TAMKANG JOURNAL OF MATHEMATICS
Volume 35, Number 1, Spring 2004

STRONGLY STARLIKE FUNCTIONS ASSOCIATED WITH THE DZIOK-SRIVASTAVA OPERATOR

JIN-LIN LIU

Abstract. In terms of the Dziok-Srivastava operator, we introduce and study some new classes of strongly starlike functions. Certain properties of these subclasses are studied.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. A function $f(z)$ belonging to the class A is said to be convex of order γ in E if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \gamma \quad (z \in E;\ 0 \le \gamma < 1). \tag{1.2}$$

A function $f(z)$ belonging to the class A is said to be starlike of order γ in E if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \gamma \quad (z \in E;\ 0 \le \gamma < 1). \tag{1.3}$$

We denote by $C(\gamma)$ the class of all functions in A which are convex of order γ in E and by $S^*(\gamma)$ the class of all functions in A which are starlike of order γ in E. From (1.2) and (1.3), we can see that $f(z) \in C(\gamma)$ if and only if $zf'(z) \in S^*(\gamma)$.

If $f(z) \in A$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta \quad (z \in E;\ 0 < \beta \le 1,\ 0 \le \gamma < 1), \tag{1.4}$$

Received March 15, 2002; revised October 9, 2002.
2000 *Mathematics Subject Classification*. Primary. 30C45, 26A33; Secondary. 33C20.
Key words and phrases. The Dziok-Srivastava operator, strongly starlike function, strongly convex function.

then we say that $f(z)$ is strongly starlike of order β and type γ in E, and we denote by $S^*(\beta,\gamma)$ the class of all such functions. If $f(z)\in A$ satisfies

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}-\gamma\right)\right|<\frac{\pi}{2}\beta \quad (z\in E;\ 0<\beta\le 1,\ 0\le\gamma<1), \tag{1.5}$$

then $f(z)$ is said to be strongly convex of order β and type γ in E, and denote by $f(z)\in C(\beta,\gamma)$. It is obvious that $f(z)\in A$ belongs to $C(\beta,\gamma)$ if and only if $zf'(z)\in S^*(\beta,\gamma)$. Further, we note that $S^*(1,\gamma)=S^*(\gamma)$ and $C(1,\gamma)=C(\gamma)$.

Very recently, Dziok and Srivastava [3] have given a systematic investigation of a linear operator $H(\alpha_1,\alpha_2,\ldots,\alpha_q;\beta_1,\beta_2,\ldots,\beta_s)$ which is defined by the generalized hypergeometric function (see, for details, [3]). They showed that

$$\begin{aligned}&zH'(\alpha_1,\alpha_2,\ldots,\alpha_q;\beta_1,\beta_2,\ldots,\beta_s)f(z)\\ &=\alpha_1H(\alpha_1+1,\alpha_2,\ldots,\alpha_q;\beta_1,\beta_2,\ldots,\beta_s)f(z)-(\alpha_1-1)H(\alpha_1,\alpha_2,\ldots,\alpha_q;\beta_1,\beta_2,\ldots,\beta_s)f(z)\end{aligned} \tag{1.6}$$

Let $q,\ s\in N$, $0<\beta\le 1$ and $0\le\gamma<1$. We now introduce the following classes in terms of the linear operator $H(\alpha_1,\alpha_2,\ldots,\alpha_q;\beta_1,\beta_2,\ldots,\beta_s)$:

$$M_{\alpha_1}(\beta,\gamma)=\left\{f(z)\in A: H_{q,s}(\alpha_1)f(z)\in S^*(\beta,\gamma),\ \frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)}\ne\gamma \quad \text{for } z\in E\right\}$$

and

$$P_{\alpha_1}(\beta,\gamma)=\left\{f(z)\in A: H_{q,s}(\alpha_1)f(z)\in C(\beta,\gamma),\ 1+\frac{z(H_{q,s}(\alpha_1)f(z))''}{(H_{q,s}(\alpha_1)f(z))'}\ne\gamma \ \text{ for } z\in E\right\},$$

where, for convenience,

$$H_{q,s}(\alpha_1)=H(\alpha_1,\alpha_2,\ldots,\alpha_q;\beta_1,\beta_2,\ldots,\beta_s).$$

For $q=s+1$ and $\alpha_1=1$, $\alpha_2=\beta_1,\ldots,\alpha_q=\beta_s$, it is easy to see that $M_1(\beta,\gamma)$ is the class of strongly starlike functions of order β and type γ whereas $P_1(\beta,\gamma)$ is the class of strongly convex functions of order β and type γ. Further, when $\beta=1$, we note that $M_1(1,\gamma)=S^*(\gamma)$ and $P_1(1,\gamma)=C(\gamma)$.

In this note, we shall study some inclusion properties of the classes $M_{\alpha_1}(\beta,\gamma)$ and $P_{\alpha_1}(\beta,\gamma)$. The basic tool of our investigation is the following lemma due to Nunokawa [16, 17].

Lemma. *Let a function $p(z)=1+b_1z+\cdots$ be analytic in E and $p(z)\ne 0$ $(z\in E)$. If there exists a point $z_0\in E$ such that*

$$|\arg p(z)|<\pi\beta/2\ (|z|<|z_0|) \quad \text{and} \quad |\arg p(z_0)|=\pi\beta/2\ (0<\beta\le 1),$$

then we have $z_0p'(z_0)/p(z_0)=ik\beta$, where

$$k\ge\frac{1}{2}\left(a+\frac{1}{a}\right) \quad (\text{where } \arg p(z_0)=\pi\beta/2),$$
$$k\le-\frac{1}{2}\left(a+\frac{1}{a}\right) \quad (\text{where } \arg p(z_0)=-\pi\beta/2),$$

and $(p(z_0))^{1/\beta} = \pm ia$ $(a > 0)$.

2. Main Results

Our first inclusion theorem is the following

Theorem 1. $M_{\alpha_1+1}(\beta,\gamma) \subset M_{\alpha_1}(\beta,\gamma)$ *for* $\alpha_1 > 1-\gamma$ *and* $0 \le \gamma < 1$.

Proof. Let $f(z) \in M_{\alpha_1+1}(\beta,\gamma)$. Suppose that

$$\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} = \gamma + (1-\gamma)p(z), \tag{2.1}$$

where $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in E and $p(z) \neq 0$ for all $z \in E$. By (2.1) and (1.6), we have

$$\begin{aligned}\frac{H_{q,s}(\alpha_1+1)f(z)}{H_{q,s}(\alpha_1)f(z)} &= \frac{1}{\alpha_1}\left[\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} + (\alpha_1-1)\right]\\ &= \frac{1}{\alpha_1}[(1-\gamma)p(z) + (\gamma+\alpha_1-1)]. \end{aligned}\tag{2.2}$$

Differentiating both sides of (2.2) logarithmically, it follows that

$$\begin{aligned}\frac{z(H_{q,s}(\alpha_1+1)f(z))'}{H_{q,s}(\alpha_1+1)f(z)} &= \frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} + \frac{(1-\gamma)zp'(z)}{(1-\gamma)p(z)+(\gamma+\alpha_1-1)}\\ &= (1-\gamma)p(z) + \gamma + \frac{(1-\gamma)zp'(z)}{(1-\gamma)p(z)+(\gamma+\alpha_1-1)},\end{aligned}$$

or

$$\frac{z(H_{q,s}(\alpha_1+1)f(z))'}{H_{q,s}(\alpha_1+1)f(z)} - \gamma = (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{(1-\gamma)p(z)+(\gamma+\alpha_1-1)}.$$

Suppose that there exists a point $z_0 \in E$ such that

$$|\arg p(z)| < \pi\beta/2 \ (|z| < |z_0|) \quad \text{and} \quad |\arg p(z_0)| = \pi\beta/2.$$

Then, by Lemma, we can write that $z_0 p'(z_0)/p(z_0) = ik\beta$ and $(p(z_0))^{1/\beta} = \pm ia$ $(a > 0)$.

Therefore, if $\arg p(z_0) = \pi\beta/2$, then

$$\begin{aligned}\frac{z(H_{q,s}(\alpha_1+1)f(z_0))'}{H_{q,s}(\alpha_1+1)f(z_0)} - \gamma &= (1-\gamma)p(z_0)\left[1 + \frac{z_0p'(z_0)/p(z_0)}{(1-\gamma)p(z_0)+(\gamma+\alpha_1-1)}\right]\\ &= (1-\gamma)a^\beta e^{i\pi\beta/2}\left[1 + \frac{ik\beta}{(1-\gamma)a^\beta e^{i\pi\beta/2}+(\gamma+\alpha_1-1)}\right].\end{aligned}$$

This implies that

$$\arg\left\{\frac{z_0(H_{q,s}(\alpha_1+1)f(z_0))'}{H_{q,s}(\alpha_1+1)f(z_0)} - \gamma\right\}$$

$$= \frac{\pi}{2}\beta + \arg\left\{1 + \frac{ik\beta}{(1-\gamma)a^{\beta}e^{i\pi\beta/2} + (\gamma+\alpha_1-1)}\right\}$$
$$= \frac{\pi}{2}\beta + \tan^{-1}\left\{\frac{k\beta[(\gamma+\alpha_1-1)+(1-\gamma)a^{\beta}\cos(\frac{\pi\beta}{2})]}{(\gamma+\alpha_1-1)^2+2(\gamma+\alpha_1-1)(1-\gamma)a^{\beta}\cos(\frac{\pi\beta}{2})+(1-\gamma)^2a^{2\beta}+k\beta(1-\gamma)a^{\beta}\sin(\frac{\pi\beta}{2})}\right\}$$
$$\geq \frac{\pi}{2}\beta \quad (\text{where } k \geq \tfrac{1}{2}(a+\tfrac{1}{a}) > 1),$$

which contradicts the hypothesis that $f(z) \in M_{\alpha_1+1}(\beta,\gamma)$.

Similarly, if $\arg p(z_0) = -\pi\beta/2$, then we obtain that

$$\arg\left\{\frac{z_0(H_{q,s}(\alpha_1+1)f(z_0))'}{H_{q,s}(\alpha_1+1)f(z_0)} - \gamma\right\} \leq -\frac{\pi}{2}\beta,$$

which also contradicts the hypothesis that $f(z) \in M_{\alpha_1+1}(\beta,\gamma)$.

Thus the function $p(z)$ has to satisfy $|\arg p(z)| < \pi\beta/2$ $(z \in E)$. This shows that

$$\left|\arg\left\{\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)} - \gamma\right\}\right| \leq \frac{\pi}{2}\beta \quad (z \in E).$$

This evidently completes the proof of Theorem 1.

By Theorem 1, we also have the following

Theorem 2. *$P_{\alpha_1+1}(\beta,\gamma) \subset P_{\alpha_1}(\beta,\gamma)$ for $\alpha_1 > 1-\gamma$ and $0 \leq \gamma < 1$.*

Proof.
$$\begin{aligned} & f(z) \in P_{\alpha_1+1}(\beta,\gamma) \Leftrightarrow H_{q,s}(\alpha_1+1)f(z) \in C(\beta,\gamma) \\ \Leftrightarrow\ & z(H_{q,s}(\alpha_1+1)f(z))' \in S^*(\beta,\gamma) \Leftrightarrow H_{q,s}(\alpha_1+1)(zf'(z)) \in S^*(\beta,\gamma) \\ \Leftrightarrow\ & zf'(z) \in M_{\alpha_1+1}(\beta,\gamma) \Rightarrow zf'(z) \in M_{\alpha_1}(\beta,\gamma) \\ \Leftrightarrow\ & H_{q,s}(\alpha_1)(zf'(z)) \in S^*(\beta,\gamma) \Leftrightarrow z(H_{q,s}(\alpha_1)f(z))' \in S^*(\beta,\gamma) \\ \Leftrightarrow\ & H_{q,s}(\alpha_1)f(z) \in C(\beta,\gamma) \Leftrightarrow f(z) \in P_{\alpha_1}(\beta,\gamma). \end{aligned}$$

We next state.

Theorem 3. *Let $v > -\gamma$ and $0 \leq \gamma < 1$. If $f(z) \in M_{\alpha_1}(\beta,\gamma)$ with $z(H_{q,s}(\alpha_1)J_vf(z))'/H_{q,s}(\alpha_1)J_vf(z) \neq \gamma$ for all $z \in E$, then $J_vf(z) \in M_{\alpha_1}(\beta,\gamma)$, where $J_vf(z)$ is given by*

$$J_vf(z) = \frac{v+1}{z^v}\int_0^z t^{v-1}f(t)dt \quad (v > -1; f(z) \in A). \tag{2.3}$$

Proof. Let
$$\frac{z(H_{q,s}(\alpha_1)J_vf(z))'}{H_{q,s}(\alpha_1)J_vf(z)} = \gamma + (1-\gamma)p(z), \tag{2.4}$$

where $p(z)$ is analytic in E, $p(0) = 1$ and $p(z) \neq 0$ $(z \in E)$. From (2.3), we have

$$z(H_{q,s}(\alpha_1)J_vf(z))' = (v+1)H_{q,s}(\alpha_1)f(z) - vH_{q,s}(\alpha_1)J_vf(z). \tag{2.5}$$

By (2.4) and (2.5), we obtain

$$\frac{H_{q,s}(\alpha_1)f(z)}{H_{q,s}(\alpha_1)J_vf(z)}=\frac{1}{v+1}[(1-\gamma)p(z)+(\gamma+v)]. \tag{2.6}$$

Differentiating (2.6) logarithmically, we have

$$\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)}-\gamma=(1-\gamma)p(z)+\frac{(1-\gamma)zp'(z)}{(1-\gamma)p(z)+(\gamma+v)}.$$

The remaining part of the proof is similar to that of Theorem 1 and so is omitted.

Finally, we derive the following.

Theorem 4. *Let $v>-\gamma$ and $0\le\gamma<1$. If $f(z)\in P_{\alpha_1}(\beta,\gamma)$ and $1+z(H_{q,s}(\alpha_1)J_vf(z))''/(H_{q,s}(\alpha_1)J_vf(z))'\neq\gamma$ for all $z\in E$, then $J_vf(z)\in P_{\alpha_1}(\beta,\gamma)$.*

Proof.
$$\begin{aligned}&f(z)\in P_{\alpha_1}(\beta,\gamma)\Leftrightarrow zf'(z)\in M_{\alpha_1}(\beta,\gamma)\\ \Leftrightarrow\ &J_v(zf'(z))\in M_{\alpha_1}(\beta,\gamma)\Leftrightarrow z(J_vf(z))'\in M_{\alpha_1}(\beta,\gamma)\\ \Leftrightarrow\ &J_vf(z)\in P_{\alpha_1}(\beta,\gamma).\end{aligned}$$

Acknowledgement

The author would like to express his gratitude to the referee for many valuable advices.

References

[1] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc. **135**(1969), 429-446.

[2] B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions*, SIAM J. Math. Anal. **15**(1984), 737-745.

[3] J. Dziok and H. M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput. **103**(1999), 1-13.

[4] A. Gangaharan and V. Ravichandran, *Radii of convexity and strong starlikeness for some classes of analytic functions*, J. Math. Anal. Appl. **211**(1997), 301-313.

[5] Yu. E. Hohlov, *Operators and operations in th class of univalent functions*, IZV. Vysš. Učebn. Zaved. Mathematika **10**(1978), 83-89.

[6] I. B. Jung, Y. C. Kim and H. M. Srivastava, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl. **176**(1993), 138-147.

[7] Y. C. Kim, S. H. Lee and H. M. Srivastava, *Some properties of convolution operators in the class $P_\alpha(\beta)$*, J. Math. Anal. Appl. **187**(1994), 498-512.

[8] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. **16**(1965), 755-758.

[9] J. L. Liu, *Note on the Ruscheweyh derivatives*, J. Math. Anal. Appl. **199**(1996), 936-940.

[10] J. L. Liu, *Noor integral operator and strongly starlike functions*, J. Math. Anal. Appl. **261**(2001), 441-447.

[11] A. E. Livingston, *On the radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc. **17**(1966), 352-357.

[12] W. Ma and D. Minda, *Coefficient inequalities for strongly close-to-convex functions*, J. Math. Anal. Appl. **205**(1997), 537-553.

[13] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49**(1975), 109-115.

[14] H. M. Srivastava and S. Owa, *Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators, and certain subcalsses of analytic functions*, Nagoya Math. J. **106**(1987), 1-28.

[15] H. M. Srivastava and S. Owa (Eds.), Univalent Functions, Fractional Calculus, and Their Applications, Halsted Press (Ellis Horwood Limited, Chichester), Wiley, New York/Chichester/Brisbane/Toronto, 1989.

[16] M. Nunokawa, *On properties of non-caratheodory functions*, Proc. Japan Acad. Ser. A Math. Sci. **68**(1992), 152-153.

[17] M. Nunokawa et al., *Some results for strongly starlike functions*, J. Math. Anal. Appl. **212**(1997), 98-106.

Department of Mathematics, Yangzhou University, Yangzhou 225002, Jiangsu, People's Republic of China.

E-mail: jlliucn@yahoo.com.cn