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PROPERTY (w) OF UPPER TRIANGULAR OPERATOR MATRICES

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Abstract. Let $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \mathscr{L}(\mathbb{X}, \mathbb{Y})$ be be an upper triangulate Banach space operator. The relationship between the spectra of M_C and M_0, and their various distinguished parts, has been studied by a large number of authors in the recent past. This paper brings forth the important role played by SVEP, the *single-valued extension property,* in the study of some of these relations. In this work, we prove necessary and sufficient conditions of implication of the type M_0 satisfies property $(w) \Leftrightarrow M_C$ satisfies property (w) to hold. Moreover, we explore certain conditions on $T \in \mathscr{L}(\mathscr{H})$ and $S \in \mathscr{L}(\mathscr{K})$ so that the direct sum $T \oplus S$ obeys property (w), where $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces.

1. Introduction

Throughout this paper, $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces and $\mathscr{L}(\mathbb{X}, \mathbb{Y})$ denotes the space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$. For $\mathbb{X} = \mathbb{Y}$ we write $\mathscr{L}(\mathbb{X}, \mathbb{Y}) = \mathscr{L}(\mathbb{X})$. For $T \in \mathscr{L}(\mathbb{X})$, let T^*, $\ker(T)$, $\Re(T)$, $\sigma(T)$, $\sigma_d(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ denote the adjoint, the null space, the range, the spectrum, the surjective spectrum, the point spectrum and the approximate point spectrum of T, respectively. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by $\alpha(T) = \dim\ker(T)$ and $\beta(T) = co\dim\Re(T)$.

For A, B and $C \in \mathscr{L}(\mathbb{X})$, let M_C denote the upper triangular operator matrix $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. A study of the spectrum, the Browder and Weyl spectra, and the Browder and Weyl theorems for the operator M_C, and the related diagonal operator $M_0 = A \oplus B$, has been carried by a number of authors in the recent past (see [6, 10, 11, 20] for further references). Of particular interest here is the relationship between the spectral, the Fredholm, the Browder and the Weyl properties.

Let $a := a(T)$ be the ascent of an operator T ; i.e., the smallest nonnegative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, let $d := d(T)$ be descent of an operator T ; i.e., the smallest nonnegative integer s such that

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$\Re(T^s) = \Re(T^{s+1})$, and if such integer does not exist we put $d(T) = \infty$. It is well known that if $a(T)$ and $d(T)$ are both finite then $a(T) = d(T)$ [17, Proposition 38.3].

In this paper, we introduce most of our notation and terminology in Section 2, Section 3 is devoted to proving a number of complementary results, sections 3 and 4 are devoted to proving our main results. In Section 3, we explore certain conditions on T and S so that the direct sum $T \oplus S$ obeys property (w). We consider property (w) for the operators M_0 and M_C in Section 4. Here we prove a necessary and sufficient for the equivalence M_0 satisfies property $(w) \Leftrightarrow M_C$ satisfies property (w) for operators M_C such that $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, which is then applied to deduce a number of known results. For operators M_0 and M_C such that $\sigma_{aw}(M_0) = \sigma_{aw}(M_C)$, we prove a sufficient condition for the implications M_0 property $(w) \Rightarrow M_C$ satisfies property (w) and M_C satisfies property $(w) \Rightarrow M_0$ satisfies property (w).

2. Notation and terminology

Let $\Phi_+(\mathbb{X}) := \{T \in \mathscr{L}(\mathbb{X}) : \alpha(T) < \infty \text{ and } T(\mathbb{X}) \text{ is closed}\}$ be the class of all *upper semi-Fredholm* operators, and let $\Phi_-(\mathbb{X}) := \{T \in \mathscr{L}(\mathbb{X}) : \beta(T) < \infty\}$ be the class of all *lower semi-Fredholm* operators. The class of all *semi-Fredholm* operators is defined by $\Phi_\pm(\mathbb{X}) := \Phi_+(\mathbb{X}) \cup \Phi_-(\mathbb{X})$, while the class of all *Fredholm* operators is defined by $\Phi(\mathbb{X})) := \Phi_+(\mathbb{X}) \cap \Phi_-(\mathbb{X})$. If $T \in \Phi_\pm(\mathbb{X})$, the *index* of T is defined by

$$ind(T) := \alpha(T) - \beta(T).$$

Recall that a bounded operator T is said *bounded below* if it injective and has closed range. Evidently, if T is bounded below then $T \in \Phi_+(\mathbb{X})$ and $ind(T) \leq 0$. Define

$$W_+(\mathbb{X}) : = \{T \in \Phi_+(\mathbb{X}) : ind(T) \leq 0\},$$

and

$$W_-(\mathbb{X}) : = \{T \in \Phi_-(\mathbb{X}) : ind(T) \geq 0\}.$$

The set of *Weyl* operators is defined by

$$W(\mathbb{X}) := W_+(\mathbb{X}) \cap W_-(\mathbb{X}) = \{T \in \Phi(\mathbb{X}) : ind(T) = 0\}.$$

The classes of operators defined above generate the following spectra. Denote by

$$\sigma_a(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}$$

the *approximate point spectrum*, and by

$$\sigma_d(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not surjective}\}$$

the *surjectivity spectrum* of $T \in \mathscr{L}(\mathbb{X})$. The *Weyl spectrum* is defined by

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin W(\mathbb{X})\},$$

the *Weyl essential approximate point spectrum* is defined by

$$\sigma_{aw}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin W_+(\mathbb{X})\},$$

while the *Weyl essential surjectivity spectrum* is defined by

$$\sigma_{lw}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin W_-(\mathbb{X})\},$$

Obviously, $\sigma_w(T) = \sigma_{aw}(T) \cup \sigma_{lw}(T)$ and from basic Fredholm theory we have

$$\sigma_{aw}(T) = \sigma_{ws}(T^*) \qquad \sigma_{ws}(T) = \sigma_{aw}(T^*).$$

Note that $\sigma_{aw}(T)$ is the intersection of all approximate point spectra $\sigma_a(T+K)$ of compact perturbations K of T, while $\sigma_{lw}(T)$ is the intersection of all surjectivity spectra $\sigma_s(T+K)$ of compact perturbations K of T, see, for instance, [1, Theorem 3.65].

The class of all *upper semi-Browder* operators is defined by

$$B_+(\mathbb{X}) := \{T \in \Phi_+(\mathbb{X}) : a(T) < \infty\},$$

while the class of all *lower semi-Browder* operators is defined by

$$B_-(\mathbb{X}) := \{T \in \Phi_+(\mathbb{X}) : d(T) < \infty\}.$$

The class of all *Browder* operators is defined by

$$B(\mathbb{X}) := B_+(\mathbb{X}) \cap B_-(\mathbb{X}) = \{T \in \Phi(\mathbb{X}) : a(T), d(T) < \infty\}.$$

We have

$$B(\mathbb{X}) \subseteq W(\mathbb{X}), \qquad B_+(\mathbb{X}) \subseteq W_+(\mathbb{X}), \qquad B_-(\mathbb{X}) \subseteq W_-(\mathbb{X}),$$

see [1, Theorem 3.4]. The *Browder spectrum* of $T \in \mathscr{L}(\mathbb{X})$ is defined by

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin B(\mathbb{X})\},$$

the *upper Browder spectrum* is defined by

$$\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin B_+(\mathbb{X})\},$$

and analogously the *lower Browder spectrum* is defined by

$$\sigma_{lb}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \notin B_-(\mathbb{X})\}.$$

Clearly, $\sigma_b(T)=\sigma_{ub}(T)\cup\sigma_{lb}(T)$ and $\sigma_w(T)\subseteq\sigma_b(T)$.

Let write K^{iso} for the set of all isolated points of $K\subseteq\mathbb{C}$. For a bounded operator $T\in\mathscr{L}(\mathbb{X})$ set $\pi_0(T):=\sigma(T)\setminus\sigma_b(T)=\{\lambda\in\sigma(T): T-\lambda I\in\mathscr{L}(\mathbb{X})\}$. Note that every $\lambda\in\pi_0(T)$ is a pole of the resolvent and hence an isolated point of $\sigma(T)$, see [17, Proposition 50.2]. Moreover, $\pi_0(T)=\pi_0(T^*)$. Define

$$E_0(T):=\{\lambda\in iso\sigma(T): 0<\alpha(T-\lambda I)<\infty\}.$$

Obviously,

$$\pi_0(T)\subseteq E_0(T) \qquad \text{for every } T\in\mathscr{L}(\mathbb{X}).$$

For a bounded operator $T\in\mathscr{L}(\mathbb{X})$ let us define

$$E_0^a(T):=\{\lambda\in iso\sigma_a(T): 0<\alpha(T-\lambda I)<\infty\},$$

and

$$\pi_0^a(T):=\sigma_a(T)\setminus\sigma_{ub}(T)=\{\lambda\in\sigma_a(T): T-\lambda I\in B_+(\mathbb{X})\}.$$

Hence we have

$\pi_0(T)\subseteq\pi_0^a(T)\subseteq E_0^a(T)$ and $E_0(T)\subseteq E_0^a(T)$.

Following Harte and W.Y. Lee [16], we shall say that T satisfies *Browder's theorem* if $\sigma_w(T)=\sigma_b(T)$, while, $T\in\mathscr{L}(\mathbb{X})$ is said to satisfy *a-Browder's theorem* if $\sigma_{aw}(T)=\sigma_{ub}(T)$. Let $\Delta(T)=\sigma(T)\setminus\sigma_w(T)$ and $\Delta_a(T)=\sigma_a(T)\setminus\sigma_{aw}(T)$. Following Coburn [8], we say that Weyl's theorem holds for $T\in\mathscr{L}(\mathbb{X})$ if $\Delta(T)=E_0(T)$. According to Rakočević [21], an operator $T\in\mathscr{L}(\mathbb{X})$ is said to satisfy a-Weyl's theorem if $\Delta_a(T)=E_0^a(T)$. We can write

$$\Delta_a(T)=\{\lambda\in\mathbb{C}: T-\lambda\in W_+(\mathbb{X}) \text{ and } \alpha(T-\lambda I)>0\}.$$

It is known (see [21]) that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem too, but the converse does not hold in general.

Recall that an operator $T\in\mathscr{L}(\mathbb{X})$ is said to satisfy property (w) if $\Delta_a(T)=E_0(T)$. In [22] the author introduce the property (w) which is a variant of Weyl's theorem.

An operator $T\in\mathscr{L}(\mathbb{X})$ has the *single-valued extension property* at $\lambda_0\in\mathbb{C}$, SVEP at λ_0, if for every open disc U_{λ_0} centered at λ_0 the only analytic function $f: U_{\lambda_0}\longrightarrow\mathbb{X}$ which satisfies $(T-\lambda I)f(\lambda)=0$ for all $\lambda\in U_{\lambda_0}$ is the function $f\equiv 0$. Trivially, every operator T has SVEP on the resolvent set $\rho(T)=\mathbb{C}\setminus\sigma(T)$; also T has SVEP at points $\lambda\in\sigma^{iso}(T)$. Let $S(T)$ denote the set of $\lambda\in\mathbb{C}$ where T does not have SVEP: we say that T has SVEP if $S(T)=\emptyset$. SVEP plays an important role in determining the relationship between the Browder and Weyl spectra, and the Browder and Weyl theorems. Thus $\sigma_b(T)=\sigma_w(T)\cup S(T)=\sigma_w(T)\cup S(T^*)$, and if T^* has SVEP then $\sigma_b(T)=\sigma_w(T)=\sigma_{ab}(T)=\sigma_{aw}(T)$ [1, Page 141- 142]; T satisfies Browder's theorem (resp., a-Browder's theorem) if and only if T has SVEP at $\lambda\notin\sigma_w(T)$ (resp., $\lambda\notin\sigma_{aw}(T)$) [12, Lemma 2.18]; and if T^* has SVEP, then $T\in\mathscr{W}$ if and only if $T\in a\mathscr{W}$.

In the following, the diagonal operator M_0 and the upper operator M_C will defined as in the introduction, and $T \in \mathscr{L}(q)$ shall denote a general Banach space operator. It is known that if either $S(A^*) = \emptyset$ or $S(B) = \emptyset$, then $\sigma(M_C) = \sigma(M_0) = \sigma(A) \cup \sigma(B)$; if $S(A) \cup S(B) = \emptyset$, then M_C has SVEP, $\sigma_b(M_c) = \sigma_w(M_C) = \sigma_w(M_0) = \sigma_b(M_0)$, and $M_C \in aB$. Browders theorem, much less Weyls theorem, does not transfer from individual operators to direct sums: for example, the forward unilateral shift and the backward unilateral shift on a Hilbert space satisfy Browder's theorem, but their direct sum does not. However, if $(S(A) \cap S(B^*)) \cup S(A^*) = \emptyset$, then : M_0 satisfies Browder's theorem (resp., a-Browder's theorem) implies M_C satisfies Browder's theorem (resp., a-Browder's theorem); if points $\lambda \in \sigma^{iso}(A)$ are eigenvalues of $A \in \mathscr{W}$, then $M_0 \in \mathscr{W}$ implies $M_C \in \mathscr{W}$ [11, Proposition 4.1 and Theorem 4.2].
It is known that from [6, 7, 9, 10, 11] that

(i) $\sigma_x(M_0) = \sigma_x(A) \cup \sigma_x(B) = \sigma_x(M_C) \cup \{\sigma_x(A) \cap \sigma_x(B)\}$, where $\sigma_x = \sigma, \sigma_b$ or σ_e;

(ii) $\sigma_w(M_0) \subseteq \sigma_w(A) \cup \sigma_w(B) = \sigma_w(M_C) \cup \{\sigma_w(A) \cap \sigma_w(B)\}$;

(iii) if $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then $\sigma(M_C) = \sigma(M_0)$ and

(iv) $\sigma_{aw}(M_0) \subseteq \sigma_{aw}(A) \cup \sigma_{aw}(B) = \sigma_{aw}(M_C) \cup \left\{S(A) \cup S(A^*)\right\}$.

Remark 2.1. Let $SP(T)$ be the spectral picture of T, it is known that: if either $SP(A)$ or $SP(B)$ has no pseudo holes, then $\sigma^{acc}(M_0) \subseteq \sigma_w(M_0) \Rightarrow \sigma^{acc}(M_C) \subseteq \sigma_w(M_C)$ [20, Theorem 2.3]; if additionally A is an isoloid (the isolated points of $\sigma(A)$ are eigenvalues of A) and A satisfies Weyl's theorem, then $M_0 \in \mathscr{W} \Rightarrow M_C \in \mathscr{W}$ [20, Theorem 2.4]. If $\{S(A) \cap S(B^*)\} \cup S(A^*) = \emptyset$, then $\sigma^{acc}(M_0) \subseteq \sigma_w(M_0) \Rightarrow \sigma^{acc}(M_C) \subseteq \sigma_w(M_C)$ [11, Proposition 4.1]. Again, if $\sigma_a(A^*)$ has empty interior, A is an a-isoloid (isolated points of $\sigma_a(A)$ are eigenvalues of A) and $A \in a\mathscr{W}$, then $M_0 \in a\mathscr{W} \Rightarrow M_C \in a\mathscr{W}$ [7, Theorem 3.3].

3. Property (w) for direct sum

Let $\mathscr{H}$ and $\mathscr{K}$ be infinite-dimensional Hilbert spaces. In this section we show that if T and S are two operators on $\mathscr{H}$ and $\mathscr{K}$ respectively and at least one of them satisfies property (w) then their direct sum $T \oplus S$ obeys property (w) under certain conditions. We have also explored various conditions on T and S so that $T \oplus S$ satisfies property (w).

Theorem 3.1. *Suppose that property (w) holds for $T \in \mathscr{L}(\mathscr{H})$ and $S \in \mathscr{L}(\mathscr{K})$. If T and S are isoloid and $\sigma_{aw}(T \oplus S) = \sigma_{aw}(T) \cup \sigma_{aw}(S)$, then property (w) holds for $T \oplus S$.*

Proof. We know that $\sigma_a(T \oplus S) = \sigma_a(T) \cup \sigma_a(S)$ for any pairs of operators. If T and S are a-isoloid, then

$$E_0(T \oplus S) = \left[E_0(T) \cap \rho_a(S)\right] \cup \left[\rho_a(T) \cap E_0(S)\right] \cup \left[E_0(T) \cap E_0(S)\right],$$

where $\rho_a(.) = \mathbb{C} \setminus \sigma_a(.)$.
If property (w) holds for T and S, then

$$\begin{aligned}&[\sigma_a(T)\cup\sigma_a(S)]\setminus[\sigma_{aw}(T)\cup\sigma_{aw}(S)]\\ &\qquad = \left[E_0(T)\cap\rho_a(S)\right]\cup\left[\rho_a(T)\cap E_0(S)\right]\cup[E_0(T)\cap E_0(S)].\end{aligned}$$

Thus, $E_0(T\oplus S) = [\sigma_a(T)\cup\sigma_a(S)]\setminus[\sigma_{aw}(T)\cup\sigma_{aw}(S)]$.

If $\sigma_{aw}(T\oplus S) = \sigma_{aw}(T)\cup\sigma_{aw}(S)$, then

$$E_0(T\oplus S) = \sigma_a(T\oplus S)\setminus\sigma_{aw}(T\oplus S).$$

Hence property (w) holds for $T\oplus S$. □

The assumption A and B are isoloid is essential in Theorem 3.1.

Example 3.2. If $A, B : \ell^2 \to \ell^2$ are defined by

$$A(x_1, x_2, \dots) = (0, x_2, x_3, \dots) \text{ and } B(x_1, x_2, \dots) = \left(0, x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots\right),$$

then we have that property (w) holds for A and B; $\sigma_a(A) = \{0,1\}$, $\sigma_{aw}(A) = \{1\}$, $\sigma_a(B) = \sigma_{aw}(B) = \{0\}$, $E_0(A) = \{0\}$, $E_0(B) = \emptyset$; $\sigma_a(A\oplus B) = \{0,1\} = \sigma_{aw}(A\oplus B)$ and $E_0(A\oplus B) = \{0\}$. Then property (w) does not holds for $A\oplus B$.

Theorem 3.3. *Suppose that $T \in \mathscr{L}(\mathscr{H})$ such that $iso\sigma_a(T) = \emptyset, \sigma(T) = \sigma_a(T)$ and $S \in \mathscr{L}(\mathscr{K})$ satisfies property (w). If $\sigma_{aw}(T\oplus S) = \sigma_a(T)\cup\sigma_{aw}(S)$, then property (w) holds for $T\oplus S$.*

Proof. We know that $\sigma_a(T\oplus S) = \sigma_a(T)\cup\sigma_a(S)$ for any pairs of operators. Then

$$\begin{aligned}\sigma_a(T\oplus S)\setminus\sigma_{aw}(T\oplus S) &= [\sigma_a(T)\cup\sigma_a(S)]\setminus[\sigma_a(T)\cup\sigma_{aw}(S)]\\ &= \sigma_a(S)\setminus[\sigma_a(T)\cup\sigma_{aw}(S)]\\ &= [\sigma_a(S)\setminus\sigma_{aw}(S)]\setminus\sigma_a(T)\\ &= E_0(S)\cap\rho_a(T)\end{aligned}$$

If $\sigma_a^{iso}(T) = \emptyset$ it implies that $\sigma_a(T) = \sigma_a^{acc}(T)$, where $\sigma_a^{acc}(T) = \sigma_a(T)\setminus\sigma_a^{iso}(T)$ is the set of all accumulation points of $\sigma_a(T)$. Thus we have

$$\begin{aligned}\sigma_a^{iso}(T\oplus S) &= \left[\sigma_a^{iso}(T)\cup\sigma_a^{iso}(S)\right]\setminus\left[\left(\sigma_a^{iso}(T)\cap\sigma_a^{acc}(S)\right)\cup\left(\sigma_a^{acc}(T)\cap\sigma_a^{iso}(S)\right)\right]\\ &= \left[\sigma_a^{iso}(T)\setminus\sigma_a^{acc}(S)\right]\cup\left[\sigma_a^{iso}(S)\setminus\sigma_a^{acc}(T)\right]\\ &= \sigma_a^{iso}(S)\setminus\sigma_a(T)\\ &= \sigma_a^{iso}(S)\cap\rho_a(T).\end{aligned}$$

We know that $\sigma_p(T\oplus S)=\sigma_p(T)\cup\sigma_p(S)$ and $\alpha(T\oplus S)=\alpha(T)+\alpha(S)$ for any pairs of operators, so that

$$\sigma_{PF}(T\oplus S)=\{\lambda\in\sigma_{PF}(T)\cup\sigma_{PF}(S):\alpha(T-\lambda I)+\alpha(S-\lambda I)<\infty\}.$$

Therefore,

$$\begin{aligned}E_0(T\oplus S)&=\sigma_a^{iso}(T\oplus S)\cap\sigma_{PF}(T\oplus S)\\&=\sigma_a^{iso}(S)\cap\rho_a(T)\cap\sigma_{PF}(S)\\&=E_0(S)\cap\rho_a(T).\end{aligned}$$

Thus $\sigma_a(T\oplus S)\setminus\sigma_{aw}(T\oplus S)=E_0(T\oplus S)$. Hence $T\oplus S$ satisfies property (w). □

Corollary 3.4. *Suppose that $T\in\mathscr{L}(\mathscr{H})$ is such that $\sigma_a^{iso}(T)=\emptyset$ and $S\in\mathscr{L}(\mathscr{K})$ satisfies property (w) with $\sigma_a^{iso}(S)\cap\sigma_p(S)=\emptyset$, and $\Delta_a(T\oplus S)=\emptyset$, then $T\oplus S$ satisfies property (w).*

Proof. Since S satisfies property (w), therefore given condition $\sigma_a^{iso}(S)\cap\sigma_p(S)=\emptyset$ implies that $\sigma_a(S)=\sigma_{aw}(S)$. Now $\Delta_a(T\oplus S)=\emptyset$ gives that $\sigma_{aw}(T\oplus S)=\sigma_a(T\oplus S)=\sigma_a(T)\cup\sigma_{aw}(S)$. Thus from Theorem 3.3, we have that $T\oplus S$ satisfies property (w). □

Corollary 3.5. *Suppose that $T\in\mathscr{L}(\mathscr{H})$ is such that $\sigma_a^{iso}(T)\cup\Delta_a(T)=\emptyset$ and $S\in\mathscr{L}(\mathscr{K})$ satisfies property (w). If $\sigma_{aw}(T\oplus S)=\sigma_{aw}(T)\cup\sigma_{aw}(S)$, then $T\oplus S$ satisfies property (w).*

Theorem 3.6. *Let $T\in\mathscr{L}(\mathscr{H})$ be an a-isoloid operator that satisfies property (w). If $S\in\mathscr{L}(\mathscr{K})$ is a normal operator satisfies property (w). Then property (w) holds for $T\oplus S$.*

Proof. If S is normal, then both S and S^* have SVEP, and $ind(S-\lambda I)=0$ for every λ such that $S-\lambda I$ is a Fredholm. Observe that $\lambda\notin\sigma_{aw}(T\oplus S)$ if and only if $S-\lambda I\in W_+(\mathscr{K})$ and $T-\lambda I\in W_+(\mathscr{H})$ and $ind(T-\lambda I)+ind(S-\lambda I)=ind(T-\lambda I)\le 0$ if and only if $\lambda\notin\Delta_a(T)\cap\Delta_a(S)$. Hence $\sigma_{aw}(T\oplus S)=\sigma_{aw}(T)\cup\sigma_{aw}(S)$. It is well known that the isolated points of the approximate point spectrum of a normal operator are simple poles of the resolvent of the operator implies that S is a-isoloid. So the result follows now from Theorem 3.1. □

4. Property (w) for M_C

In the following, let

$$\begin{aligned}\Phi_+(T)&=\left\{\lambda\in\mathbb{C}:T-\lambda I\quad\text{is upper semi-Fredholm}\right\},\\\Phi_+^-(T)&=\{\lambda\in\mathbb{C}:ind(T-\lambda I)\le 0\},\\\Phi_-(T)&=\{\lambda\in\mathbb{C}:T-\lambda I\quad\text{is lower semi-Fredholm}\},\\\Phi_-^+(T)&=\{\lambda\in\mathbb{C}:ind(T-\lambda I)\ge 0\},\end{aligned}$$

$$\Phi(T) = \Phi_+(T) \cap \Phi_-(T), \quad \text{and}$$
$$\Phi^0(T) = \{\lambda \in \Phi(T) : ind(T - \lambda I) = 0\}.$$

Then the upper semiFredholm spectrum $\sigma_{SF_+}(T)$, the lower semiFredholm spectrum $\sigma_{SF_-}(T)$, the (Fredholm) essential spectrum $\sigma_e(T)$ and the Weyl essential surjectivity spectrum $\sigma_{sw}(T)$ of T are the sets

$$\sigma_{SF_+}(T) = \{\lambda \in \sigma(T) : \lambda \notin \Phi_+(T)\},$$
$$\sigma_{SF_-}(T) = \{\lambda \in \sigma(T) : \lambda \notin \Phi_-(T)\},$$
$$\sigma_e(T) = \{\lambda \in \sigma(T) : \lambda \notin \Phi(T)\} \quad \text{and}$$
$$\sigma_{sw}(T) = \left\{\lambda \in \sigma(T) : \lambda \notin \Phi_-^+(T)\right\}.$$

It is easily verified, see [25, Exercise 7, Page 293], that

$$a(A - \lambda I) \leq a(M_C - \lambda I) \leq a(A - \lambda I) + a(B - \lambda I);$$
$$d(A - \lambda I) \leq d(M_C - \lambda I) \leq d(A - \lambda I) + d(B - \lambda I)$$

for every $\lambda \in \mathbb{C}$.

Remark 4.1. The following implications hold [1, Theorem 3.4]: $a(T - \lambda I) < \infty \Rightarrow \alpha(T - \lambda I) \leq \beta(T - \lambda I))$; $d(T - \lambda I) < \infty \Rightarrow \beta(T - \lambda I) \leq \alpha(T - \lambda I)$; if $\alpha(T - \lambda I) = \beta(T - \lambda I)$, then either of $a(T - \lambda I) < \infty$ and $d(T - \lambda I) < \infty \Rightarrow a(T - \lambda I) = d(T - \lambda I) < \infty$. If $\lambda \in \Phi_+^-(T)$, then T has SVEP at $\lambda \Leftrightarrow a(T - \lambda I) < \infty$ and T^* has SVEP at $\lambda \Leftrightarrow a(T - \lambda I) < \infty$ [1, Theorems 3.16, 3.17]. From this it follows that if both T and T^* have SVEP at $\lambda \in \Phi_+^-(T)$, then $\lambda \in \Phi^0(T)$ and $\lambda \in \pi_0(T)$. If $\lambda \in \pi_0(T)$ and either of $a(T - \lambda I)$ and $d(T - \lambda I)$ is finite (equivalently, either T or T^* has SVEP at λ), then $\lambda \in \pi_0(T)$. Again, if $\lambda \in \Phi_+^-(T)$ and T has SVEP at λ, then $\lambda \in \pi_0^a(T)$ [1, Theorem 3.23].

For an operator $S \in \mathscr{L}(\mathbb{X})$ and $\sigma_x(T)$ a subset of $\sigma(T)$, let

$$S_{\sigma_x(T)}(S) = \{\lambda \in \sigma(T) \setminus \sigma_x(T) : S \quad \text{does not have SVEP at } \lambda\}.$$

Remark 4.2. From [6, 7, 14, 15]. The Following relations hold:

(i)
$$\sigma(M_0) = \sigma(A) \cup \sigma(B) = \sigma(M_C) \cup \{\sigma(A) \cap \sigma(B)\}$$
$$= \sigma(M_C) \cup \left\{S_{\sigma_a(A)}(A^*) \cap S_{\sigma_a(B)}(B)\right\}.$$

(ii)
$$\sigma_b(M_0) = \sigma_b(A) \cup \sigma_b(B) = \sigma_b(M_C) \cup \{\sigma_b(A) \cap \sigma_b(B)\}$$
$$= \sigma_b(M_C) \cup \left\{S_{\sigma_b(M_C)}(A^*) \cap S_{\sigma_b(M_C)}(B)\right\}.$$

(iii)
$$\sigma_w(A) \cup \sigma_w(B) \subseteq \sigma_w(M_C) \cup \left\{S_{\sigma_w(M_C)}(P) \cup S_{\sigma_w(M_C)}(Q)\right\},$$
$$\text{where} \quad (P, Q) = (A, A^*), (B, B^*), (A, B), \quad \text{or } (A^*, B^*).$$

Lemma 4.3. *If either A^* or B has SVEP and $\lambda \in \sigma_a^{iso}(M_C)$, then $\lambda \in \sigma_a^{iso}(A) \cup \sigma_a^{iso}(B)$.*

Proof. The hypothesis A^* or B has SVEP implies that

$$\sigma_a(M_C) = \sigma_a(A) \cup \sigma_a(B).$$

Hence $\lambda \in (\sigma_a(A) \cup \sigma_a(B))^{iso} \subset \sigma_a^{iso}(A) \cup \sigma_a^{iso}(B)$. □

Theorem 4.4. *Let A and B have SVEP, and let $\dim \chi_B(\{\lambda\}) < \infty$ for all $\lambda \in \sigma_a^{iso}(B)$. If a-Weyl's theorem holds for M_0, then a-Weyl's theorem holds for M_C for every $C \in \mathscr{L}(\mathbb{Y}, \mathbb{X})$.*

Proof. Since A and B have SVEP, M_C has SVEP [18, Proposition 3.1], and so M_C obeys a-Browder's theorem. Hence,

$$\sigma_a(M_C) \setminus \sigma_{aw}(M_C) = \pi_0^a(M_C) \subseteq E_0^a(M_C).$$

Let $\lambda \in E_0^a(M_C)$. Then $\lambda \in \sigma_a^{iso}(M_C)$. By Lemma 4.3, $\lambda \in \sigma_a^{iso}(A) \cup \sigma_a^{iso}(B)$. Hence $\lambda \in \sigma_a^{iso}(M_0)$. Since $\ker(A - \lambda I) \oplus \{0\} \subset \ker(M_C - \lambda I)$, $\dim \ker(A - \lambda I) < \infty$ in the case in which $\lambda \in \sigma_a^{iso}(A) \cup \rho_a(A)$. Again, if $\lambda \in \sigma_a^{iso}(B)$, or $\lambda \in \rho_a(B)$, then the assumption that $\dim \chi_B(\{\lambda\}) < \infty$ implies (by [19, Proposition 1.2.16]) that $\dim \ker(B - \lambda I) < \infty$, and hence that

$$\dim(\ker(A - \lambda I) \oplus \ker(B - \lambda I)) < \infty$$

Evidently, the non-triviality of $\ker(M_C - \lambda I)$ implies that $\ker(A - \lambda I) \cup \ker(B - \lambda I) \neq \{0\}$, i.e., $0 < \dim(\ker(A - \lambda I) \oplus \ker(B - \lambda I))$. Hence, $\lambda \in \sigma_a^{iso}(M_0)$ and

$$0 < \dim(\ker(A - \lambda I) \oplus \ker(B - \lambda I)) < \infty,$$

i.e., $\lambda \in \pi_0^a(M_0) = \sigma_a(M_0) \setminus \sigma_{aw}(M_0)$. By [7, Theorem 3.1] this implies that $\lambda \notin \sigma_{aw}(M_C)$. □

Recall that an operator $T \in \mathscr{L}(\mathbb{X})$ is said to be polaroid (resp., isoloid) at $\lambda \in \sigma^{iso}(T)$ if $a(T - \lambda I) = d(T - \lambda I) < \infty$ (resp., λ is an eigenvalue of T). Trivially, T polaroid at λ implies T isoloid at λ. We say that T is a-polaroid if T is polaroid at $\lambda \in \sigma_a^{iso}(T)$.

Lemma 4.5. *Let $A \in \mathscr{L}(\mathbb{X})$ and $B \in \mathscr{L}(\mathbb{Y})$ have SVEP. If A and B are polaroid, then M_C is polaroid for every $C \in \mathscr{L}(\mathbb{Y}, \mathbb{X})$.*

Proof. Suppose that $\lambda \in \sigma^{iso}(M_C)$. If B has SVEP then $\sigma(B)$ coincides with the defect spectrum of B. It follows from [10, Theorem 2.3] that $\sigma(M_C) = \sigma(A) \cup \sigma(B)$. Therefore $\lambda \in (\sigma(A) \cup \sigma(B))^{iso}$. Suppose that $\lambda \in \sigma(A)$. Then $\lambda \in \sigma^{iso}(A)$. Since A is isoloid, $\ker(A - \lambda I) \neq \{0\}$. Observe that $\ker(A - \lambda I) \oplus \{0\} \subseteq \ker(M_C - \lambda I)$, and hence $\ker(M_C - \lambda I) \neq \{0\}$. Since $\ker(M_C - \lambda I) \neq \{0\}$ then $\ker(M_0 - \lambda I) \neq \{0\}$; also, $\dim(\ker(M_C - \lambda I)) < \infty$ implies $\dim(\ker(A - \lambda I)) < \infty$. We claim that $\dim(\ker(B-\lambda I))<\infty$. For suppose to the contrary that $\dim(\ker(B-\lambda I))$ is infinite. Since

$$(M_C-\lambda I)(x\oplus y)=\{(A-\lambda I)x+Cy)\oplus(B-\lambda I)y\},$$

either $\dim(C(\ker(B-\lambda I))<\infty$ or $\dim(C(\ker(B-\lambda I))=\infty$. If $\dim(C(\ker(B-\lambda I))<\infty$, then $\ker(B-\lambda I)$ contains an orthonormal sequence $\{y_j\}$ such that $(M_C-\lambda I)(0\oplus y_j)=0$ for all $j=1,2,\ldots$. But then $\dim\ker(M_C-\lambda I)=\infty$, a contradiction. Assume now that $\dim(C(\ker(B-\lambda I))=\infty$. Since $\lambda\in\rho(A)\cup\sigma^{iso}(A)$, A satisfies Browder's theorem, A is polaroid and $\alpha(A-\lambda I)<\infty$, $\beta(A-\lambda I)<\infty$. Hence $\dim\{C(\ker(B-\lambda I))\cap\Re(A-\lambda I)\}=\infty$ implies the existence of a sequence $\{x_j\}$ such that $(A-\lambda I)x_j=Cy_j$ for all $j=1,2,\ldots$. But then $(M_C-\lambda I)(x_j\oplus -y_j)=0$ for all $j=1,2,\ldots$. Thus $\dim\ker(M_C-\lambda I)=\infty$, again a contradiction. Our claim having been proved, we conclude that $\lambda\in\pi(M_0)$. Thus $\pi(M_C)\subseteq\pi(M_0)$. □

Remark 4.6. If $S(A^*)\cup S(B^*)=\emptyset$, then M_C^* has SVEP. Hence

$$\sigma(M_0)=\sigma(M_C),\sigma_{aw}(M_C)=\sigma_w(M_C)=\sigma_w(M_0)\,\text{and}\,\pi_0(M_C)=\pi_0^a(M_C).$$

Evidently, both M_0 and M_C satisfy a-Browder's theorem. Since

$$E_0(M_0)=\big(E_0(A)\cap\rho(B)\big)\cup\big(\rho(A)\cap E_0(B)\big)\cup(E_0(A)\cap E_0(B))$$

if M_0 is polaroid at $\lambda\in E_0(M_0)$, then either A or B is polaroid at λ; in particular, A and B are polaroid at $\lambda\in E_0(A)\cap E_0(B)$. Conversely, if A is polaroid at $\lambda\in E_0(A)$ and B is polaroid at $\mu\in E_0(B)$, then M_0 is polaroid at $\nu\in E_0(M_0)$.

Theorem 4.7. *If $S(A^*)\cup S(B^*)=\emptyset$, A is polaroid at $\lambda\in E_0^a(M_C)$ (or, A is isoloid and satisfies Weyl's theorem) and B is polaroid at $\mu\in E_0^a(B)$, then M_C satisfies property (w).*

Proof. Since A^* and B^* have SVEP, both M_0^* and M_C^* have SVEP. Hence M_C (also, M_0) satisfies Browder's theorem, which implies that $\sigma(M_C)\setminus\sigma_w(M_C)=\pi_0(M_C)\subseteq E_0(M_C)$. Apparently, $\sigma(M_0)=\sigma(M_C)=\sigma_a(M_C),\sigma_w(M_0)=\sigma_w(M_C)=\sigma_{aw}(M_C)$, $E_0(M_C)=E_0^a(M_C)$ and $\sigma^{iso}(M_C)=\sigma^{iso}(M_0)$. Following (part of) the argument of the proof of the sufficiency part of Theorem 3.7 of [14], it follows that if $\lambda\in E_0(M_C)$, then $\lambda\in E_0(A)\cap E_0(B)$. By assumption, both A and B are polaroid at λ. Hence M_0 is polaroid at λ, which implies that $\lambda\in\pi_0(M_0)$. Since M_0 satisfies Browder's theorem, $\lambda\notin\sigma_w(M_0)=\sigma_w(M_C)$, which in view of the fact that M_C satisfies Browder's theorem implies that $\lambda\in\pi_0(M_C)$. Hence $\sigma(M_C)\setminus\sigma_w(M_C)=E_0(M_C)$ implies $\sigma_a(M_C)\setminus\sigma_{aw}(M_C)=E_0^a(M_C)=E_0(M_C)$, i.e., M_C satisfies property (w). □

Example 4.8. Let A,B and $C\in\mathscr{L}(\ell^2)$ be the operators

$$A(x_1,x_2,\ldots)=\left(0,x_1,0,\frac{1}{2}x_2,\frac{1}{3}x_3,\ldots\right),$$

$$B(x_1, x_2, \dots) = (0, x_2, 0, x_4, 0, \dots),$$

and

$$C(x_1, x_2, \dots) = (0, 0, x_2, 0, x_3, \dots).$$

Then A, A^*, B and B^* have SVEP, $\sigma(A) = \sigma_a(A) = \sigma_w(A) = \sigma_{aw}(A) = \{0\}$, $\pi_0 = E_0(A) = \emptyset$, and A satisfies property (w). Since $\sigma_a(M_0) = \sigma_{aw} = \{0,1\}$ and $E_0(M_0) = \pi_0(M_0) = \emptyset$, M_0 satisfies property (w). However, since $\sigma_a(M_C) = \sigma_{aw} = \{0,1\}$ and $E_0(M_C) = \{0\}$, M_C does not satisfy property (w). Observe that A is not polaroid on $E_0(M_C)$.

Remark 4.9. If the operators A and B have SVEP, then M_0 and M_C have SVEP, $\sigma(M_0) = \sigma(M_C) = \sigma(M_C^*) = \sigma_a(M_C^*)$, $\sigma^{iso}(M_0^*) = \sigma^{iso}(M_C^*) = \sigma_a^{iso}(M_C^*), E_0(M_C^*) = E_0^a(M_C^*)$ and $\sigma_w(M_0) = \sigma_w(M_C) = \sigma_w(M_C^*) = \sigma_{aw}(M_C^*)$. Evidently, A^*, B^*, M_0^* and M_C^* satisfy Browder's theorem; in particular, $\pi_0(M_0^*) = \pi_0(M_C^*) \subseteq E_0(M_C^*)$.

Theorem 4.10. *If the polaroid operators A and B have SVEP, then M_C^* satisfies property (w).*

Proof. Since the polaroid hypothesis on A and B implies that A^* and B^* are polaroid, an argument similar to that in the proof of Theorem 4.7 to M_C^* implies that if $\lambda \in E_0(M_C^*)$, then $\lambda \in E_0(A^*) \cap E_0(B^*)$ implies $\lambda \in \pi_0(A^*) \cap \pi_0(B^*)$. So $\lambda \notin \sigma_w(M_0^*) = \sigma_w(M_C^*)$ implies M_C^* satisfies Weyl's theorem. Hence it follows from Remark 4.9 that $\sigma(M_C^*) \setminus \sigma_w(M_C^*) = E_0(M_C^*) = \sigma_a(M_C^*) \setminus \sigma_{aw}(M_C^*)$. That is, M_C^* satisfies property (w). ☐

Let $H(K)$ denote the space of functions holomorphic on an open neighborhood of $K \subset \mathbb{C}$.

Lemma 4.11. *Let $A \in \mathscr{L}(\mathbb{X})$ and $B \in LB(\mathbb{Y})$ have SVEP. Then*

$$\sigma_{aw}(f(M_C)) = f(\sigma_{aw}(M_C)) \qquad \textit{for every } f \in H(\sigma(M_C)).$$

Proof. Since A and B have SVEP, M_C also has SVEP. Then $f(M_C)$ has SVEP by Corollary 2.40 of [1]. Then it follows from [7, Theorem 3.1] that $f(M_C)$ satisfies a-Browder's theorem. That is, $\sigma_{ab}(f(M_C)) = \sigma_{aw}(f(M_C))$. The proof is follows now from Theorem 3.71 of [1]. ☐

Theorem 4.12. *If $A^* \in \mathscr{L}(\mathbb{X})$ and $B^* \in \mathscr{L}(\mathbb{Y})$ are each polaroid, and have the single valued extension property, then property (w) holds for $f(M_C)$ for arbitrary $f \in H(\sigma(M_C))$ and for arbitrary bounded operators $C \in \mathscr{L}(\mathbb{Y}, \mathbb{X})$.*

Proof. Since M_C is polaroid by Lemma 4.5, then it is a-isoloid. Then

$$f(\sigma_a(M_C) \setminus E_0(M_C)) = \sigma_a(f(M_C)) \setminus E_0(f(M_C)) \qquad \text{for every } f \in H(\sigma(M_C)).$$

It from From Theorem 4.10 and Lemma 4.11 that

$$f(\sigma_a(M_C) \setminus E_0(M_C)) = \sigma_a(f(M_C)) \setminus E_0(f(M_C)) = \sigma_{aw}(f(M_C)) = f(\sigma_{aw}(M_C))$$

for every $f \in H(\sigma(M_C))$. □

An operator $T \in \mathscr{L}(\mathbb{X})$ is said to be a-isoloid if all isolated points of $\sigma_a(T)$ are eigenvalues of T, and $T \in \mathscr{L}(\mathbb{X})$ is called finite a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T of finite multiplicity. Note that finite-a-isoloid implies a-isoloid but the converse is not true.

Theorem 4.13. *Suppose that $\sigma_d(A)$ has no interior points. If A is finite-a-isoloid and property (w) holds for A, then for every $B \in \mathscr{L}(\mathbb{Y})$ and $C \in \mathscr{L}(\mathbb{Y}, \mathbb{X})$, property (w) holds for M_0 implies property (w) holds for M_C.*

Proof. It follows from Theorem 3.1 of [7] that M_C satisfies a-Browder's theorem, i.e.,

$$\sigma_a(M_C) \setminus \sigma_{aw}(M_C) = \pi_0^a(M_C) \subseteq E_0^a(M_C).$$

Conversely, suppose that $\lambda_0 \in E_0^a(M_C)$. Then $M_C - \lambda I$ is bounded below if $|\lambda - \lambda_0|$ is sufficiently small and hence λ is not in $\sigma_a(M_C)$. Since $\sigma_d(A)$ has no interior points, by [7, Corollary 2.4], $\sigma_a(M_C) = \sigma_a(A) \cup \sigma_a(B) = \sigma_a(M_0)$. Then λ is not in $\sigma_a(M_0)$ if $|\lambda - \lambda_0|$ is sufficiently small, that is $\lambda_0 \in \sigma_a^{iso}(M_0)$. Without loss of generality, we suppose that $\lambda_0 \in \sigma_a(A)$, then $\lambda_0 \in \sigma_a^{iso}(A)$. Since $\ker(A - \lambda_0 I) \oplus \{0\} \subseteq \ker(M_C - \lambda_0 I)$, we know that $\alpha(A - \lambda_0 I) < \infty$. A is finite-a-isoloid, then $\lambda_0 \in E_0(A)$. Since property (w) holds for A, it follows that $A - \lambda_0 I \in \Phi_+(\mathbb{X})$ and $a(A - \lambda_0 I) < \infty$. The condition $\sigma_d(A)$ has no interior points asserts that λ_0 is not in $\sigma_d(A)$ or $\lambda_0 \in \partial\sigma_d(A)$. Then in any neighborhood U of λ_0, there exists $\lambda_1 \in U$ such that $\Re(A - \lambda_1 I) = \mathbb{X}$. By perturbation theory of upper semi-Fredholm operator $A - \lambda_0 I$, we get that $A - \lambda I$ is invertible and $ind(A - \lambda_0 I) = ind(A - \lambda I) = 0$ if $|\lambda - \lambda_0|$ is sufficiently small, which means that $A - \lambda_0 I$ is Weyl with finite ascent. [24, Theorem 4.5] asserts that $A - \lambda_0 I$ is Browder. Using the same way in Theorem 2.4 in [20], we get that $0 < \dim[\ker(A - \lambda_0 I) \oplus \ker(B - \lambda_0 I)] < \infty$, which implies that $\lambda_0 \in E_0(M_0)$. Since property (w) theorem holds for M_0, it follows that $M_0 - \lambda_0 I \in \Phi_+^-(\mathbb{X} \oplus \mathbb{Y})$. Hence $M_C - \lambda_0 I \in \Phi_+^-(\mathbb{X} \oplus \mathbb{Y})$, then $\lambda_0 \in \sigma_a(M_C) \setminus \sigma_{aw}(M_C)$. Now we have proved that $\sigma_a(M_C) \setminus \sigma_{aw}(M_C) = E_0(M_C)$, which means that property (w) holds for M_C for every $C \in \mathscr{L}(\mathbb{X}, \mathbb{Y})$. □

Similar to the proof in Theorem 4.13, we can prove that:

Theorem 4.14. *Suppose that $\sigma_d(A) \cap \sigma_{ab}(B)$ has no interior points. If $SP(A)$ has no pseudo-holes (or $\sigma_e(A) = \sigma_{ab}(A)$), where $SP(A)$ denote the spectral picture of A and if A is finite-a-isoloid operator for which property (w) holds, then for every $C \in \mathscr{L}(\mathbb{X}, \mathbb{Y})$, then property (w) holds for M_0 implies property (w) holds for M_C.*

Theorem 4.15. *Let A and B have SVEP. If A is finite-a-isoloid, and if property (w) holds for both A and M_0, then property (w) holds for M_C for every $C \in \mathscr{L}(\mathbb{Y}, \mathbb{X})$.*

Proof. Since A and B have SVEP, M_C also has SVEP and M_C obeys a-Browder's theorem, i.e. $\sigma_a(M_C)\setminus\sigma_{aw}(M_C)=\pi_0^a(M_C)\subseteq E_0^a(M_C)$.

Suppose now that $\lambda\in E_0^a(M_C)$. Then, it follows from Lemma 4.3 that $\lambda\in\sigma_a^{iso}(A)\cup\sigma_a^{iso}(B)$, then $\lambda\in\sigma_a^{iso}(M_0)$, $\dim\ker(M_0-\lambda I)>0$ and $\dim\ker(A-\lambda I)>0$. If $\lambda\in\sigma_a(A)$, then λ is an isolated point and (by the finite-a-isoloid hypothesis) $\lambda\in E_0(A)=\sigma_a(A)\setminus\sigma_{aw}(A)$. If $\lambda\notin\sigma_a(A)$, then again $\lambda\notin\sigma_{aw}(A)$. Hence, in either case, $\lambda\notin\sigma_{aw}(A)$, $\Re(A-\lambda I)$ is closed and $0\le\alpha(A-\lambda I)=\beta(A-\lambda I)<\infty$.

Next, we prove that $\dim\ker(B-\lambda I)$ is finite. Suppose to the contrary that $\dim\ker(B-\lambda I)$ is infinite. Then there exists an infinite sequence $\{u_2^n\}_{n=1}^{\infty}$ of linearly independent vectors in $\ker(B-\lambda I)$. Since $\dim\ker(M_C-\lambda I)<\infty$, there exists a natural number n_0 such that $Cu_2^n\neq 0$ for every natural number $n>n_0$. (For if not, then $(M_C-\lambda I)(0\oplus u_2^n)=0$ for all n, and then $\dim\ker(M_C-\lambda I)=\infty$.) Without loss of generality we may assume that $Cu_2^n\neq 0$ for all n. Since $\beta(A-\lambda I)<\infty$, there exists a natural number n_1 such that $Cu_2^n\in\Re(A-\lambda I)$ for every $n>n_1$, i.e. there exists a sequence $\{u_1^n\}_{n=1}^{\infty}$ in $\mathbb{X}$ such that $(A-\lambda I)(-u_1^n)=Cu_2^n$. Then $(M_C-\lambda I)(u_1^n\oplus u_2^n)=0$ for every $n>n_1$, i.e. $\dim\ker(M_C-\lambda I)=\infty$.

The conclusion that $\dim\ker(B-\lambda I)<\infty$ implies that $0<\dim\ker(M_0-\lambda I)<\infty$ and $\lambda\in\sigma_a^{iso}(M_C)$. Moreover, since $\lambda\in E_0(M_0)=\sigma_a(M_0)\setminus\sigma_{aw}(M_0)$, $\lambda\notin\sigma_{aw}(M_C)$. Hence property (w) holds for M_C. □

Theorem 4.16. *If the finite-a-isoloid operators A and B have SVEP, and if property (w) holds for both A and B, then property (w) holds for M_C for every $C\in\mathscr{L}(\mathbb{Y},\mathbb{X})$.*

Proof. If the hypotheses of the theorem are satisfied then it follows from the argument of the proof of Theorem 4.15 that M_C obeys a-Browder's theorem, and if $\lambda\in E_0(M_C)$ then (by the finite-a-isoloid property of A) $\lambda\notin\sigma_{aw}(A)$, $\lambda\in\sigma_a^{iso}(B)\cup\rho_a(B)$ and $\dim\ker(B-\lambda I)<\infty$. Since B is finite-a-isoloid operator for which property (w) holds, $\lambda\in E_0(B)=\sigma_a(B)\setminus\sigma_{aw}(B)$. Hence, $\lambda\notin\sigma_{aw}(A)\cup\sigma_{aw}(B)$ and so $\lambda\notin\sigma_{aw}(M_C)$. □

We consider now necessary and(/or) sufficient conditions for the implications M_0 satisfies property (w) $\Leftrightarrow$ M_C satisfies property (w). As one would expect, M_0 satisfies property (w) does not imply M_C satisfies property (w). For example, if $A,B,C\in\mathscr{L}(\ell^2\oplus\ell^2)$ are the operators $A=U\otimes I$, $B=U^*\otimes I$ and C is the diagonal operator with entries $(0,I-UU^*,I-UU^*,\dots)$, where $U\in\mathscr{L}(\ell^2)$ is the forward unilateral shift, then $\sigma_a(M_0)=\sigma_{aw}(M_0)$, $\pi_0^a(M_0)=\emptyset=E_0(M_0)$ and M_0 satisfies property (w); however, $\sigma(M_C)$ is the closed unit disc $\mathbf{D}$, $\sigma_w(M_C)$ is the boundary $\partial\mathbf{D}$ of $\mathbf{D}$, $\pi_0(M_C)=\emptyset$, and M_C does not satisfy Browder's theorem (much less property (w)). Conversely, M_C satisfies property (w) does not imply M_0 satisfies property (w), as the example of the operator $\begin{pmatrix} U & I-UU^* \\ 0 & U^* \end{pmatrix}$ shows. Recall, however, that M_0 satisfies a-Browder's theorem if and only if A and B have SVEP on $\Delta_a(M_0)$; hence, if M_C has SVEP on $\sigma_{aw}(M_0)\setminus\sigma_{aw}(M_C)$, then, since M_0 satisfies a-Browder's theorem implies M_C has SVEP on $\Delta_a(M_C)$, M_C satisfies a-Browder's theorem.

Theorem 4.17.

(a) *If* $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, *and either* A^* *or* B *has SVEP on* $\Delta_a(M_C)$. *Then the equivalence*

$$M_0 \quad \textit{satisfies property}\,(w) \Leftrightarrow M_C \quad \textit{satisfies property}\,(w)$$

holds if and only if $E_0(M_0) = E_0(M_C)$.

(b) *If* A *and* A^*, *or* A^* *and* B^*, *have SVEP on* $\Delta_a(M_C)$. *Then the equivalence*

$$M_0 \quad \textit{satisfies property}\,(w) \Leftrightarrow M_C \quad \textit{satisfies property}\,(w)$$

holds if and only if $E_0(M_0) = E_0(M_C)$.

(c) *If* A *and* A^* *have SVEP on* $\Delta_a(M_C) \setminus \sigma_{SF_+}(A)$, *or* A^* *has SVEP on* $\Delta_a(M_C) \setminus \sigma_{SF_+}(A)$ *and* B^* *has SVEP on* $\Delta_a(M_C) \setminus \sigma_{SF_+}(B)$. *Then the equivalence*

$$M_0 \quad \textit{satisfies property}\,(w) \Leftrightarrow M_C \quad \textit{satisfies property}\,(w)$$

holds if and only if $E_0(M_0) = E_0(M_C)$.

Proof.

(a) The hypothesis $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ implies that $\sigma_{aw}(M_0) = \sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. If M_0 satisfies property (w) then M_0 satisfies a-Browder's theorem, so A and B have SVEP on $\Delta_a(M_0)$ implies that M_C has SVEP on $\Delta_a(M_C)$, and so M_C satisfies a-Browder's theorem. Hence

$$\sigma_a(M_C) \setminus \sigma_{aw}(M_C) = \sigma_a(M_0) \setminus \sigma_{aw}(M_0) = E_0(M_0) = \pi_0^a(M_0) = \pi_0^a(M_C) \subseteq E_0(M_C)$$

Assume now that M_C satisfies property (w), then M_C satisfies a-Browder's theorem $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, and either A^* or B has SVEP on $\Delta_a(M_C)$. Since M_C satisfies a-Browder's theorem implies A has SVEP on $\Delta_a(M_C)$, if B has SVEP on $\Delta_a(M_C)$, then M_0 has SVEP on $\Delta_a(M_0) = \Delta_a(M_C)$, and so M_0 satisfies a-Browder's theorem. Assume now that A^* has SVEP on $\Delta_a(M_C)$: we prove that $\sigma_a(M_0) = \sigma_a(A) \cup \sigma_a(B) = \sigma_a(M_C)$. If $\mu \notin \sigma_a(M_C)$, then $M_C - \mu I$ and $A - \mu I$ are left invertible, $\mu \in \Delta_a(M_C)$. The left invertibility of $A - \mu I$ implies the right invertibility of $A^* - \mu I^*$; hence, since A^* has SVEP on $\Delta_a(M_C)$, $A^* - \mu I^*$ is invertible. But then the invertibility of $A - \mu I$, taken along with the left invertibility of $M_C - \mu I$, implies that $B - \mu I$ is left invertible. Hence $\mu \notin \sigma_a(A) \cup \sigma_a(B)$. Since $\sigma_a(M_C) \subseteq \sigma_a(A) \cup \sigma_a(B)$ always, $\sigma_a(M_C) = \sigma_a(A) \cup \sigma_a(B) = \sigma_a(M_0)$. Assume now that M_C satisfies a-Browder's theorem. Then $\lambda \in \Delta_a(M_C)$ implies that $\lambda \in \sigma_a^{iso}(M_C) = \sigma_a^{iso}(A) \cup \sigma_a^{iso}(B)$;

hence (A and B have SVEP on $\Delta_a(M_0) = \Delta_a(M_C)$ implies) M_0 has SVEP on $\Delta_a(M_0)$, and so M_0 satisfies a-Browder's theorem, and so

$$\sigma_a(M_0) \setminus \sigma_{aw}(M_0) = \sigma_a(M_C) \setminus \sigma_{aw}(M_C) = E_0(M_C) = \pi_0^a(M_C) = \pi_0^a(M_0) \subseteq E_0(M_0).$$

Thus, the statements of the theorem are equivalent if and only if $E_0(M_0) = E_0(M_C)$.

(b) Let $\lambda \in \Delta_a(M_C)$. Then the hypothesis that A and A^* have SVEP on $\Delta_a(M_C)$ implies that $\lambda \in \Phi^0(A) \cap \Phi_+^-(B) \subseteq \Phi_+^-(A) \cup \Phi_+^-(B)$. Consequently, $\sigma_{aw}(A) \cup \sigma_{aw}(B) \subset \sigma_{aw}(M_C)$; hence $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. Again, if A^* and B^* have SVEP on $\Delta_a(M_C)$, then $\lambda \in \Delta_a(M_C)$ $\Rightarrow \lambda \in \Phi_+(A)$, $ind(A-\lambda I) + ind(B-\lambda I) \le 0$ $\beta(A-\lambda I) \le \alpha(A-\lambda I)$, $\beta(B-\lambda I) \le \alpha(B-\lambda I)$. Hence, in view of Proposition 3.3 of [15], $\lambda \in \Phi^0(A) \cap \Phi^0(B) \subseteq \Phi_+^-(A) \cup \Phi_+^-(B)$, which (once again) leads to the conclusion that $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. Applying part (a), the proof follows.

(c) Let $\lambda \in \Delta_a(M_C)$. Then $\lambda \in \Phi_+(A)$ and $ind(A-\lambda I) + ind(B-\lambda I) \le 0$. If A and A^* have SVEP on $\Delta_a(M_C) \setminus \sigma_{SF_+}(A)$, then $\lambda \in \Phi^0(A)$ (is isolated in $\sigma_a(A)$), and this forces $\lambda \in \Phi_+^-(B)$. Hence $\lambda \notin \sigma_{aw}(A) \cup \sigma_{aw}(B)$, which leads us to the equality $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. Again, if A^* has SVEP on $\Delta_a(M_C) \setminus \sigma_{SF_+}(A)$, then $\lambda \in \Phi_+^-(A)$ and this implies $\lambda \in \Phi_+(B)$; thus, if B^* has SVEP on $\Delta_a(M_C) \setminus \sigma_{SF_+}(B)$, then $\lambda \in \Phi_+^-(B)$, which forces $\lambda \in \Phi^0(A) \cap \Phi^0(B)$ and $\lambda \in \sigma^{iso}(A) \cup \sigma^{iso}(B)$. Once again, we conclude that $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. The proof now follows from an application of part (b) (since both A and A^* have SVEP on $\Delta_a(M_C)$). □

Theorem 4.18. *(a). If $\sigma_{aw}(M_0) = \sigma_{aw}(M_C)$, then M_0 satisfies property (w) implies M_C satisfies property (w) if and only if $E_0(M_C) \subseteq E_0(M_0)$.*
(b). If $\sigma_{aw}(M_0) = \sigma_{aw}(M_C)$ and A^* has SVEP on $\Delta_a(M_C)$, then then M_C satisfies property (w) implies M_0 satisfies property (w) if and only if $E_0(M_0) \subseteq E_0(M_C)$.

Proof.

(a) Since M_0 satisfies property (w) implies M_0 satisfies a-Browder's theorem, A and B have SVEP on $\Delta_a(M_C)$. $(\sigma_{aw}(M_C) = \sigma_{aw}(M_0) = \sigma_{aw}(A) \cup \sigma_{aw}(B))$. Hence M_C satisfies a-Browder's theorem by Theorem 4.12 of [15]. Thus $\lambda \in \pi_0^a(M_C)$ if and only if $\lambda \in \Delta_a(M_C) = \Delta_a(M_0) = \pi_0^a(M_0) = E_0(M_0)$. It follows that

$$\sigma_a(M_C) \setminus \sigma_{aw}(M_C) = \pi_0^a(M_C) = \pi_0^a(M_0) = E_0(M_0) \subseteq E_0(M_C),$$

which proves that M_C satisfies property (w) if and only if $E_0(M_C) \subseteq E_0(M_0)$.

(b) The argument of th proof of Theorem 4.12 part(i) of [15] shows that if $\sigma_{aw}(M_0) = \sigma_{aw}(M_C)$ and A^* has SVEP on $\Delta_a(M_C)$, then $\sigma_a(M_C) = \sigma_a(M_0) = \sigma_a(A) \cup \sigma_a(B)$. Thus, if M_C satisfies property (w), then M_0 satisfies a-Browder's theorem, i.e., $\Delta_a(M_0) = \pi_0^a(M_0)$ and

$$\Delta_a(M_0) = \Delta_a(M_C) = \pi_0^a(M_C) = \pi_0^a(M_0) = E_0(M_0) \subseteq E_0(M_C),$$

where the equality $\pi_0^a(M_0) = \pi_0^a(M_C)$ follows from the implications

$$\lambda \in \pi_0^a(M_C) \Leftrightarrow \lambda \in \Delta_a(M_C) = \Delta_a(M_0) \Leftrightarrow \lambda \in \pi_0^a(M_0).$$

hence M_0 satisfies property (w) if and only if $E_0(M_C) \subseteq E_0(M_0)$. □

Theorem 4.19. *If $\sigma_{aw}(A) = \sigma_{SF_+}(B)$, A is a-isoloid and property (w), then M_0 satisfies property (w) implies M_C satisfies property (w).*

Proof. Start by observing that if $\lambda \in \Phi_+^-(M_C)$ and $ind(A-\lambda I) > 0$, then $\lambda \in \Phi(A) \cap \Phi_+(B)$ and $ind(A-\lambda I)) + ind(B-\lambda I)) \leq 0$; if, instead, $ind(A-\lambda I) \leq 0$, then $\sigma_{aw}(A) = \sigma_{SF_+}(B)$ and $\lambda \in \Phi_+^-(M_C)$ imply that $\lambda \in \Phi_+^-(A) \cap \Phi_+(B)$ and $ind(A-\lambda I) + ind(B-\lambda I) \leq 0$. In either case, $\lambda \in \Phi_+^-(M_C)$ implies $\lambda \in \Phi_+^-(M_0)$; hence$\sigma_{aw}(M_C) = \sigma_{aw}(M_0)$. In view of Theorem 4.18, we are thus left to prove that $E_0(M_C) \subseteq E_0(M_0)$. If $\lambda \in E_0(M_C)$, then $\lambda \in \sigma_a^{iso}(A) \cup \sigma_a^{iso}(B)$, and so $\lambda \in E_0(A) = \Delta_a(A) = \sigma_a(B) \setminus \sigma_{SF_+}(B)$ (since A is a-isoloid, A satisfies property (w) and $\sigma_{aw}(A) = \sigma_{SF_+}(B)$. But then, since M_0 satisfies a-Browder's theorem implies B has SVEP at λ, $\lambda \in \pi_0^a(B)$. Hence $\lambda \in \pi_0^a(M_0) = E_0(M_0)$. □

Remark 4.20. If A^* has SVEP, then $\lambda \in \Delta_a(M_C)$ implies $\lambda \in \Phi(A) \cap \Phi_+^-(B)$, $ind(A-\lambda I)) \geq 0$ and $ind(A-\lambda I) + ind(B-\lambda I) \leq 0$; this in turn implies that $\lambda \notin \sigma_{aw}(A) \cup \sigma_{aw}(B)$. Thus, if A^* has SVEP and M_0 satisfies a-Browder's theorem, then $\sigma_{aw}(M_0) = \sigma_{aw}(A) \cup \sigma_{aw}(B) = \sigma_{aw}(M_C)$.

Theorem 4.21. *If $\sigma_a(A^*)$ has empty interior, A is a-isoloid and property (w), then M_0 satisfies property (w) implies M_C satisfies property (w).*

Proof. Evidently, A^* has SVEP, M_0 satisfies a-Browder's theorem and $\sigma_{aw}(M_0) = \sigma_{aw}(M_C)$. Now argue as in the (latter part of the) proof of Theorem 4.19. □

For an operator $T \in \mathscr{L}(\mathbb{X})$ such that T^* has SVEP, T satisfies Weyl's theorem if and only if T property (w) [3, Theorem 2.16]. Thus, if A^* and B^* have SVEP, then $M_X^* = M_0^*$ or M_C^* has SVEP, and the (two way) implication M_X satisfies Weyl's theorem if and only if M_X satisfies property (w). The following theorem, proves more.

Theorem 4.22. *If $S_{\sigma_{SF_+}(A)}(A^*) \cup S_{\sigma_{SF_+}(B)}(B^*) = \emptyset$, then M_C satisfies Weyl's theorem if and only M_C satisfies a-Weyl's theorem if and only if M_C satisfies property (w).*

Proof. The implication M_C satisfies a-Weyl's theorem or M_C satisfies property (w) implies M_C satisfies Weyl's theorem being clear, we prove the reverse implication. For this, it would suffice to prove that $\sigma(M_C) = \sigma_a(M_C)$ (which would then imply $E_0(M_C) = E_0^a(M_C)$ and $\sigma_w(M_C) = \sigma_{aw}(M_C)$).

Evidently, $\sigma_a(M_C) \subseteq \sigma(M_C)$. Let $\lambda \notin \sigma_a(M_C)$. Then $M_C - \lambda I$ and $A - \lambda I$ are left invertible.

The left invertibility of $A-\lambda I$ implies $\lambda \in \Phi_+(A)$. Since A^* has SVEP at points $\lambda \in \Phi_+(A)$, it follows that $A-\lambda I$ is invertible. But then $B-\lambda I$ is left invertible, which (because B^* has SVEP at points $\lambda \in \Phi_+(B)$ implies that $B-\lambda I$ is invertible. Thus, $\lambda \notin \sigma(A)\cup\sigma(B)$, i.e., $\sigma(M_C) \subseteq \sigma(A)\cup \sigma(B) \subseteq \sigma_a(M_C)$. Next, we prove that $\sigma_w(M_C) \subseteq \sigma_{aw}(M_C)$: this would then imply the equality $\sigma_w(M_C) = \sigma_{aw}(M_C)$. Let $\lambda \notin \sigma_{aw}(M_C)$; then $\lambda \in \Phi_+(A)$ (and $ind(A-\lambda I)+ind(B-\lambda I) \leq 0$). Since A^* has SVEP at points $\lambda \in \Phi_+(A)$, it follows that $ind(A-\lambda I) \geq 0$) implies $\lambda \in \Phi(A)$ (with $ind(A-\lambda I) \geq 0$). Since this forces $\lambda \in \Phi_+(B)$, it follows (from the hypothesis B^* has SVEP on the set of $\lambda \in \Phi_+(B)$) that $\lambda \in \Phi(B)$ and $ind(B-\lambda I) \geq 0$. Since $ind(A-\lambda I)+ind(B-\lambda I) \leq 0$, we conclude that $\lambda \in \Phi^0(A)\cap\Phi^0(B)$. Hence $\sigma_w(M_C) \subseteq \sigma_w(A)\cup\sigma_w(B) \subseteq \sigma_{aw}(M_C)$, and the proof is achieved. □

Two important T-invariant subspaces of T are defined as follows. The *quasinilpotent part* $H_0(T-\lambda I)$ and the *analytic core* $K(T-\lambda I)$ of $T-\lambda I$ are defined by

$$H_0(T-\lambda I) := \{x \in \mathbb{X} : \lim_{n\longrightarrow\infty} \|(T-\lambda I)^n x\|^{\frac{1}{n}} = 0\}.$$

and

$$\begin{aligned} K(T-\lambda I) = \{x \in \mathbb{X} : & \text{ there exists a sequence } \{x_n\} \subset \mathbb{X} \text{ and } \quad \delta > 0 \quad \text{for which} \\ & x = x_0, (T-\lambda I)x_{n+1} = x_n \text{and} \quad \|x_n\| \leq \delta^n \|x\| \text{for all} \quad n = 1,2,\ldots\}. \end{aligned}$$

Note that $(T-\lambda I)K(T-\lambda I) = K(T-\lambda I)$ [1, Chapter 1], Moreover (see [2]),

$$H_0(T-\lambda I) \text{ is closed} \Longrightarrow T \text{ has SVEP at } \lambda.$$

Lemma 4.23. *([4]) Suppose that for a bounded linear operator $T \in \mathscr{L}(\mathbb{X})$ there exists $\lambda_0 \in \mathbb{C}$ such that $K(T-\lambda_0 I) = \{0\}$ and $\ker(T-\lambda_0 I) = \{0\}$. Then $\sigma_p(T) = \emptyset$.*

Proof. Since $\ker(T-\lambda I) \subseteq \ker(T-\lambda_0 I)$ for all $\lambda \neq \lambda_0$, so that $\ker(T-\lambda I) = \{0\}$ for all $\lambda \in \mathbb{C}$. □

Theorem 4.24. *Suppose that there exists $\lambda_0 \in \mathbb{C}$ such that*

$$K(A-\lambda_0 I) = \{0\}, \ker(A-\lambda_0 I) = \{0\}, K(B-\lambda_0 I) = \{0\} \quad \textit{and} \quad \ker(B-\lambda_0 I) = \{0\}.$$

Then property (w) holds for $f(M_C)$ for all $f \in H(\sigma(f(M_C))$.

Proof. It follows from Lemma 4.23 that $\sigma_p(A) = \sigma_p(B) = \emptyset$, so A and B have SVEP and hence M_C has SVEP. We show that also $\sigma_p(f(M_C)) = \emptyset$. Let $\mu \in \sigma(f(M_C)))$ and write $f(\lambda)-\mu = p(\lambda)g(\lambda)$, where g is analytic on an open neighborhood U containing $\sigma(M_C)$ and without zeros in $\sigma(M_C)$, p a polynomial of the form $p(\lambda) = \prod_{k=1}^n (\lambda-\lambda_k)^{\nu_k}$, with distinct roots $\lambda_1,\ldots,\lambda_n$ lying in $\sigma(M_C)$. Then

$$f(M_C)-\mu I = \prod_{k=1}^n (M_C-\lambda_k I)^{\nu_k} g(M_C).$$

Since $g(M_C)$ is invertible, $\sigma_p(M_C) \subseteq \sigma_p(A) \cup \sigma_p(B) = \emptyset$ implies that $\ker(f(M_C) - \mu I) = \{0\}$ for all $\mu \in \mathbb{C}$. Since M_C has SVEP then $f(M_C)$ has SVEP, see Theorem 2.40 of [1], so that $f(M_C) \in aB$ [5]. To prove that property (w) holds for $f(M_C)$, by Theorem 2.7 of [3] it then suffices to prove that

$$\pi_0^a(f(M_C)) = E_0(f(M_C)).$$

Obviously, the condition $\sigma_p(f(M_C)) = \emptyset$ entails that

$$E_0(f(M_C)) = E_0^a(f(M_C)) = \emptyset.$$

On the other hand, the inclusion $\pi_0^a(f(M_C)) \subseteq E_0^a(f(M_C))$ holds for every operator $M_C \in \mathscr{L}(\mathbb{X}, \mathbb{Y})$, so also $\pi_0^a(f(M_C))$ is empty. Hence, the result follows. □

Recall that an operator $T \in \mathscr{L}(\mathbb{X})$ is an a-polaroid if $\sigma_a^{iso}(T) \subseteq \pi(T)$. Since $\pi_0(T) \subseteq E_0^a(T)$, then if T is a-polaroid then $\pi_0(T) = E_0^a(T)$.

Theorem 4.25. *Let A and B be a-polaroid with the SVEP. Then M_C obeys property (w) for every $C \in \mathscr{L}(\mathbb{Y}, \mathbb{X})$.*

Proof. A and B are a-polaroid hence $\pi_0(A) = E_0^a(A)$ and $\pi_0(B) = E_0^a(B)$. Since A and B have the SVEP, we have by by [3] that A and B satisfy property (w). Therefore,

$$E_0(M_0) = \Delta_a(M_0) = \Delta_a(M_C).$$

Hence it is enough to show that $E_0(M_0) = E_0(M_C)$. Let $\lambda \in E_0(M_C)$. Then $\lambda \in \sigma_p(M_C) \subseteq \sigma_p(A) \cup \sigma_p(B)$. Hence $\lambda \in \sigma_p(M_0)$. Since $\lambda \in \sigma^{iso}(M_C) = \sigma^{iso}(M_0)$ we have $\lambda \in E_0(M_0)$. Now let $\lambda \in E_0(M_0)$. If $\lambda \in \sigma_a(A)$ then $\lambda \in \sigma_a^{iso}(A)$. Since A is a-isoloid, we have $\lambda \in \sigma_p(A) \subseteq \sigma_p(M_C)$. Hence $\lambda \in E_0(M_C)$. If $\lambda \sigma_a(B) \setminus \sigma_a(A)$, then $\lambda \in \sigma_p(B)$. Since A is invertible, we conclude that $\lambda \in \sigma_p(M_C)$. Thus $\lambda \in E_0(M_C)$. So the proof of the theorem is achieved. □

Theorem 4.26. *Let A be an a-isoloid. Assume that A and B (or A^* and B^*) have the SVEP. If A and M_0 satisfy property (w) then M_C satisfies property (w) for every $C \in \mathscr{L}(\mathbb{Y}, \mathbb{X})$.*

Proof. Let $\lambda \in \Delta_a(M_C)$. Then $\sigma_a(M_C) = \sigma_a(M_0)$ and hence $\Delta_a(M_C) = \Delta_a(M_0) = E_0(M_0)$ since M_0 satisfies property (w). Thus $\lambda \in \sigma_a^{iso}(M_0) = \sigma_a^{iso}(M_C)$. If $\lambda \in \sigma_a^{iso}(A)$, since A is a-isoloid then $\lambda \in \sigma_p(A)$. Hence $\lambda \in \sigma_p(M_C)$. Then $\lambda \in E_0(M_C)$. Now assume that $\lambda \in \sigma_a^{iso}(B) \setminus \sigma_a^{iso}(A)$. If $\lambda \notin \sigma_a(A)$ then it is not difficult to see that $\lambda \in \sigma_p(M_C)$. Also if $\lambda \in \sigma_p(A)$ then $\lambda \in \sigma_p(M_C)$, so assume that $\lambda \in \sigma_p(B) \setminus \sigma_p(A)$. Then $\lambda \notin E_0(A)$. Since A satisfies property (w), then $\lambda \in \sigma_{aw}(A)$. This is impossible. Therefore, $\lambda \in E_0(M_C)$. Conversely, assume that $\lambda \in E_0(M_C)$. Then $\lambda \in \sigma_a^{iso}(M_C) = \sigma_a^{iso}(M_0)$. On the other hand, $\lambda \in \sigma_p(M_C) \subseteq \sigma_p(A) \cup \sigma_p(B)$. Hence $\lambda \in \sigma_p(M_0)$. Thus

$$\lambda \in E_0(M_0) = \Delta_a(M_0) = \Delta_a(M_C).$$ □

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