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STURM-LIOUVILLE DIFFERENTIAL OPERATORS WITH DEVIATING ARGUMENT

S. A. BUTERIN, M. PIKULA AND V. A. YURKO

Abstract. Non-selfadjoint second-order differential operators with a constant delay are studied. We establish properties of the spectral characteristics and investigate the inverse problem of recovering operators from their spectra. For this inverse problem the uniqueness theorem is proved.

1. Introduction

We study an inverse spectral problem for non-selfadjoint Sturm-Liouville differential operators on a finite interval with a constant delay and with complex-valued potentials. Inverse spectral problems consist in recovering operators from their spectral characteristics. The greatest success in the inverse spectral theory has been achieved for the classical Sturm–Liouville operator (see the monographs [1–5] and the references therein) and afterwards for higher-order differential operators and other classes of differential operators and systems [4]−[7]. The classical methods of inverse spectral theory (transformation operator method [1]−[4] and method of spectral mappings [3]−[6]), which allow obtaining global solutions of inverse problems for differential operators, are not applicable for differential operators with deviating argument as well as for other classes of nonlocal operators such as integro-differential, integral and other operators. Therefore, the general inverse spectral theory for nonlocal operators has not yet been constructed and there are only isolated results in this direction not forming the general picture [8]−[21].

In the present paper we consider the boundary value problems $L_j = L_j(q,h)$, $j = 0,1$, of the form

$$-y''(x) + q(x)y(x-a) = \lambda y(x), \quad x \in (0,\pi), \tag{1}$$

$$y'(0) - hy(0) = y^{(j)}(\pi) = 0, \tag{2}$$

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Corresponding author: S. A. Buterin.

where λ is the spectral parameter, $a \in (0,\pi)$, h is a complex number, $q(x)$ is a complex-valued function, $q(x) \in L(a,\pi)$, and $q(x) = 0$ for $x \in [0,a]$.

The following inverse problem is studied: given the spectra of the problems L_j, $j = 0,1$, find the potential $q(x)$ and the coefficient h. Differential equations with delay arise in various problems of mathematics as well as in applications (see the monographs [22]–[25] and the references therein). Some results on the spectral theory of differential operators with delay can be found in [10, 18, 24, 26] and other works. The presence of delay in a mathematical model produces serious qualitative changes in the study of spectral problems. Therefore, up to now there are no comprehensive results in the inverse problem theory for operators with delay.

In the next section we study spectral properties of the boundary value problems (1)–(2), in particular, properties of the characteristic functions and the eigenvalues of L_j. In Section 3 we consider the inverse spectral problem of recovering the potential $q(x)$ and the coefficient h from the given two spectra of the boundary value problems L_0 and L_1. We provide a uniqueness result for this inverse problem. More precisely, we prove that if the eigenvalues of $L_j(q,h)$ are the same as for the zero potential, then q can be only zero.

2. Characteristic functions and spectra

Let $N \in \mathbb{N}$ be such that $aN < \pi \le a(N+1)$, i.e. $a \in [\pi/(N+1), \pi/N)$. Let $C(x,\lambda)$, $S(x,\lambda)$ and $\varphi(x,\lambda)$ be solutions of Eq. (1) under the initial conditions

$$C(0,\lambda) = S'(0,\lambda) = \varphi(0,\lambda) = 1, \quad S(0,\lambda) = C'(0,\lambda) = 0, \quad \varphi'(0,\lambda) = h.$$

For each fixed x, and $\nu = 0,1$ the functions $C^{(\nu)}(x,\lambda)$, $S^{(\nu)}(x,\lambda)$ and $\varphi^{(\nu)}(x,\lambda)$ are entire in λ of order $1/2$, and

$$\varphi(x,\lambda) = C(x,\lambda) + hS(x,\lambda). \tag{3}$$

Let $\lambda = \rho^2$ and $\rho = \sigma + i\tau$, i.e. $\sigma = Re\,\rho$, $\tau = Im\,\rho$.

Lemma 1. *The following representations hold*

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \varphi_1(x,\lambda) + \ldots + \varphi_N(x,\lambda),\ S(x,\lambda) = S_0(x,\lambda) + S_1(x,\lambda) + \ldots + S_N(x,\lambda), \tag{4}$$

where

$$\varphi_0(x,\lambda) = \cos\rho x + h\frac{\sin\rho x}{\rho},\ S_0(x,\lambda) = \frac{\sin\rho x}{\rho}, \quad x \ge 0,$$

$$\left.\begin{aligned} \varphi_k(x,\lambda) &= \int_{ka}^{x} \frac{\sin\rho(x-t)}{\rho} q(t)\varphi_{k-1}(t-a,\lambda)\,dt, \\ S_k(x,\lambda) &= \int_{ka}^{x} \frac{\sin\rho(x-t)}{\rho} q(t)S_{k-1}(t-a,\lambda)\,dt, \end{aligned}\right\} \tag{5}$$

for $x \ge ka$, *and* $\varphi_k(x,\lambda) = S_k(x,\lambda) = 0$ *for* $x \le ka$. *Moreover, for* $|\rho| \to \infty$, $\nu = 0, 1$, *uniformly in* x *the following estimates hold:*

$$\varphi_k^{(\nu)}(x,\lambda) = O(\rho^{\nu-k}\exp(|\tau|(x-ka))), \quad S_k^{(\nu)}(x,\lambda) = O(\rho^{\nu-k-1}\exp(|\tau|(x-ka))). \tag{6}$$

Proof. The functions $\varphi(x,\lambda)$ and $S(x,\lambda)$ are the solutions of the integral equations

$$\left.\begin{aligned} \varphi(x,\lambda) &= \cos\rho x + h\frac{\sin\rho x}{\rho} + \int_0^x \frac{\sin\rho(x-t)}{\rho} q(t)\varphi(t-a,\lambda)\,dt, \\ S(x,\lambda) &= \frac{\sin\rho x}{\rho} + \int_0^x \frac{\sin\rho(x-t)}{\rho} q(t) S(t-a,\lambda)\,dt. \end{aligned}\right\} \tag{7}$$

Solving integral equations (7) by the method of successive approximations we arrive at (4), where the functions $\varphi_k(x,\lambda)$ and $S_k(x,\lambda)$, $k = \overline{1,N}$, are defined by (5). Moreover, it follows from (5) that

$$\left.\begin{aligned} \varphi_k'(x,\lambda) &= \int_{ka}^x \cos\rho(x-t) q(t)\varphi_{k-1}(t-a,\lambda)\,dt, \\ S_k'(x,\lambda) &= \int_{ka}^x \cos\rho(x-t) q(t) S_{k-1}(t-a,\lambda)\,dt, \end{aligned}\right\} \tag{8}$$

for $x \ge ka$. Clearly, $\varphi_0^{(\nu)}(x,\lambda) = O(\rho^{\nu}\exp(|\tau|x))$, $S_0^{(\nu)}(x,\lambda) = O(\rho^{\nu-1}\exp(|\tau|x))$. Using (5) and (8), we arrive at (6) by induction. □

Corollary 1. *The following representation holds*

$$C(x,\lambda) = C_0(x,\lambda) + C_1(x,\lambda) + \ldots + C_N(x,\lambda), \tag{9}$$

$$\begin{aligned} C_0(x,\lambda) &= \cos\rho x, \quad x \ge 0, \\ C_k(x,\lambda) &= \int_{ka}^x \frac{\sin\rho(x-t)}{\rho} q(t) C_{k-1}(t-a,\lambda)\,dt, \end{aligned} \tag{10}$$

for $x \ge ka$, *and* $C_k(x,\lambda) = 0$ *for* $x \le ka$. *Moreover, for* $|\rho| \to \infty$, $\nu = 0, 1$, *uniformly in* x *the following estimate holds:*

$$C_k^{(\nu)}(x,\lambda) = O(\rho^{\nu-k}\exp(|\tau|(x-ka))). \tag{11}$$

Relations (9), (10) and (11) follow from (4), (5) and (6), respectively, with $h = 0$.

Remark 1. The functions $C_1(x,\lambda), S_1(x,\lambda)$ and $\varphi_1(x,\lambda)$ depend linearly on the potential q. In particular, taking (5) and (10) into account, we calculate for $x \geq a$:

$$\left.\begin{aligned} C_1(x,\lambda) &= \frac{\sin\rho(x-a)}{2\rho}\int_a^x q(t)\,dt + \frac{1}{2\rho}\int_a^x q(t)\sin\rho(x-2t+a)\,dt, \\ C_1'(x,\lambda) &= \frac{\cos\rho(x-a)}{2}\int_a^x q(t)\,dt + \frac{1}{2}\int_a^x q(t)\cos\rho(x-2t+a)\,dt, \\ S_1(x,\lambda) &= -\frac{\cos\rho(x-a)}{2\rho^2}\int_a^x q(t)\,dt + \frac{1}{2\rho^2}\int_a^x q(t)\cos\rho(2t-x-a)\,dt, \\ S_1'(x,\lambda) &= \frac{\sin\rho(x-a)}{2\rho}\int_a^x q(t)\,dt + \frac{1}{2\rho}\int_a^x q(t)\sin\rho(2t-x-a)\,dt. \end{aligned}\right\} \tag{12}$$

Denote $\Delta_j(\lambda) := \varphi^{(j)}(\pi,\lambda),\ j=0,1$. The functions $\Delta_j(\lambda)$ are entire in λ of order $1/2$. The zeros of $\Delta_j(\lambda)$ coincide with the eigenvalues of the boundary value problems $L_j(q,h)$ (counting with multiplicities). The function $\Delta_j(\lambda)$ is called the characteristic function for $L_j(q,h)$.

Lemma 2. *For $|\rho|\to\infty$ the following asymptotical formulae are valid:*

$$\Delta_0(\lambda) = \cos\rho\pi + h\frac{\sin\rho\pi}{\rho} + \frac{\sin\rho(\pi-a)}{2\rho}\int_a^\pi q(t)\,dt + o\Big(\frac{1}{\rho}\exp(|\tau|(\pi-a))\Big), \tag{13}$$

$$\Delta_1(\lambda) = -\rho\sin\rho\pi + h\cos\rho\pi + \frac{\cos\rho(\pi-a)}{2}\int_a^\pi q(t)\,dt + o\Big(\exp(|\tau|(\pi-a))\Big). \tag{14}$$

Proof. Using (9), (11) and (12), we obtain

$$\begin{aligned} C(\pi,\lambda) &= \cos\rho\pi + \frac{\sin\rho(\pi-a)}{2\rho}\int_a^\pi q(t)\,dt + \frac{1}{2\rho}\int_a^\pi q(t)\sin\rho(\pi-2t+a)\,dt \\ &\quad + O(\rho^{-2}\exp(|\tau|(\pi-2a))). \end{aligned}$$

Since

$$\int_a^\pi q(t)\sin\rho(\pi-2t+a)\,dt = \frac{1}{2}\int_{-(\pi-a)}^{(\pi-a)} q((\pi+a-\xi)/2)\sin\rho\xi\,d\xi = o(\exp(|\tau|(\pi-a))),$$

it follows that

$$C(\pi,\lambda) = \cos\rho\pi + \frac{\sin\rho(\pi-a)}{2\rho}\int_a^\pi q(t)\,dt + o(|\rho|^{-1}\exp(|\tau|(\pi-a))). \tag{15}$$

Analogously we calculate

$$\left.\begin{aligned} C'(\pi,\lambda) &= -\rho\sin\rho\pi + \frac{\cos\rho(\pi-a)}{2}\int_a^\pi q(t)\,dt + o(\exp(|\tau|(\pi-a))), \\ S(\pi,\lambda) &= \frac{\sin\rho\pi}{\rho} - \frac{\cos\rho(\pi-a)}{2\rho^2}\int_a^\pi q(t)\,dt + o(|\rho|^{-2}\exp(|\tau|(\pi-a))), \\ S'(\pi,\lambda) &= \cos\rho\pi + \frac{\sin\rho(\pi-a)}{2\rho}\int_a^\pi q(t)\,dt + o(|\rho|^{-1}\exp(|\tau|(\pi-a))). \end{aligned}\right\} \tag{16}$$

By virtue of (3), one has

$$\Delta_j(\lambda) = C^{(j)}(\pi,\lambda) + hS^{(j)}(\pi,\lambda), \quad j=0,1. \tag{17}$$

Substituting (15)−(16) into (17), we arrive at (13)−(14). □

Lemma 3. *The boundary value problem L_j has a countable set of eigenvalues $\{\lambda_{nj}\}_{n\ge 0}$, $j=0,1$ (counting with multiplicities) and for $n\to\infty$:*

$$\rho_{n0} := \sqrt{\lambda_{n0}} = \Big(n+\frac{1}{2}\Big) + \frac{h}{\pi n} + \frac{\cos(n+1/2)a}{2\pi n}\int_a^\pi q(t)\,dt + o\Big(\frac{1}{n}\Big), \tag{18}$$

$$\rho_{n1} := \sqrt{\lambda_{n1}} = n + \frac{h}{\pi n} + \frac{\cos na}{2\pi n}\int_a^\pi q(t)\,dt + o\Big(\frac{1}{n}\Big). \tag{19}$$

Proof. It follows from (14) that

$$\Delta_1(\lambda) = -\rho\sin\rho\pi + g(\lambda), \quad |g(\lambda)| \le C\exp(|\tau|\pi). \tag{20}$$

Here and below, the symbol "C" denotes various positive constants in estimates. Fix $\delta>0$. Denote $G_\delta := \{\rho : |\rho-k|\ge\delta,\ k\in\mathbb{Z}\}$. Since $|\sin\rho\pi| \ge C\exp(|\tau|\pi)$ for $\rho\in G_\delta$, it follows from (20) that

$$|\rho\sin\rho\pi| > |g(\lambda)|, \quad \rho\in G_\delta, \quad |\rho|\ge\rho^*, \tag{21}$$

for sufficiently large ρ^*. Let $\Gamma_n := \{\lambda : |\lambda| = (n+1/2)^2\}$. Using (20), (21) and Rouché's theorem [27, p.125], we conclude that the number of zeros of $\Delta_1(\lambda)$ inside Γ_n is equal to $n+1$. Thus, in the circle $|\lambda| < (n+1/2)^2$ there exist exactly $n+1$ eigenvalues of the boundary value problem L_1: $\lambda_{01},\dots,\lambda_{n1}$. Applying now Rouche's theorem to the circle $\gamma_n(\delta) := \{\rho : |\rho-n|\le\delta\}$, we conclude that for sufficiently large n, in γ_n there is exactly one zero of $\Delta(\rho^2)$, namely $\rho_{n1} = \sqrt{\lambda_{n1}}$. Since $\delta>0$ is arbitrary, it follows that

$$\rho_{n1} = n + \varepsilon_n, \quad \varepsilon_n = o(1),\ n\to\infty. \tag{22}$$

Since $\Delta_1(\rho_{n1}^2)=0$, it follows from (14) and (22) that

$$\rho_{n1}\sin\rho_{n1}\pi = h\cos\rho_{n1}\pi + \frac{\cos\rho_{n1}(\pi-a)}{2}\int_a^\pi q(t)\,dt + o(1), \quad n\to\infty,$$

and, consequently,

$$n\sin\varepsilon_n\pi = h + \frac{\cos na}{2}\int_a^\pi q(t)\,dt + o(1), \quad n\to\infty.$$

This yields

$$\varepsilon_n = \frac{h}{\pi n} + \frac{\cos na}{2\pi n}\int_a^\pi q(t)\,dt + o\Big(\frac{1}{n}\Big), \quad n\to\infty,$$

and we arrive at (19). Relation (18) can be obtained by similar arguments. □

Lemma 4. *The specification of the spectrum* $\{\lambda_{nj}\}_{n\geq 0}$ *uniquely determines the characteristic function* $\Delta_j(\lambda)$ *by the formulae*

$$\Delta_0(\lambda) = \prod_{n=0}^{\infty} \frac{\lambda_{n0} - \lambda}{(n+1/2)^2}, \quad \Delta_1(\lambda) = \pi(\lambda_{01} - \lambda) \prod_{n=1}^{\infty} \frac{\lambda_{n1} - \lambda}{n^2}. \tag{23}$$

Proof. By Hadamard‘s factorization theorem [27, p.289], $\Delta_1(\lambda)$ is uniquely determined up to a multiplicative constant by its zeros:

$$\Delta_1(\lambda) = C \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_{n1}}\right) \tag{24}$$

(the case when $\Delta_1(0) = 0$ requires minor modifications). Consider the function

$$\tilde{\Delta}_1(\lambda) = -\rho \sin \rho\pi = -\lambda\pi \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{n^2}\right).$$

Then

$$\frac{\Delta_1(\lambda)}{\tilde{\Delta}_1(\lambda)} = \frac{C(\lambda - \lambda_{01})}{\lambda_{01}\pi\lambda} \prod_{n=1}^{\infty} \frac{n^2}{\lambda_{n1}} \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_{n1} - n^2}{n^2 - \lambda}\right).$$

Taking (14) and (19) into account we calculate

$$\lim_{\lambda\to-\infty} \frac{\Delta_1(\lambda)}{\tilde{\Delta}_1(\lambda)} = 1, \quad \lim_{\lambda\to-\infty} \prod_{n=1}^{\infty} \left(1 + \frac{\lambda_{n1} - n^2}{n^2 - \lambda}\right) = 1,$$

and hence

$$C = \pi\lambda_{01} \prod_{n=1}^{\infty} \frac{\lambda_{n1}}{n^2}.$$

Substituting this into (24) we arrive at (23) for $\Delta_1(\lambda)$. For the function $\Delta_0(\lambda)$, the arguments are similar. □

Denote

$$L(\rho) := \Delta_1(\lambda) + i\rho\Delta_0(\lambda). \tag{25}$$

The function $L(\rho)$ is entire in ρ of exponential type, and $L(\rho)$ is the characteristic function for the Redge-type boundary value problem for Eq. (1) with the boundary conditions $y'(0) - hy(0) = y'(\pi) + i\rho y(\pi) = 0$. Clearly,

$$L(\rho) = L_0(\rho) + L_1(\rho) + \ldots + L_N(\rho), \quad L_k(\rho) = \varphi_k'(\pi,\rho) + i\rho\varphi_k(\pi,\rho). \tag{26}$$

In particular,

$$L_0(\rho) = (-\rho \sin \rho\pi + h \cos \rho\pi) + (i\rho \cos \rho\pi + ih \sin \rho\pi),$$

and consequently,

$$L_0(\rho) = (i\rho + h) \exp(i\rho\pi).$$

Let us calculate $L_1(\rho)$. For this purpose we use (5), (10), (12) and (26):

$$\begin{aligned}L_1(\rho) &= (C_1'(\pi,\lambda)+hS_1'(\pi,\lambda))+i\rho(C_1(\pi,\lambda)+hS_1(\pi,\lambda))\\ &= \frac{\cos\rho(\pi-a)}{2}\int_a^\pi q(t)\,dt+\frac{1}{2}\int_a^\pi q(t)\cos\rho(\pi-2t+a)\,dt\\ &\quad+\frac{h\sin\rho(\pi-a)}{2\rho}\int_a^\pi q(t)\,dt+\frac{h}{2\rho}\int_a^\pi q(t)\sin\rho(2t-\pi-a)\,dt\\ &\quad+\frac{i\sin\rho(\pi-a)}{2}\int_a^\pi q(t)\,dt+\frac{i}{2}\int_a^\pi q(t)\sin\rho(\pi-2t+a)\,dt\\ &\quad-\frac{ih\cos\rho(\pi-a)}{2\rho}\int_a^\pi q(t)\,dt+\frac{ih}{2\rho}\int_a^\pi q(t)\cos\rho(2t-\pi-a)\,dt.\end{aligned}$$

Therefore,

$$\begin{aligned}L_1(\rho) &= \Big(\frac{1}{2}+\frac{h}{2i\rho}\Big)\exp(i\rho(\pi-a))\int_a^\pi q(t)\,dt\\ &\quad+\Big(\frac{1}{2}-\frac{h}{2i\rho}\Big)\exp(i\rho(\pi+a))\int_a^\pi q(t)\exp(-2i\rho t)\,dt.\end{aligned} \tag{27}$$

Lemma 5. *For $\tau\ge 0$, $|\rho|\to\infty$, $k\ge 1$, the following estimate holds*

$$L_k(\rho)=O\Big(\frac{1}{\rho^{k-1}}\int_{ka}^\pi |q(t)\exp(-i\rho(2t-\pi-ka))|\,dt\Big). \tag{28}$$

Proof. Using (26), (5) and (8), we calculate

$$L_k(\rho)=\int_{ka}^\pi\Big(\cos\rho(\pi-t)+i\sin\rho(\pi-t)\Big)q(t)\varphi_{k-1}(t-a,\lambda)\,dt,$$

and consequently,

$$L_k(\rho)=\int_{ka}^\pi \exp(i\rho(\pi-t))q(t)\varphi_{k-1}(t-a,\lambda)\,dt,\quad k\ge 1. \tag{29}$$

On the other hand, it follows from (6) that

$$|\varphi_{k-1}(t-a,\lambda)|\le C|\rho|^{1-k}|\exp(-i\rho(t-ka))|,\ \tau\ge 0. \tag{30}$$

Substituting (30) into (29), we arrive at (28). □

3. The inverse problem

In this section we consider the following inverse problem: given two spectra $\{\lambda_{nj}\}_{n\ge 0}$, $j=0,1$, find $q(x)$ and h. In order to formulate a uniqueness result for this inverse problem, we consider together with L_j the boundary value problems $\tilde L_j := L_j(\tilde q,\tilde h)$ of the same form but with different $\tilde q$ and $\tilde h$. We agree that if a certain symbol β denotes an object related to L_j, then $\tilde\beta$ will denote the analogous object related to $\tilde L_j$.

Let $\{\tilde{\lambda}_{nj}\}_{n\ge 0}$, $j=0,1$, be the eigenvalues of the boundary value problems $\tilde{L}_j$ with $\tilde{q}(x)\equiv 0$. Let $\tilde{\Delta}_j(\lambda)$ be the characteristic functions of $\tilde{L}_j$, and

$$\tilde{L}(\rho)=\tilde{\Delta}_1(\lambda)+i\rho\tilde{\Delta}_0(\lambda). \tag{31}$$

Theorem 1. *If $\lambda_{nj}=\tilde{\lambda}_{nj}$ for all $n\ge 0$, $j=0,1$, then $q(x)=\tilde{q}(x)$ a.e. on (a,π) and $h=\tilde{h}$.*

Proof. 1) By virtue of Lemma 4, one has

$$\Delta_0(\lambda)=\tilde{\Delta}_0(\lambda),\quad \Delta_1(\lambda)=\tilde{\Delta}_1(\lambda).$$

Using (25) and (31) we get

$$L(\rho)=\tilde{L}(\rho). \tag{32}$$

Since $\tilde{q}(x)\equiv 0$, it follows from (18)–(19) and (29) that

$$\int_a^\pi q(t)\,dt=0,\quad h=\tilde{h},\quad \tilde{L}(\rho)\equiv L_0(\rho)=(i\rho+h)\exp(i\rho\pi). \tag{33}$$

In particular, (33) and (27) yield

$$L_1(\rho)=\Big(\frac{1}{2}-\frac{h}{2i\rho}\Big)\exp(i\rho(\pi+a))\int_a^\pi q(t)\exp(-2i\rho t)\,dt. \tag{34}$$

Denote $L^+(\rho):=L_2(\rho)+\ldots+L_N(\rho)$ for $N\ge 2$, and $L^+(\rho)\equiv 0$ for $N=1$. Using (26), (32) and (33), we obtain

$$L_1(\rho)\equiv -L^+(\rho). \tag{35}$$

2) First of all, we note that if $q(x)=0$ a.e. on $(2a,\pi)$, then $q(x)=0$ a.e. on (a,π). Indeed, under this assumption we have $L^+(\rho)\equiv 0$, and according to (35) we infer $L_1(\rho)\equiv 0$. Using (34) we obtain

$$\int_a^\pi q(t)\exp(-2i\rho t)\,dt\equiv 0,$$

and consequently, $q(x)=0$ a.e. on (a,π). In particular, this finishes the proof for $N=1$, since in this case we have $2a\ge\pi$ and automatically $L^+(\rho)\equiv 0$.

3) Let $N\ge 2$. Fix $\nu=\overline{0,2N-3}$. Let us prove that

$$if\ \ q(x)=0\ \ a.e.\ on\ \Big(\pi-\frac{\nu a}{2},\pi\Big),\ \ then\ \ q(x)=0\ \ a.e.\ on\ \Big(\pi-\frac{(\nu+1)a}{2},\pi\Big). \tag{36}$$

Indeed, it follows from (28) that

$$L_2(\rho)=O\Big(\frac{1}{\rho}\int_{2a}^{\pi-\nu a/2}|q(t)\exp(-i\rho(2t-\pi-2a))|\,dt\Big),\quad \tau\ge 0,\ |\rho|\to\infty.$$

Let $\pi - \nu a/2 > 2a$, otherwise we arrive at the situation in 2) and the proof is finished. Then in the integral we have $2a - \pi \le 2t - \pi - 2a \le \pi - (\nu+2)a$, which yields

$$L_2(\rho) = O\Big(\frac{1}{\rho}\exp(-i\rho(\pi - (\nu+2)a))\Big), \quad \tau \ge 0, \ |\rho| \to \infty. \tag{37}$$

For $k > 2$, the functions $L_k(\rho)$ have less growth than in (37). This means that

$$L^+(\rho) = O\Big(\frac{1}{|\rho|}\exp(-i\rho(\pi - (\nu+2)a))\Big), \quad \tau \ge 0, \ |\rho| \to \infty. \tag{38}$$

It follows from (34), (35) and (38) that

$$\exp(i\rho(\pi + a))\int_a^{\pi-\nu a/2} q(t)\exp(-2i\rho t)\,dt = O\Big(\frac{1}{\rho}\exp(-i\rho(\pi-(\nu+2)a))\Big), \ \tau \ge 0, \ |\rho| \to \infty,$$

or, which is the same,

$$\exp(i\rho(2\pi - (\nu+1)a))\int_a^{\pi-\nu a/2} q(t)\exp(-2i\rho t)\,dt = O\Big(\frac{1}{\rho}\Big), \quad \tau \ge 0, \ |\rho| \to \infty. \tag{39}$$

Moreover, one has

$$\int_a^{\pi-(\nu+1)a/2} q(t)\exp(-2i\rho t)\,dt = O\Big(\exp(-i\rho(2\pi - (\nu+1)a)\Big), \quad \tau \ge 0, \ |\rho| \to \infty. \tag{40}$$

Let us introduce the function

$$F(\rho) := \exp(i\rho(2\pi - (\nu+1)a))\int_{\pi-(\nu+1)a/2}^{\pi-\nu a/2} q(t)\exp(-2i\rho t)\,dt.$$

The function $F(\rho)$ is entire in ρ. Clearly, $F(\rho) = O(1)$ for $\tau \le 0$. On the other hand, it follows from (39) and (40) that $F(\rho) = O(1)$ for $\tau \ge 0$. By Liouville's theorem [27, p.77], $F(\rho) \equiv C - const$. Since $F(\rho) = o(1)$ for real ρ, $|\rho| \to \infty$, it follows that $F(\rho) \equiv 0$, i.e.

$$\int_{\pi-(\nu+1)a/2}^{\pi-\nu a/2} q(t)\exp(-2i\rho t)\,dt \equiv 0.$$

This yields $q(x) = 0$ a.e. on the interval $(\pi - (\nu+1)a/2, \pi - \nu a/2)$, i.e. (36) is proved.

Applying proposition (36) successively for $\nu = 0, 1, \ldots, 2N-3$, we obtain $q(x) = 0$ a.e. on the interval $(\pi - (N-1)a, \pi) \supset (2a, \pi)$. According to 2) we get $q(x) = 0$ a.e. on (a, π). Theorem 1 is proved.

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Department of Mathematics, Saratov University, Astrakhanskaya 83, Saratov 410012, Russia.

E-mail: ButerinSA@info.sgu.ru

Department of Mathematics, Saratov University, Astrakhanskaya 83, Saratov 410012, Russia.

E-mail: YurkoVA@info.sgu.ru

Department of Mathematics, Informatics and Physics, University East Sarajevo, Alekse Šantica 1, East Sarajevo, Bosnia and Herzegovina

E-mail: pikulam1947@gmail.com