TAMKANG JOURNAL OF MATHEMATICS
Volume 35, Number 1, Spring 2004

# BOUNDARY VALUES OF ANALYTIC FUNCTIONS WITHOUT DISTRIBUTIONAL POINT VALUES

RICARDO ESTRADA

**Abstract**. We give a method to construct distributions that are boundary values of analytic functions which have non-tangential limits at points where the distributional point value does not exist.

## 1. Introduction.

Let $f \in \mathcal{D}'(\mathbb{R})$ be a distribution that is the boundary value of an analytic function defined in the upper-half plane, $f(x) = F(x+i0)$, distributionally. Then it is well-known that if the distributional point value $f(x_0) = L$ exists in the sense of Łojasiewicz [7], then $F(x_0 + iy) \to L$ as $y \to 0^+$ [10, 11].

The purpose of this note is to show how one can construct a counterexample to the reciprocal result. Namely, we will show that there are functions $F$, analytic in the upper-half plane, with distributional boundary values $f(x) = F(x + i0)$, $f \in \mathcal{D}'(\mathbb{R})$, for which the limit $\lim_{y\to 0^+} F(x_0+iy)$ exists, but the distributional value $f(x_0)$ does not. Actually, counterexamples where even the non-tangential limit of $F(z)$ as $z \to x_0$ exists can be constructed.

Our construction is based on the Baire theorem, and so we start by giving a useful variant of this result in Section 2. Next, in Section 3 we show the existence of series $\sum_{n=0}^{\infty} a_n$ that are Abel but not Cesàro summable and satisfy the additional condition $a_n = O(n^\beta)$, $n \to \infty$, for some $\beta > -1$. The existence of such series is used in Section 4 to prove the existence of the announced counterexamples.

## 2. A variant of the Baire Theorem

Our construction is based on the well-known Baire theorem [5]. The theorem of Baire says that a complete metric space is of the second category. Sets of the second category are those that are not of the first category, i.e., countable unions of nowhere dense sets. A nowhere dense set is one whose closure has empty interior.

Received May 29, 2002.
2000 *Mathematics Subject Classification*. Primary 30B30, 46F12.
*Key words and phrases*. Boundary values of analytic functions, distributional point values.

Our argument uses a variant of the Baire theorem, that although really useful, does not seem to be stated explicitly in the texts; thus, it is convenient to start the article by considering this result.

If $E$ is a closed proper linear subspace of a topological vector space $F$, then it is nowhere dense, since the only linear subspace with non-empty interior is $F$ itself. Thus is $\{E_n\}_{n=1}^{\infty}$ is a sequence of proper *closed* linear subspaces of a Fréchet space $F$, then $\bigcup_{n=1}^{\infty} E_n \neq F$. In the present situation we need to conclude that $\bigcup_{n=1}^{\infty} E_n \neq F$, where the $E_n$'s are proper linear subspaces of $F$, but not closed (in fact, each of them is dense in $F$!).

That $\bigcup_{n=1}^{\infty} E_n$ may be equal to $F$, where $F$ is any infinite dimensional Fréchet space, can be seen from the following construction. Let $\{x_a\}_{a\in A}$ be an algebraic (or Hamel) basis of $F$ over the field ($\mathbb{R}$ or $\mathbb{C}$). Since $A$ is infinite, there is an increasing sequence of sets $\{A_n\}_{n=1}^{\infty}$ whose union is $A$, $A_n \nearrow A$. Let $E_n$ be the linear subspace spanned (algebraically) by $\{x_a\}_{a\in A_n}$. Then the $E_n$ are proper linear subspaces of $F$ and $\bigcup_{n=1}^{\infty} E_n = F$. As we show in the Theorem 2, however, the conclusion $\bigcup_{n=1}^{\infty} E_n \neq F$ can be obtained under additional hypothesis on the $E_n$'s.

**Theorem 1.** *Let $E$ and $F$ be Fréchet spaces. Suppose $E \subset F$ and the inclusion is continuous. If $E \neq F$ then $E$ is of the first category in $F$.*

**Proof.** Denote as $\mathfrak{T}_E$ and $\mathfrak{T}_{F\to E}$ the topologies of $E$ as a Fréchet space and as a subspace of $F$, respectively. We are assuming that the identity map $Id : (E, \mathfrak{T}_E) \to (E, \mathfrak{T}_{F\to E})$ is continuous.

If $Id$ is also open, then the two topologies coincide. It then follows that $(E, \mathfrak{T}_{F\to E})$ is complete and, consequently, that $E$ is a closed subspace of $F$. Since $E \neq F$ it follows that $E$ is nowhere dense in $F$ and thus of the first category.

Let us now consider the situation when $Id$ is not open. Let $d_E$ and $d_F$ be invariant metrics for $E$ and $F$, respectively. Consider the "balls"

$$B_{E,r}(0) = \{x \in E : d_E(x, 0) \leq r\}. \tag{2.1}$$

Let $W_r$ be the closure of $B_{E,r}(0)$ in $F$. Then either (i) $W_r$ is a neighborhood of zero in $F$ for each $r > 0$, or (ii) there exists $r_0 > 0$ such that the origin is not in the interior of $W_r$ for $r \leq r_0$.

If (ii) holds, then the interior of $W_r$ is empty if $r \leq r_0/2$, since if $x \in \operatorname{int}(W_r)$ then $0 \in \operatorname{int}(W_{2r})$. Therefore $B_{E,r}(0)$ is nowhere dense in $F$ for $r \leq r_0/2$ and consequently $E = \bigcup_{n=1}^{\infty} nB_{E,r}(0)$ is of the first category in $F$.

Let us show that (i) cannot hold, since it would imply that $E = F$. Indeed, when (i) holds, then $\forall r > 0\ \exists \rho = \rho(r) > 0$ such that $B_{F,\rho}(0) \subset W_r$. Fix $r = r_0$ and $\rho = \rho(r)$. Let $y \in B_{F,\rho}(0)$. Let $r_n \searrow 0$ be a decreasing sequence such that $\sum_{n=1}^{\infty} r_n < \infty$, and let $\rho_n = \rho(r_n)$, chosen in such a way that $\rho_n \searrow 0$. Then a sequence $\{x_n\}_{n=1}^{\infty}$ of $E$ can be constructed recursively by requiring that

$$d_F\left(y, \sum_{j=1}^{n} x_j\right) < \rho_n, \tag{2.2}$$

$$d_E(x_n, 0) < r_{n-1}. \tag{2.3}$$

Then (2.3) implies that the partial sums of the series $\sum_{n=1}^{\infty} x_n$ form a Cauchy sequence in $E$ and, since $E$ is complete, $\sum_{n=1}^{\infty} x_n$ converges in $E$ to some $x \in E$. But this yields that $\sum_{n=1}^{\infty} x_n = x$ in the topology of $F$ too, while (2.2) gives that $\sum_{n=1}^{\infty} x_n = y$ in $F$. We conclude that $y = x \in E$. Thus $B_{F,\rho}(0) \subset E$ and so $F = \bigcup_{n=1}^{\infty} nB_{F,\rho}(0) \subset E$.

Using the Theorem 2.1 we immediately obtain our variant of the Baire theorem.

**Theorem 2.** *Let $F$ be a Fréchet space. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of Fréchet spaces such that $E_n \subset F$, inclusions being continuous. If $E_n \neq F$ $\forall n$ then $\bigcup_{n=1}^{\infty} E_n \neq F$.*

It is worth remarking that local convexity is not used in these results, so that they hold for complete metrizable topological vector spaces, not just Fréchet spaces.

## 3. Series that are Abel but not Cesàro Summable

Let $\sum_{n=0}^{\infty} a_n$ be a possibly divergent series. We say that the series is Abel summable to $S$, and write

$$\sum_{n=0}^{\infty} a_n = S \quad (\mathrm{A})\,, \tag{3.1}$$

if the series $\sum_{n=0}^{\infty} a_n r^n$ converges for $0 \le r < 1$ and

$$\lim_{r\to 1} \sum_{n=0}^{\infty} a_n r^n = S\,. \tag{3.2}$$

Notice that to each Abel summable series there corresponds an analytic function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ which is defined in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and that has a radial limit at $z = 1$, and conversely. The space of Abel summable series $F_{\mathrm{Abel}}$ has a natural structure of a Fréchet space, corresponding to the intersection topology of the Fréchet space $H(\Delta)$ of analytic functions in $\Delta$ [9], and the Banach space $C[0,1]$, with its usual supremum norm.

On the other hand, the Cesàro summability of the series $\sum_{n=0}^{\infty} a_n$ is defined as follows [3, 4]. Set

$$A_n^0 = a_0 + \cdots + a_n\,, \tag{3.3}$$

$$A_n^\kappa = A_0^{\kappa-1} + \cdots + A_n^{\kappa-1}\,. \tag{3.4}$$

Then the series is $(\mathrm{C}, \kappa)$ summable to $S$, written as

$$\sum_{n=0}^{\infty} a_n = S \quad (\mathrm{C}, \kappa)\,, \tag{3.5}$$

if

$$\lim_{n\to\infty} \frac{A_n^\kappa}{n^\kappa} = \frac{s}{\kappa!}\,. \tag{3.6}$$

The notation $\sum_{n=0}^{\infty} a_n = S$ (C) means that (3.5) holds for some $\kappa$. The formula

$$A_n^\kappa = \sum_{q=0}^{n} \binom{q+\kappa}{\kappa} a_{n-q}\,, \tag{3.7}$$

that makes sense for any $\kappa \in \mathbb{R}$, $\kappa > -1$, allows us to define the Cesàro summability of non-integral order. Since when $\kappa_1 > \kappa_2$, then $(\mathrm{C}, \kappa_1)$ summability implies $(\mathrm{C}, \kappa_2)$ summability, for our present purposes one may consider only summability of intergral order.

Convergence implies $(\mathrm{C}, \kappa)$ summability of any order, while Cesàro summability, in turn, implies Abel summability.

It is very simple to construct series that are Abel but not Cesàro summable. Indeed, if $\sum_{n=0}^{\infty} a_n$ is the series obtained by setting $x = 1$ in the Taylor series $\sum_{n=0}^{\infty} a_n x^n = e^{(1+x)^{-1}}$, then $\sum_{n=0}^{\infty} a_n = e^{1/2}$ (A), but the series is not (C) summable since the terms are not of the order $O\left(n^\beta\right)$ for any $\beta \in \mathbb{R}$. Similarly, the series $\sum_{n=0}^{\infty} (-1)^n e^{c\sqrt{n}}$, where $c > 0$, is also (A) but not (C) summable [3, 4].

The counterexample of Section 4, however, requires the construction of a series $\sum_{n=0}^{\infty} a_n$ that is Abel but not Cesàro summable and whose terms satisfy a bound of the type $a_n = O\left(n^\beta\right)$, $n \to \infty$, for some $\beta \in \mathbb{R}$. That this is not possible if $\beta = -1$ follows from Littlewood's tauberian theorem [6], since if $\sum_{n=0}^{\infty} a_n = S$ (A) and $a_n = O\left(n^{-1}\right)$, $n \to \infty$, then $\sum_{n=0}^{\infty} a_n$ actually converges to $S$. Nevertheless, such examples exist for any $\beta > -1$.

**Theorem 3.** *Let $\beta > -1$. Then there exists a series $\sum_{n=0}^{\infty} a_n$ with $a_n = O\left(n^\beta\right)$, $n \to \infty$, that is Abel but not Cesàro summable.*

**Proof.** Let $F_\beta$ be the Banach space of series $\sum a_n$ of complex terms that satisfy $a_n = O\left(n^\beta\right)$, $n \to \infty$, with the norm

$$\|\{a_n\}\|_{F_\beta} = \max\left\{|a_0|\,, \sup_{n\geq 1} n^{-\beta}\,|a_n|\right\}\,. \tag{3.8}$$

Let $F_{\mathrm{Abel}}$ be the Fréchet space of (A) summable series, introduced before, and let $F = F_\beta \cap F_{\mathrm{Abel}}$, with the intersection topology; interestingly, $F$ is not only a Fréchet space but actually a Banach space.

For any $\kappa > 1$, let $E_{(\mathrm{C},\kappa)}$ be the space of series that are $(\mathrm{C}, \kappa)$ summable, with the seminorms

$$\|\{a_n\}\|_{E_{(\mathrm{C},\kappa)}} = \sup_{n\geq 1} n^{-\kappa}\,|A_n^\kappa|\;\Gamma(\kappa+1)\,, \tag{3.9}$$

where the Cesàro means of order $\kappa$ are given by (3.7). Let $E_\kappa = E_{(\mathrm{C},\kappa)} \cap F_\beta$ with the intersection topology. Since $E_\kappa$ is a is Banach space, and the inclusion $E_\kappa \to F$ is continuous, if we show that $E_\kappa \neq F \forall\kappa$, it will follow that $E_\kappa$ is of the first category in $F$ and, consequently, that $\bigcup_{\kappa>-1} E_\kappa \neq F$, which is precisely what we are required to show.

But to show that $E_\kappa \neq F$ for any $\kappa$, since the $E_\kappa$ are increasing with $\kappa$, it is enough to show that $E_{\kappa_1} \neq E_{\kappa_2}$ if $\kappa_1 \neq \kappa_2$. But [4, Section 6.12] the series

$$\sum_{n=1}^{\infty} n^{b+ic} e^{Ain^a} , \tag{3.10}$$

where $0 < a < 1$, $b > -1$, is $(\mathrm{C}, \kappa)$ summable if and only if

$$(\kappa + 1)\, a - b > 1 . \tag{3.11}$$

If $\kappa_1 < \kappa_2$, by choosing the parameters $a$ and $b$, $0 < a < 1$, $b > -1$ in a way that

$$b \leq \beta , \quad b < \kappa_2 \ , \quad \frac{b+1}{\kappa_1 + 1} > \alpha > \frac{b+1}{\kappa_2 + 1} , \tag{3.12}$$

we obtain that the series (3.10) belongs to $E_{\kappa_1} \setminus E_{\kappa_2}$.

Notice that the same procedure produces series that are (A) but not (C) summable and satisfy other extra conditions. For instance, we may ask $\{a_n\} \in l^p$ for $1 < p < \infty$.

## 4. Distributions without Point Values

Let $f \in \mathcal{D}'(\mathbb{R})$ be a distribution. Then [3, 7] $f$ has a distributional point value in the sense of Łojasiewicz at $x = x_0$, written as

$$f(x_0) = L \ \text{ distributionally,} \tag{4.1}$$

if

$$\lim_{\varepsilon \to 0} f(x_0 + \varepsilon x) = L , \tag{4.2}$$

in the topology of $\mathcal{D}'(\mathbb{R})$ , that is, if

$$\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \phi(x) \rangle = L \int_{-\infty}^{\infty} \phi(x)\, dx \ , \quad \forall \phi \in \mathcal{D}'(\mathbb{R}) \ . \tag{4.3}$$

It can be shown that $f(x_0) = L$ distributionally if and only if there exist $N \in \mathbb{N}$ and a primitive of order $N$ of $f$, $F^{(N)} = f$, that is continuous in a neighborhood of $x = x_0$ and satisfies

$$\lim_{x \to x_0} \frac{F(x)}{(x - x_0)^N} = \frac{L}{N!} \ . \tag{4.4}$$

Let now $f \in \mathcal{D}'(\mathbb{R})$ be a periodic distribution of period $2\pi$, with Fourier series expansion

$$f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} . \tag{4.5}$$

Then [1] $f(\theta_0) = L$ distributionally if and only if certain Cesàro averages of the Fourier series for $\theta = \theta_0$ converge to $L$, namely, if and only if

$$\lim_{x\to\infty} \sum_{-x\le n\le ax} a_n e^{in\theta} = L \quad (\mathrm{C}) \quad \forall a > 0. \tag{4.6}$$

In particular [1, 10], if the Fourier series of $f$ is of the power series type, $f(\theta) = \sum_{n=0}^{\infty} a_n e^{in\theta}$, then

$$f(\theta_0) = L \quad \text{distributionally} \quad \Leftrightarrow \quad \sum_{n=0}^{\infty} a_n e^{in\theta_0} = L \quad (\mathrm{C})\ . \tag{4.7}$$

Let now $F(z)$ be an analytic function defined in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Let

$$F(z) = \sum_{n=0}^{\infty} a_n z^n\,, \tag{4.8}$$

be its Taylor series at $z = 0$. Then $F$ has distributional boundary limits at $S = \{\xi \in \mathbb{C} : |\xi| = 1\}$, that is,

$$f(\xi) = \lim_{r\to 1^-} F(r\xi)\ , \tag{4.9}$$

exists in $\mathcal{D}'(S)$ if and only if one of the following two equivalent conditions is satisfied [2]:

**1.** $\exists M > 0,\ \alpha \in \mathbb{R}$ such that $|F(z)| \le M(1-|z|)^{-\alpha} \quad \forall z \in \Delta$;

**2.** $\exists \beta \in \mathbb{R}$ such that $a_n = O\left(n^{\beta}\right)$ as $n \to \infty$.

Combining these results with those of the previous section, we immediately obtain.

**Theorem 4.** *There exists an analytic function defined in* $|z| < 1$, *that has distributional boundary values at* $|z| = 1$, $f(\xi) = \lim_{r\to 1^-} F(r\xi)$, *where* $f \in \mathcal{D}'(S)$, *such that*

(a) $\lim_{r\to 1^-} F(r)$ *exists*

(b) $f\left(e^{i\theta}\right)$ *does not have a distributional value at* $\theta = 0$.

**Proof.** Let $\sum_{n=0}^{\infty} a_n$ be a series that is Abel but not Cesàro summable and that satisfies $a_n = O\left(n^{\beta}\right)$, $n \to \infty$ for some $\beta \in \mathbb{R}$. Then $F(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfies all the required conditions.

Observe that we may ask the boundary function $f$ to belong, for instance, to smaller spaces such as $H^2$. Notice also that we may construct a function $F$ as in Theorem 4 for which not only the radial limit but the non-tangential limit exists as $z = 1$.

Using a conformal map we see that a corresponding result holds at any point of any smooth closed contour. In particular, there exists an analytic function $G(z)$ defined in the upper half-plane $\mathrm{Im} z > 0$ that has distributional boundary values in $\mathbb{R}$, $g(x) = G(x + i0)$, $g \in \mathcal{D}'(\mathbb{R})$, such that

(a) $\lim_{y\to 0} G(x_0 + iy)$ exists (or more generally $\lim_{\substack{z\to x_0 \\ \text{N.T.}}} G(z)$ exists.)

(b) $g(x)$ does not have a distributional point value at $x = x_0$.

It is interesting to observe that another application of the Baire theorem argument allows us to construct a function $f \in H^2$ such that the distributional point values $f(x)$ do not exists for $x \in X$, where $X$ is a dense $G_\delta$ subset of $S$. On the other hand, it is well-known [8] that if $f \in H^2$ is the boundary value of the analytic function $F(z)$ then $\lim_{r\to 1} F(r\xi)$ exists almost everywhere, that is, for all $\xi \in Y$ where $Y$ is a subset of full measure of $S$. However, this does not provide the counterexample of Theorem 4, since even though $X$ and $Y$ are very big subsets of $S$, in their own way, it could be that $X \cap Y = \emptyset$.

To construct such $f \in H^2$ and such dense $G_\delta$ set $X$ we proceed as follows. If $f \in H^2$, let $f(\theta) = \sum_{n=0}^{\infty} a_n e^{in\theta}$ be its Fourier series expansion. Define

$$\alpha_{\theta,k}(f) = \sup_{n\in\mathbb{N}} |A_n^\kappa(\theta)| \ , \tag{4.10}$$

where

$$A_n^\kappa(\theta) = \sum_{q=0}^{n} \binom{q+\kappa}{q} a_{n-q} e^{i(n-q)\theta} \ , \tag{4.11}$$

are the $(C, \kappa)$ means of the Fourier series at $\theta$. Fix $\theta$. Use of the series (3.10) shows that for each $\kappa$ there exists $f \in H^2$ with $\alpha_{\phi,\kappa}(f) = \infty$. Employing the Banach-Steinhaus theorem, we obtain that there exists a dense $G_\delta$ subset $B_\theta$ of $H^2$ such that $\alpha_{\theta,\kappa}(f) = \infty \ \forall\kappa$ if $f \in B_\theta$. Let $\{\theta_n\}_{n=1}^{\infty}$ be a dense countable subset of $[0, 2\pi]$ and let $B = \bigcap_{n=1}^{\infty} B_{\theta_n}$. Then $B$ is also a dense $G_\delta$ subset of $H^2$, and $\alpha_{\theta_n,\kappa}(f) = \infty \ \forall n, \kappa$ and $\forall f \in B$. But observe that for each fixed $f$ and $\kappa$ then $\alpha_{\theta,\kappa}(f)$ is an upper semicontinuous function of $\theta$. Therefore $X = \{\theta : \alpha_{\theta,\kappa}(f) = \infty \text{ for some } \kappa\}$ is a dense $G_\delta$ subset of $[0, 2\pi]$.

## References


[1] Estrada, R., *Characterization of the Fourier series of distributions having a value at a point*, Proc. Amer. Math. Soc. **124**(1996), 1205-1212.

[2] Estrada, R. and Kanwal, R. P., *Distributional boundary values of analytic and harmonic functions*, J. Math. Anal. Appl. **89**(1982), 262-289.

[3] Estrada, R. and Kanwal, R. P., A Distributional Approach to Asymptotics. Theory and Applications, Second Edition, Birkhäuser, Boston, 2002.

[4] Hardy, G. H., Divergent Series, Clarendon Press, Oxford, 1949.

[5] Kelley, J. L. and Namioka, I., Linear Topological Spaces, Springer-Verlag, New York, 1976.

[6] Littlewood, J. E., *The converse of Abel's theorem on power series*, Proc. London. Math. Soc. **9**(1911), 434-448.

[7] Łojasiewicz, S., *Sur la valeur et la limite d'une distribution en un point*, Studia Math. **16**(1957), 1-36.

[8] Rudin, W., Real and Complex Analysis, McGraw Hill, New York, 1966.

[9] Treves, F., Topological Vector Spaces, Distributions and Kernels, Academic Press, New York, 1967.

[10] Walter, G., *Pointwise convergence of distribution expansions*, Studia Math. **26**(1966), 143-154.
[11] Walter, G., *Local behavior of harmonic functions*, Illinois J. Math. **16**(1970), 491-501.

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, U.S.A.
E-mail: restrada@math.lsu.edu