and any insertion of the edges of one of the graphs in $ER(12,4,1)$ on the vertices of the P_j, $j=1,\dots,k+1$, the resulting graph on $15+16k$ will be regular of degree 6, any two adjacent vertices will be joined by one or two edges (possibly one from the scaffold and one inserted) and will have one or two neighbors in common (possibly one common neighbor in the scaffold and one in the imposed graph from $ER(12,4,1)$). We need to make arrangements so that there are no doubled edges in the completed graph and no two adjacent vertices in the completed graph have two common neighbors in that graph.

We posit the following requirements on $P_1,\dots,P_{k+1}$ and on the graphs $H_j \in ER(12,4,1)$ obtained by inserting edges among the vertices of P_j, $j=1,\dots,k+1$:

Each P_j must be partitionable into 3 sets Q_{1j},Q_{2j},Q_{3j} of 4 vertices each such that if $u\in Q_{ij}$, $v\in Q_{tj}$, $1\le i<t\le 3$, then u, v are distant at least 3 from each other in the scaffold. Explanation: Each graph in $ER(12,4,1)$ has chromatic number 3 and vertex independence number 4. The Q_{ij} will be independent sets of vertices in H_j, so pairs of vertices adjacent in H_j will be from different Q_{ij}. Therefore, because vertices in Q_{ij} and Q_{tj} for $t\ne i$ are distant at least 3 from each other in the scaffold, there will be no chance that an edge of the imposed H_j will double an edge of the scaffold. It is now sufficient to take care that no two vertices adjacent in H_j have a common neighbor in the scaffold and that no two vertices in P_j adjacent in the scaffold have a common neighbor in H_j.

Since two vertices adjacent in H_j are in Q_{ij} for different values of i, they are distant at least 3 from each other in the scaffold, and therefore have no common neighbor in the scaffold. Now suppose that $u,v\in P_j$ are adjacent in the scaffold. Then they must belong to the same Q_{ij}, since no two vertices in different Q_{ij} can be adjacent in the scaffold.

Since any 4 unfinished vertices in the scaffold induce one of $4K_1$, $2K_1+K_2$, or $2K_2$ in the scaffold, we can require that each Q_{ij} be partitioned into two 2-element sets, R_{1ij} and R_{2ij}, such that no vertex in R_{1ij} is adjacent to any vertex in R_{2ij} in the scaffold. Then form $H_j\simeq L(K_{4,4})-M$ with $R_{1,1,j}$, $R_{2,1,j}$,…, $R_{1,3,j}$, $R_{2,3,j}$ playing the roles of $\{x_1,x_2\}$, $\{x_3,x_4\}$,…, $\{z_1,z_2\}$, $\{z_3,z_4\}$, respectively, in a copy of $L(K_{4,4}-M)$ which completes the primary scaffold depicted in Figure 2 to $K(6,2)$.

Observe that the recommendation above for taking $H_j\simeq L(K_{4,4}-M)$ under certain circumstances is not an iron-clad requirement. It may well be that $L(K_{4,4}-M)$ may be successfully imposed upon P_j in other ways than the recommended way, or that $L(G_2)$ (see Corollary

6) may be successfully imposed, even if some Q_{ij}, $i \in \{1,2,3\}$, contains a pair of vertices adjacent in the scaffold. $L(G_2)$ may certainly be used if P_j is an independent set of vertices in the scaffold.

In Figure 3 are depicted two very different scaffolds of order 31, each with 24 unfinished vertices partitioned into two sets of 12 vertices, each of which is partitioned into 3 sets of 4 vertices each, satisfying the requirement that two vertices from different partition sets of 4 within either set of 12 are distant at least 3 from each other in the scaffold. In neither circumstance is the partition into 12-vertex subsets unique. In the top example, if the partition of the unfinished vertices into 12-vertex sets is as given, then the partition within each into 4-element sets is forced. This is not true in the other example. In each case, we intend $L(K_{4,4}-M)$ to be the graph imposed on each 12-vertex partition set for the completion of the given scaffold to a graph in $ER(31,6,1)$. We are certain that this is the only possible choice of an imposed graph from $ER(12,4,1)$ no matter what the partition choices for the top scaffold, and we are pretty sure that the same holds for the bottom scaffold. As k goes up, the $12(k+1)$ unfinished vertices in scaffolds of order $15+16k$ become more numerous and "spaced away" from each other, offering many more choices for admissible partitions into 12-vertex sets. It becomes easier to make arrangements so that $L(G_2)$ (see Cor. 6) can be used in the construction.

For non-negative integers d and λ, let $S_\lambda(d) = \{n \mid ER(n,d,\lambda) \neq \emptyset\}$ and $S^c_\lambda(d) = \{n \mid ER(n,d,\lambda)$ contains a connected graph $\}$. Observe that $S_\lambda(d)$ is closed under addition, since if $G_i \in ER(n_i,d,\lambda)$, $i=1,2$, then $G_1+G_2 \in ER(n_1+n_2,d,\lambda)$.

To find $S_\lambda(d)$ it suffices to find $S^c_\lambda(d)$. We know from Corollary 4 that $S^c_1(6) = S_1(6) = \{15,17,18,\ldots\}$. In what follows we will verify by direct construction all but finitely many of the elements of $S^c_1(6)$.

Variations in scaffold-building

In all of this, we start with the primary scaffold, depicted in Figure 2.

Let the scaffold-building operation described previously, in which an unfinished vertex is joined to a new $2K_2$, be called Method 1, or M1 for short. Here are two other scaffold-building operations.

M2: Take two unfinished vertices, a distance ≥ 3 from each other in the current scaffold; join them and join each to a new vertex. Finish each by joining it to a K_2 — the K_2s being disjoint and formed from new vertices.

Note that the number of vertices has increased by 5 and the number of unfinished vertices has increased by 3.

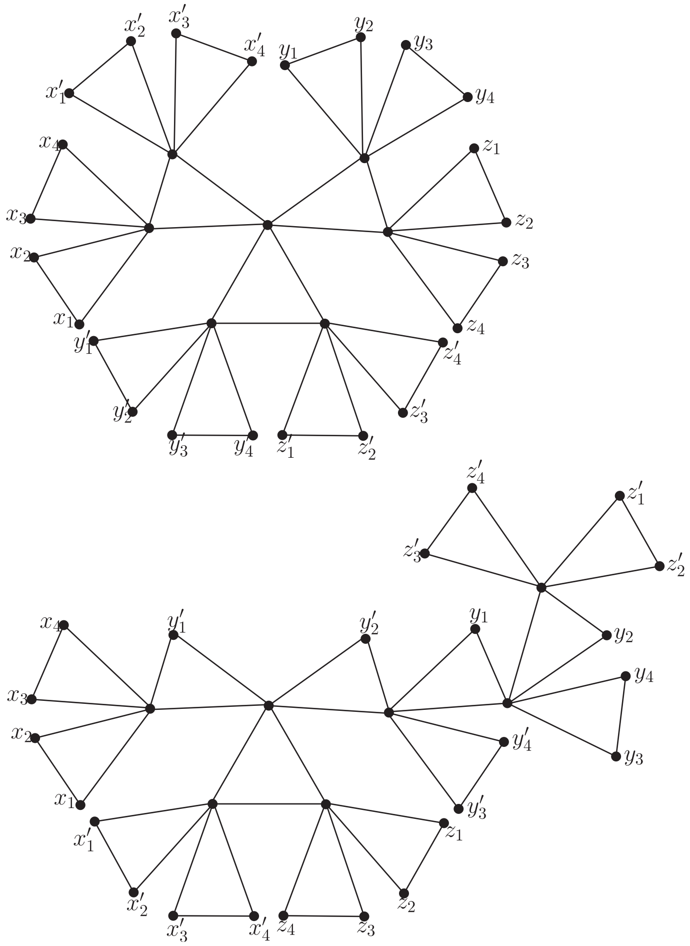


Figure 3: Two different scaffolds of order 31, with admissible partitions of the unfinished vertices into two 12-vertex sets: $\{x_1, x_2, \ldots, z_3, z_4\}$, $\{x'_1, x'_2, \ldots, z'_3, z'_4\}$.

M3: Take 3 unfinished vertices, any two distant at least 3 from each other in the current scaffold. Make them the vertices of a K_3, and then join each up to its own K_2, whose vertices are new and unfinished in the new scaffold.

The number of vertices has increased by 6, and the number of unfinished vertices has increased by 3.

If, starting with the primary scaffold, M_1 is applied a times, M_2 b times, and M_3 c times, in any order, then the resulting scaffold has $15+4a+5b+6c$ vertices, and $12+3(a+b+c)$ of them are unfinished. We leave it as an entertainment for the reader to verify that $\{15+4a+5b+6c \mid a, b, c$ are non-negative integers and $a+b+c \equiv 0 \mod 4\} = \{15\} \cup \{31, 32, \ldots\} \setminus \{40, \ldots, 46\}$, and so these are the orders of edge regular graphs with $d = 6$ and $\lambda = 1$ that can be constructed by this method proposed here. The meticulous reader will rightly worry that, when $a+b+c \equiv$

$0 \mod 4$, there may not be a partition of the unfinished vertices into 12-sets satisfying the requirements described above. Applications of M_1 alone results in an expanding universe of unfinished vertices, and there seems to be no problem, but in M_2 and M_3 unfinished vertices a distance ≥ 3 from each other are made adjacent in constructing the new scaffold. However, note that any application of M_2 or M_3 will increase the supply of pairs of unfinished vertices a distance ≥ 3 from each other, as each vertex which is being finished is made adjacent to the vertices of a new K_2. We won't bore you with a long explanation; we are sure that this consideration will ultimately satisfy the skeptical.

Since we already know $S_1^c(6)$ from Corollary 4, what is the use of all this construction activity? That's a question for mathematical philosophers. We are liberals, in favor of exposing all reasonably interesting mathematical discoveries, and letting natural selection take its course. Very likely these constructions are headed straight for the dustbin of history; on the other hand, it just might happen that one fine day an astrophysicist will badly need to estimate the greatest and least diameters of graphs in $ER(1028,6,1)$, in which case these constructions could come in handy.

There are further variations available in these constructions. Most obviously, we can allow two kinds of unfinished vertices, one kind with closed neighborhood K_3 in the scaffold, as before, and a new kind with closed neighborhood $K_1 \vee 2K_2$ in the scaffold. We are not sure if edge regular graphs with $d = 6$, $\lambda = 1$ can be constructed under this relaxation that could not be constructed using M_1, M_2, and M_3 only.

4.2. $ER(n,d,1)$ for d even, $d \geq 8$

By remarks preceding Corollary 5 and by Corollary 7, $S_1^c(2) = \{3\}$, $S_1(2) = \{3m \mid m$ is a positive integer$\}$ and $S_1^c(4) = S_1(4) = \{3t \mid 3 \leq t$ is an integer$\}$.

As mentioned previously, by Corollary 4 we have $S_1^c(6) = S_1(6) = \{15\} \cup \{n \geq 17 \mid n$ is an integer$\}$. By Corollary 7, for integers $q \geq 2$, if $d = 4+2q$ then $3^{q+1}t \in S_1^c(d)$ for each integer $t \geq 3$, so $S_1^c(d)$ is infinite for every even $d \geq 8$.

Obviously Proposition 6 can be used to improve Corollary 7 by enlarging our knowledge of $S_1^c(d)$, d even, $d \geq 8$. For example, if $d = 14 = 4+10$, Corollary 7 says that $3^6 t \in S_1^c(14)$ for all integers $t \geq 3$; this comes from Proposition 6, which implies that $(K_3)^5 \square G \in ER(3^6 t, 14, 1)$ for any $G \in ER(3t,4,1) \neq \emptyset$. [As before, all graph powers are taken with respect to the Cartesian product.] We also have that for $G \in ER(n_1,6,1)$, and $X \in ER(3x,4,1)$, $Y \in ER(3y,4,1)$, $x, y \geq 3$, $G \square X \square Y \in ER(9n_1xy,14,1)$, by Proposition 6; therefore $9n_1xy \in S_1^c(14)$ for all $n_1 \in S_1^c(6)$ and integers $x, y \geq 3$. From $14 = 6+6+2$ we get $3n_1n_2 \in S_1^c(14)$ for all $n_1, n_2 \in S_1^c(6)$. And there's more! But you get the idea. For every $n \in S_1^c(d)$ there may be quite a number of graphs in $ER(n,d,1)$—and for each of these a (v_r, b_3) configuration, $v = n$, $r = d/2$, $b = nd/6$, by Proposition 3.

But in the midst of this plenty, there are two nagging questions that we cannot answer. Recall that, by Proposition 5 (2), if $n \in S_1(d)$ and $3 \nmid d$ then $3 \mid n$. Our questions, concerning arbitrary even $d > 6$:

1. Suppose $3 \nmid d$; does $S_1^c(d)$ necessarily contain all sufficiently large multiples of 3?
2. Suppose $6 \mid d$: does $S_1^c(d)$ necessarily contain all sufficiently large integers?

We have that the answer to question 1 is yes when $d = 4$, and the answer to question 2 is yes when $d = 6$. We can also get an answer to question 1 when $d = 8$.

Proposition 7. $\{45\} \cup \{3n \mid n \in \{17, 18, \ldots\}\} \subseteq S_1^c(8)$.

Proof. By Proposition 6, if $G \in ER(n,6,1)$ then $G \square K_3 \in ER(3n,8,1)$. By Corollary 4, $\{n \mid ER(n,6,1)$ contains a connected graph$\} = \{15, 17, 18, \ldots\}$. □

We shall shortly have a similar result for $d = 10$. But, first, we exploit the fact that, for each d, $S_1(d)$ is closed under addition to answer questions analogous to 1 and 2 above for $S_1(d)$.

Theorem 4. *Suppose that $d > 2$ is an even integer. If $6 \mid d$ then $S_1(d)$ contains all sufficiently large integers. Otherwise, $S_1(d)$ contains all sufficiently large integer multiples of 3.*

Proof. We may as well suppose that $d > 8$, by previous results. The proof hinges on a well-known theorem of Frobenius, which asserts that if a and b are relatively prime positive integers, then every integer $z \geq (a-1)(b-1)$ is representable as $z = ma + nb$, m, n non-negative integers; in other words, every such z is a sum of a's and b's.

If $d = 6q$, $q \geq 2$, then, because $15, 17 \in S_1^c(6)$, $15^q, 17^q \in S_1^c(d)$ (by Proposition 6) and so $\{n \in \mathbb{N} \mid n \geq (15^q - 1)(17^q - 1)\} \subseteq \{m15^q + n17^q \mid m, n \in \mathbb{N}\} \subseteq S_1(d)$.

If $d = 6q + 2$, $q \geq 2$, take $G \in ER(15,6,1)$, and $H \in ER(17,6,1)$; then $G^q \square K_3 \in ER(3 \cdot 15^q, 6q+2, 1)$, $H^q \square K_3 \in ER(3 \cdot 17^q, 6q+2, 1)$, by Proposition 6, again. So $3 \cdot 15^q, 3 \cdot 17^q \in S_1(d)$. Therefore $\{3n \mid n \in \mathbb{N}$ and $n \geq (15^q - 1)(17^q - 1)\} \subseteq S_1(d)$.

If $d = 6q + 4$, $q \geq 1$, then, taking G and H as above, $X = L(K_{3,3}) \in ER(9,4,1)$, and $Y \in ER(12,4,1)$, then $G^q \square X \in ER(9 \cdot 15^q, 6q+4, 1)$, $H^q \square Y \in ER(12 \cdot 17^q, 6q+4, 1)$, and so, because $3 \cdot 15^q, 4 \cdot 17^q$ are relatively prime, we find that $\{3n \mid n \in \mathbb{N}$ and $n \geq (3 \cdot 15^q - 1)(4 \cdot 17^q - 1)\}$ $\subseteq S_1(d)$. □

Another query on the question of edge regular graphs with $\lambda = 1$: For which even $d > 6$ can we build edge regular graphs with degree d and $\lambda = 1$ from a scaffold consisting of $N[\{u, v, w\}]$, where uvw is a K_3 in the graph, as we did in the case $d = 6$? To be able to proceed

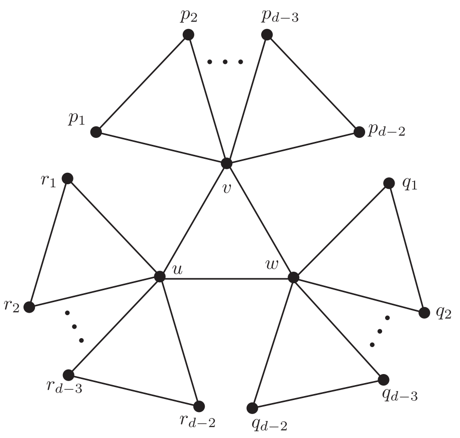


Figure 4: A spanning subgraph of $G[N[\{u,v,w\}]]$, $G \in ER(n,d,1)$ for some n.

as in that case, we would need $N[\{u,v,w\}] = V(G)$, where G is the graph containing the primary scaffold $N[\{u,v,w\}]$. In short, we need $ER(3(d-1),d,1)$ to be non-empty. (Recall that $3(d-\lambda)$ is a universal lower bound for the orders of edge regular graphs with degree d and λ triangles on each edge.)

Lemma 3. *If $d > 2$ then $ER(3(d-1),d,1) = SR(3(d-1),d,1,\frac{d}{2})$.*

Proof. Clearly $ER(3(d-1),d,1) \supseteq SR(3(d-1),d,1,\mu)$ for any μ. If $G \in ER(3(d-1),d,1)$ then d is even and for every triangle uvw in G, there is a spanning subgraph of G as depicted in Figure 4. The proof now proceeds by the argument in the proof of Theorem 3, about the case $d = 6$:

In Figure 4, p_1 and p_2 can have no common neighbor but v, neither is adjacent to any p_i, $i > 2$, and neither can be adjacent to two adjacent vertices among the q_i, nor among the r_i; it follows that each has $\frac{d-2}{2}$ neighbors among the q_i and among the r_i. Therefore, p_1 and u have $1 + \frac{d-2}{2} = \frac{d}{2}$ common neighbors. Since p_1 and u could be any two vertices not adjacent in G, it follows that G is strongly regular with $\mu = \frac{d}{2}$. □

Corollary 10. *$ER(3(d-1),d,1) \neq \emptyset$ if and only if $d \in \{2,4,6,10\}$.*

Proof. The second-best-known necessary condition for $SR(n,d,\lambda,\mu) \neq \emptyset$, the *integrality condition* ([1, 2]), is that each of $\frac{1}{2}\left[(n-1) \pm \frac{(n-1)(\mu-\lambda)-2d}{\sqrt{(\mu-\lambda)^2+4(d-\mu)}}\right]$ is a non-negative integer. Plugging $n = 3(d-1)$, $\lambda = 1$, $\mu = \frac{d}{2}$, and simplifying, we find that $\frac{1}{2}(3d-4 \pm (3d-20+\frac{48}{d+2}))$ must be non-negative integers.

Among even integers greater than 2, the possibilities for d are 4, 6, 10, and 22. Spence's website [13] shows a graph, and only one graph, in $SR(27,10,1,5)$. That $SR(63,22,1,11) =$

$\emptyset$ can be shown using a less well-known necessary condition for the existence of a strongly regular graph, the *absolute bound*. See [9], Theorem 21.4. ☐

Corollary 11. $24 \in S_1^c(8)$.

Proof. Consider $G \in ER(27, 10, 1)$, and any one of the spanning "scaffolds" depicted in Figure 4, with $d = 10$. The edges of G not pictured in Figure 4 induce $H \in ER(24, 8, 1)$. If H were not connected then one of its components would be edge-regular with $d = 8$, $\lambda = 1$, on no more than 12 vertices. Since $12 < 21 = 3(8-1)$, this is impossible. ☐

Corollary 12. $S_1^c(10)$ *contains all sufficiently large multiples of 3.*

Proof. Starting with the scaffold in Figure 4, with $d = 10$, apply the scaffold-building analogs of Methods 1, 2, 3 that were used in the case $d = 6$ to build new scaffolds in which the number of unfinished vertices is a multiple of 24. Finish these off to make connected edge-regular graphs with $d = 10$ and $\lambda = 1$ by the method analogous to that used in the case $d = 6$, using the graph H referred to in the proof of Corollary 11 as $L(K_{4,4} - M)$ was used in the $d = 6$ constructions. There follows the verification that graphs in $ER(3q, 10, 1)$ can be so constructed for all sufficiently large integers q.

In the case $d = 10$, each instance of Method 1 increases the number of vertices by 8 and the number of unfinished vertices by 7; Method 2 increases the number of vertices by 13 and the number of unfinished vertices by 11; and Method 3 increases the number of vertices by 18 and the number of unfinished vertices by 15. As in the case $d = 6$, it is obvious that the distance requirements to be satisfied in applying these methods and in finishing scaffolds to edge-regular graphs with $d = 10$, $\lambda = 1$, are not a problem. Therefore, since we are starting with a scaffold with 27 vertices, 24 of them unfinished, it suffices to show that $T = \{8a + 13b + 18c \mid a, b, c \in \mathbb{N} \text{ and } 7a + 11b + 15c \equiv 0 \mod 24\}$ contains all sufficiently large multiples of 3.

Clearly T is closed under addition; therefore, T is closed under taking non-negative integer combinations. We have that $8 \cdot 1 + 13 \cdot 1 + 18 \cdot 2 = 57 \in T$ and $8 \cdot 6 + 13 \cdot 0 + 18 \cdot 2 = 84 \in T$. Therefore, for all $d, e \in \mathbb{N}$, $57d + 84e = 3(19d + 28e) \in T$. By the famous theorem of Frobenius mentioned earlier, T contains $3t$ for all $t \geq (19-1)(28-1)$. ☐

One last open question, arising from Corollaries 7 and 10: For an even positive integer $d \notin \{2, 4, 6, 10\}$, what is the smallest element of $S_1^c(d)$, as a function of d?

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