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FURTHER RESULTS ON DIFFERENTIAL INEQUALITY OF A CLASS OF SECOND ORDER NEUTRAL TYPE

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Abstract. In this paper, we develop several new results related to the nonexistence criteria for eventually positive solutions of a class of second order neutral differential inequalities with distributed deviating arguments. The work generalizes various existing result.

1. Introduction

We consider the following second order neutral equations with distributed deviating arguments

$$[x(t)+c(t)x(t-\tau)]''+\int_a^b p(t,\xi)f(x[g(t,\xi)])d\sigma(\xi)\le 0, \tag{1}$$

$$[x(t)+c(t)x(t-\tau)]''+\int_a^b p(t,\xi)f(x[g(t,\xi)])d\sigma(\xi)\ge 0, \tag{2}$$

where $\tau>0$ is a constant; $c(t)\in C([t_0,\infty),I)$, $I=[0,1]$; $f(x)\in C(R,R)$ and $xf(x)>0$, for $x\neq 0$, $p(t,\xi)\in C([t_0,\infty)\times[a,b],R_+)$, and $p(t,\xi)$ is not eventually zero on any ray $[t_\mu,\infty)\times[a,b]$, $t_\mu\ge t_0$, $R_+=[0,\infty)$; $g(t,\xi)\in C([t_0,\infty)\times[a,b],R)$, $\frac{d}{dt}g(t,a)$ exists, $g(t,\xi)\le t$, $\xi\in[a,b]$; $g(t,\xi)$ is nondecreasing with respect to t and ξ respectively; and $\liminf_{t\to\infty,\xi\in[a,b]}\{g(t,\xi)\}=\infty$; $\sigma(\xi)\in([a,b],R)$ is nondecreasing, the integral of equation (1) is a Stieltjes one.

Recently, there has been an increasing interest in delay differential inequalities, and a number of results have been obtained. For more details, we refer the reader to the literature [1-5]. In this paper, we establish some general nonexistence criteria of eventually positive solutions for inequality (1).

As is customary, the solution $x(t)\in C([t_0,\infty),R)$ of inequality (1) is said to be eventually positive if there exists a sufficiently large positive number μ such that the inequality $x(t)>0$ holds for $t\ge\mu$.

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2. Nonexistence Criteria

The following theorems provide the sufficient conditions leading to nonexistence of eventually positive solutions for inequality (1).

Theorem 1. *Assume that* $f(-x) = -f(x)$, $x \in (0, \infty)$, *and*

$$\frac{f(x)}{x} \geq \lambda, \quad x \in (0, \infty), \quad \textit{for some constant} \quad \lambda > 0. \tag{3}$$

If for any integer $m > 2$,

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} \left[\lambda(t-s)^m \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\} d\sigma(\xi) - \frac{m^2(t-s)^2}{4g'(s,a)} \right] ds = \infty, \tag{4}$$

then inequality (1) *has no eventually positive solutions.*

Proof. Assume the contrary, without loss of generality, $x(t)$ is an eventually positive solution of inequality (1). Then from $\liminf_{t\to\infty, \xi\in[a,b]}\{g(t,\xi)\} = \infty$, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(t-\tau) > 0$ and $x[g(t,\xi)] > 0$, $t \geq t_1$, $\xi \in [a,b]$. Set

$$y(t) = x(t) + c(t)x(t-\tau), \tag{5}$$

then, we have $y(t) \geq x(t) > 0$, $y''(t) \leq 0$, $t \geq t_1$, and we can claim that $y'(t) \geq 0$, $t \geq t_1$. In fact, assume that it is not true, then there exists a $t_2 \geq t_1$ such that $y'(t_2) < 0$. From the fact that $y'(t)$ is decreasing, there exists a $t_3 \geq t_2$ such that $y'(t_3) < 0$, and $y'(t) \leq y'(t_3) < 0$, $t \geq t_3$. Integrating from t_3 to t, we have $y(t) \leq y(t_3) + y'(t_3)(t - t_3)$. Thus, we conclude that $\lim_{t\to\infty} y(t) = -\infty$. This contradicts $y(t) > 0$. From (1) and the condition of Theorem 1, we obtain

$$\begin{aligned} 0 &\geq y''(t) + \int_a^b p(t,\xi) f(x[g(t,\xi)]) d\sigma(\xi) \\ &\geq y''(t) + \lambda \int_a^b p(t,\xi)\{y[g(t,\xi)] - c[g(t,\xi)]x[g(t,\xi)-\tau]\} d\sigma(\xi). \end{aligned} \tag{6}$$

Using $y'(t) \geq 0$, and $y(t) \geq x(t)$, $t \geq t_1$, we have $y[g(t,\xi)] \geq y[g(t,\xi)-\tau] \geq x[g(t,\xi)-\tau]$, thus

$$y''(t) + \lambda \int_a^b p(t,\xi)\{1 - c[g(t,\xi)]\} y[g(t,\xi)] d\sigma(\xi) \leq 0, \quad t \geq t_1. \tag{7}$$

Furthermore, as $g(t,\xi)$ is nondecreasing with respect to ξ, we have

$$y''(t) + \lambda y[g(t,a)] \int_a^b p(t,\xi)\{1 - c[g(t,\xi)]\} d\sigma(\xi) \leq 0, \quad t \geq t_1. \tag{8}$$

Set

$$z(t) = \frac{y'(t)}{y[g(t,a)]}. \tag{9}$$

Then $z(t) \geq 0$. From the fact that there exists a $\frac{d}{dt}g(t,a)$, we obtain $y'[g(t,a)] = \frac{dy}{dg}\frac{d}{dt}g(t,a)$. Further, by noting that $g(t,\xi)$ is nondecreasing with respect to ξ, $g(t,\xi) \leq t$, $\xi \in [a,b]$, and $y''(t) \leq 0$, we obtain $y'(t) \leq y'[g(t,a)]$. Thus

$$\begin{aligned} z'(t) &= \frac{y''(t)}{y[g(t,a)]} - \frac{y'(t)y'[g(t,a)]g'(t,a)}{y^2[g(t,a)]} \\ &\leq -\lambda \int_a^b p(t,\xi)\{1 - c[g(t,\xi)]\}d\sigma(\xi) - g'(t,a)z^2(t), \quad t \geq t_1. \end{aligned} \tag{10}$$

Integrating by parts for any $t > T \geq t_1$, we have

$$\begin{aligned} &\int_T^t \lambda(t-s)^m \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}d\sigma(\xi)ds \\ &\leq -\int_T^t (t-s)^m z'(s)ds - \int_T^t (t-s)^m g'(s,a)z^2(s)ds \\ &= -\int_T^t (t-s)^m dz(s) - \int_T^t (t-s)^m g'(s,a)z^2(s)ds \\ &= (t-T)^m z(T) - m\int_T^t (t-s)^{m-1}z(s)ds - \int_T^t (t-s)^m g'(s,a)z^2(s)ds \\ &= (t-T)^m z(T) - \int_T^t \left[\sqrt{g'(s,a)(t-s)^m}z(s) + \frac{m(t-s)^{\frac{m}{2}-1}}{2\sqrt{g'(s,a)}}\right]^2 ds + \int_T^t \frac{m^2(t-s)^{m-2}}{4g'(s,a)}ds, \end{aligned}$$

which implies that for $t > T \geq t_0$

$$\begin{aligned} &\int_T^t \left[\lambda(t-s)^m \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}d\sigma(\xi) - \frac{m^2(t-s)^{m-2}}{4g'(s,a)}\right] ds \\ &\leq (t-T)^m z(T) - \int_T^t \left[\sqrt{g'(s,a)(t-s)^m}z(s) + \frac{m(t-s)^{\frac{m}{2}-1}}{2\sqrt{g'(s,a)}}\right]^2 ds. \end{aligned} \tag{11}$$

Furthermore, we have

$$\begin{aligned} &\int_{t_1}^t \left[\lambda(t-s)^m \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}d\sigma(\xi) - \frac{m^2(t-s)^{m-2}}{4g'(s,a)}\right] ds \\ &\leq (t-t_1)^m z(t_1) - \int_{t_1}^t \left[\sqrt{g'(s,a)(t-s)^m}z(s) + \frac{m(t-s)^{\frac{m}{2}-1}}{2\sqrt{g'(s,a)}}\right]^2 ds \\ &\leq (t-t_1)^m z(t_1) \leq (t-t_0)^m z(t_1). \end{aligned}$$

Thus, we have that

$$\frac{1}{t^m}\int_{t_0}^t \left[\lambda(t-s)^m \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}d\sigma(\xi) - \frac{m^2(t-s)^{m-2}}{4g'(s,a)}\right] ds$$

$$
\begin{aligned}
&= \frac{1}{t^m}\left[\int_{t_0}^{t_1} + \int_{t_1}^{t}\right]\left[\lambda(t-s)^m \int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi) - \frac{m^2(t-s)^{m-2}}{4g'(s,a)}\right]ds \\
&\le \left(1-\frac{t_0}{t}\right)^m z(t_1) + \int_{t_0}^{t_1} \lambda\left(1-\frac{s}{t}\right)^m \int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds \\
&\le z(t_1) + \lambda \int_{t_0}^{t_1}\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds, \qquad (12)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^m}\int_{t_0}^{t}\left[\lambda(t-s)^m \int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi) - \frac{m^2(t-s)^{m-2}}{4g'(s,a)}\right]ds \\
&\le z(t_1) + \lambda \int_{t_0}^{t_1}\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds < \infty, \qquad (13)
\end{aligned}
$$

this contradicts (4). Therefore, the proof of Theorem 1 is completed.

From the proof of Theorem 1, we have the following corollary.

Corollary 1. *If the condition* (4) *of Theorem* 1 *is replaced by*

$$
\limsup_{t\to\infty} \frac{1}{t^m}\int_{t_0}^{t}(t-s)^m \int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds = \infty, \qquad (14)
$$

$$
\limsup_{t\to\infty} \frac{1}{t^m}\int_{t_0}^{t}\frac{(t-s)^{m-2}}{g'(s,a)}ds < \infty, \qquad (15)
$$

then inequality (1) *has no eventually positive solutions.*

Theorem 2. *Assume that the condition of* (3) *holds, and there exists a constant* $m \ge 2$ *and function* $\rho(t) \in C'([t_0,\infty),(0,\infty))$ *such that*

$$
\limsup_{t\to\infty} \frac{1}{t^m}\int_{t_0}^{t}\frac{(t-s)^{m-2}}{\rho(s)g'(s,a)}[m\rho(s)-(t-s)\rho'(s)]^2 ds < \infty \qquad (16)
$$

$$
\limsup_{t\to\infty} \frac{1}{t^m}\int_{t_0}^{t}(t-s)^m\rho(s) \int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds = \infty, \qquad (17)
$$

then inequality (1) *has no eventually positive solutions.*

Proof. Assume the contrary, without loss of generality, that $x(t)$ is an eventually positive solution of inequality (1). Then proceeding as Theorem 1, there exists a $t_1 \ge t_0$ such that

$$
z'(t) \le -\lambda \int_a^b p(t,\xi)\{1-c[g(t,\xi)]\}d\sigma(\xi) - g'(t,a)z^2(t), \quad t \ge t_1. \qquad (10)
$$

Thus

$$\begin{aligned}
&\int_{t_1}^{t}\lambda(t-s)^m\rho(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds\\
&\le -\int_{t_1}^{t}(t-s)^m\rho(s)z'(s)ds-\int_{t_1}^{t}(t-s)^m\rho(s)g'(s,a)z^2(s)ds\\
&=(t-t_1)^m\rho(t_1)z(t_1)-\int_{t_1}^{t}(t-s)^{m-1}[m\rho(s)-(t-s)\rho'(s)]z(s)ds\\
&\quad-\int_{t_1}^{t}(t-s)^m\rho(s)g'(s,a)z^2(s)ds. \qquad (18)
\end{aligned}$$

Furthermore, we conclude that

$$\begin{aligned}
&\int_{t_1}^{t}\lambda(t-s)^m\rho(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds\\
&\le (t-t_1)^m\rho(t_1)z(t_1)\\
&\quad-\int_{t_1}^{t}\left\{\sqrt{\rho(s)g'(s,a)}(t-s)^{\frac{m}{2}}z(s)+\frac{(t-s)^{\frac{m}{2}-1}}{2\sqrt{\rho(s)g'(s,a)}}[m\rho(s)-(t-s)\rho'(s)]\right\}^2 ds\\
&\quad+\frac{1}{4}\int_{t_1}^{t}\frac{(t-s)^{m-2}}{\rho(s)g'(s,a)}[m\rho(s)-(t-s)\rho'(s)]^2ds\\
&\le (t-t_1)^m\rho(t_1)z(t_1)+\frac{1}{4}\int_{t_1}^{t}\frac{(t-s)^{m-2}}{\rho(s)g'(s,a)}[m\rho(s)-(t-s)\rho'(s)]^2ds. \qquad (19)
\end{aligned}$$

From (19), for $t>T\ge t_1$, we obtain

$$\begin{aligned}
&\frac{1}{t^m}\int_{t_1}^{t}\lambda(t-s)^m\rho(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds\\
&=\frac{1}{t^m}\left[\int_{t_1}^{T}+\int_{T}^{t}\right]\left[\lambda(t-s)^m\rho(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\right]d\sigma(\xi)ds\\
&\le \frac{1}{t^m}\int_{t_1}^{T}\lambda(t-s)^m\rho(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds+\frac{1}{t^m}(t-t_1)^m\rho(t_1)z(t_1)\\
&\quad+\frac{1}{4t^m}\int_{t_1}^{t}\frac{(t-s)^{m-2}}{\rho(s)g'(s,a)}[m\rho(s)-(t-s)\rho'(s)]^2ds, \qquad (20)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_1}^{t}\lambda(t-s)^m\rho(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds\\
&\le L+\frac{1}{4}\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_1}^{t}\frac{(t-s)^{m-2}}{\rho(s)g'(s,a)}[m\rho(s)-(t-s)\rho'(s)]^2ds, \qquad (21)
\end{aligned}$$

where $L = \rho(t_1)z(t_1)$. Thus, from condition (16), we conclude that

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} (t-s)^m \rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\} d\sigma(\xi) ds < \infty, \tag{22}$$

which contradicts (17). Therefore, the proof of Theorem 2 is completed.

Theorem 3. *Assume that the condition of* (3) *holds, and there exists a constant* $m \geq 2$ *and function* $\rho(t) \in C'([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} (t-s)^m \rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\} d\sigma(\xi) ds < \infty, \tag{23}$$

and there exists a function $\varphi(t) \in C([t_0,\infty), R)$ *satisfying*

$$\liminf_{t\to\infty} \frac{1}{t^m} \int_u^t \Bigg[\lambda(t-s)^m \rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\} d\sigma(\xi) - \frac{(t-s)^{m-2}}{4\rho(s)g'(s,a)} [m\rho(s) - (t-s)\rho'(s)]^2 \Bigg] ds \geq \varphi(u), \quad u \geq t_0, \tag{24}$$

$$\lim_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} \frac{(t-s)^m g'(s,a) \varphi_+^2(s)}{\rho(s)} ds = \infty, \quad \varphi_+(s) = \max_{s\geq t_0}\{\varphi(s), 0\}, \tag{25}$$

then inequality (1) *has no eventually positive solutions.*

Proof. Assume the contrary, without loss of generality, that $x(t)$ is an eventually positive solution of inequality (1). Then proceeding as for Theorem 2, there exists a $t_1 > u \geq t_0$ such that

$$\int_u^t \lambda(t-s)^m \rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\} d\sigma(\xi) ds \leq (t-u)^m \rho(u) z(u) + \frac{1}{4} \int_u^t \frac{(t-s)^{m-2}}{\rho(s) g'(s,a)} [m\rho(s) - (t-s)\rho'(s)]^2 ds. \tag{21}$$

Furthermore, for $t > u \geq t_0$, we have

$$\frac{1}{t^m} \int_u^t \Bigg[\lambda(t-s)^m \rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\} d\sigma(\xi) - \frac{(t-s)^{m-2}}{4\rho(s)g'(s,a)} [m\rho(s) - (t-s)\rho'(s)]^2 \Bigg] ds \leq \frac{1}{t^m} (t-u)^m \rho(u) z(u). \tag{26}$$

From (24), we conclude that

$$\varphi(u) \leq \frac{1}{t^m} \int_u^t \Bigg[\lambda(t-s)^m \rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\} d\sigma(\xi)$$

$$-\frac{(t-s)^{m-2}}{4\rho(s)g'(s,a)}[m\rho(s)-(t-s)\rho'(s)]^2\Bigg]ds$$
$$\leq \rho(u)z(u), \tag{27}$$

which implies that
$$\varphi_+^2(u) \leq \rho^2(u)z^2(u). \tag{28}$$

Let

$$v(t) = \frac{1}{t^m}\int_{t_1}^{t} z(s)(t-s)^{m-1}[m\rho(s)-(t-s)\rho'(s)]ds$$
$$w(t) = \frac{1}{t^m}\int_{t_1}^{t} \rho(s)g'(s,a)(t-s)^m z^2(s)ds,$$

then, from (20), we have

$$v(t)+w(t) \leq \frac{1}{t^m}(t-t_1)^m\rho(t_1)z(t_1) - \lambda\int_{t_1}^{t}(t-s)^m\rho(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds. \tag{29}$$

Further, from (24), we have

$$\liminf_{t\to\infty}\frac{1}{t^m}\int_u^t \lambda(t-s)^m\rho(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds \geq \varphi(u),$$
$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_1}^t \lambda(t-s)^m\rho(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds$$
$$-\liminf_{t\to\infty}\frac{1}{t^m}\int_{t_1}^t \frac{(t-s)^{m-2}}{4\rho(s)g'(s,a)}[m\rho(s)-(t-s)\rho'(s)]^2ds \geq \varphi(t_1). \tag{30}$$

From (30) and (23), we conclude that

$$\liminf_{t\to\infty}\frac{1}{t^m}\int_{t_1}^t \frac{(t-s)^{m-2}}{4\rho(s)g'(s,a)}[m\rho(s)-(t-s)\rho'(s)]^2ds < \infty.$$

Thus, there exists a sequence $\{t_n\}_1^\infty$ in (t_1,∞) such that $\lim_{n\to\infty} t_n = \infty$ and

$$\lim_{n\to\infty}\frac{1}{t_n^m}\int_{t_1}^{t_n} \frac{(t_n-s)^{m-2}}{4\rho(s)g'(s,a)}[m\rho(s)-(t_n-s)\rho'(s)]^2ds < \infty, \tag{31}$$

which implies that

$$\limsup_{t\to\infty}\{v(t)+w(t)\}$$
$$\leq \rho(t_1)z(t_1) - \liminf_{t\to\infty}\frac{1}{t^m}\int_{t_1}^t \lambda(t-s)^m\rho(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}d\sigma(\xi)ds$$
$$\leq \rho(t_1)z(t_1) - \varphi(t_1) \stackrel{\Delta}{=} M. \tag{32}$$

Hence, for any sufficiently large n, we have

$$u(t_n) + v(t_n) < M_1, \tag{33}$$

where $M_1 > M$, M and M_1 are constant. From the definition of $w(t)$, we have

$$w'(t) = \int_{t_1}^{t} \frac{ms\rho(s)g'(s,a)}{t^2} \left(1 - \frac{s}{t}\right)^{m-1} z^2(s)ds > 0.$$

Therefore, $w(t)$ is an increasing function, and $\lim_{t\to\infty} w(t) = l$ exists, where is either finite or infinite. In the case $l = \infty$, $\lim_{n\to\infty} w(t_n) = \infty$, which implies, from (33), that

$$\lim_{n\to\infty} v(t_n) = -\infty, \tag{34}$$

and

$$\frac{v(t_n)}{w(t_n)} + 1 > \frac{M_1}{w(t_n)}.$$

Thus, for any $0 < \varepsilon < 1$, and for any sufficiently large n, we have

$$\frac{v(t_n)}{w(t_n)} < \varepsilon - 1 < 0. \tag{35}$$

On the other hand, by using the Schwartz inequality, for $t \geq t_1$, we obtain

$$\begin{aligned}
0 \leq v^2(t_n) &= \frac{1}{t_n^{2m}} \left\{ \int_{t_1}^{t_n} z(s)(t_n - s)^{m-1}[m\rho(s) - (t_n - s)\rho'(s)]ds \right\}^2 \\
&\leq \left\{ \frac{1}{t_n^m} \int_{t_1}^{t_n} \rho(s)g'(s,a)(t_n - s)^m z^2(s)ds \right\} \\
&\quad \times \left\{ \frac{1}{t_n^m} \int_{t_1}^{t_n} \frac{(t_n - s)^{m-2}}{\rho(s)g'(s,a)}[m\rho(s) - (t_n - s)\rho'(s)]^2 ds \right\} \\
&= w(t_n) \frac{1}{t_n^m} \int_{t_1}^{t_n} \frac{(t_n - s)^{m-2}}{\rho(s)g'(s,a)}[m\rho(s) - (t_n - s)\rho'(s)]^2 ds.
\end{aligned}$$

Then

$$0 \leq \frac{v^2(t_n)}{w(t_n)} \leq \frac{1}{t_n^m} \int_{t_1}^{t_n} \frac{(t_n - s)^{m-2}}{\rho(s)g'(s,a)}[m\rho(s) - (t_n - s)\rho'(s)]^2 ds. \tag{36}$$

It follows from (31) that

$$0 \leq \lim_{n\to\infty} \frac{v^2(t_n)}{w(t_n)} < \infty. \tag{37}$$

From (35), we have

$$\lim_{n\to\infty} \frac{v(t_n)}{w(t_n)} = \lim_{n\to\infty} \frac{v'(t_n)}{w'(t_n)} \leq \varepsilon - 1 < 0,$$

then

$$\lim_{n\to\infty} \frac{v^2(t_n)}{w(t_n)} = \lim_{n\to\infty} \frac{2v(t_n)v'(t_n)}{w'(t_n)} \geq 2 \lim_{n\to\infty} v(t_n)(\varepsilon - 1) = \infty,$$

which contradicts (37). Thus, we have $\lim_{t\to\infty} w(t) = c < \infty$. Furthermore, according to (28), we conclude that

$$\lim_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} \frac{(t-s)^m g'(s,a)\varphi_+^2(s)}{\rho(s)} ds \le \lim_{t\to\infty} \frac{1}{t^m} \int_{t_1}^{t} \rho(s) g'(s,a)(t-s)^m z^2(s) ds = \lim_{t\to\infty} w(t) < \infty, \tag{38}$$

which implies that

$$\lim_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} \frac{(t-s)^m g'(s,a)\varphi_+^2(s)}{\rho(s)} ds = \lim_{t\to\infty} \frac{1}{t^m} \left[\int_{t_0}^{t_1} + \int_{t_1}^{t}\right] \frac{(t-s)^m g'(s,a)\varphi_+^2(s)}{\rho(s)} ds \le \int_{t_0}^{t_1} \frac{(t-s)^m g'(s,a)\varphi_+^2(s)}{\rho(s)} ds + \lim_{t\to\infty} w(t) < \infty,$$

which contradicts (25). Therefore, the proof of Theorem 3 is completed.

Similar to the above results on inequality (1), we can also obtain some results on inequality (2).

Theorem 4. *Suppose that the conditions of Theorem* 1 *hold, then inequality* (2) *has no eventually negative solutions.*

Theorem 5. *Suppose that the conditions of Theorem* 2 *hold, then inequality* (2) *has no eventually negative solutions.*

Theorem 6. *Suppose that the conditions of Theorem* 3 *hold, then inequality* (2) *has no eventually negative solutions.*

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