TAMKANG JOURNAL OF MATHEMATICS
Volume 35, Number 1, Spring 2004

A NOTE ON OSTROWSKI AND GRÜSS TYPE DISCRETE INEQUALITIES

B. G. PACHPATTE

Abstract. The aim of the present note is to establish some new discrete inequalities of the Ostrowski and Grüss type involving two functions and their forward differences.

1. Introduction

In 1938, Ostrowski (see, [2, p.468]) proved the following interesting inequality.

Let $f : [a, b] \to R$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to R$ is bounded on (a, b) i.e. $|f'(x)| \leq M < \infty$. Then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)M, \tag{O}$$

for all $x \in [a, b]$, where M is a constant.

In 1935, Grüss (see, [1, p.296]) proved the following inequality.

Let $f,\ g : [a, b] \to R$ be two integrable functions such that $p \leq f(x) \leq P$ and $q \leq g(x) \leq Q$ for all $x \in [a, b]$, p, P, q, Q are constants. Then

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right) \right| \leq \frac{1}{4}(P-p)(Q-q). \tag{G}$$

Over the years the inequalities (O) and (G) evoked considerable interest and many generalizations, extensions and variants have appeared in the literature, see [1-5] and some of the references given therein. The main objective of the present note is to establish new discrete inequalities of the Ostrowski and Grüss type involving functions and their forward differences. The analysis used in the proofs is fairly elementary and our results provide new estimates on these types of inequalities.

Received August 8, 2002.
2000 *Mathematics Subject Classification*. 26D10, 26D15.
Key words and phrases. Ostrowski and Grüss type, discrete inequalities, forward differences, empty sum, summation by parts.

2. Statement of Results

In what follows, R and N denote the set of real numbers and natural numbers and $N_{a,b} = \{a, a+1, \dots, a+n = b\}$ for $a \in R$, $n \in N$. For any function $u(n)$, $n \in N_{a,b}$, we define the operator Δ by $\Delta u(n) = u(n+1) - u(n)$. We use the usual convention that empty sum is taken to be zero.

Our main resutls are given in the following theorems.

Theorem 1. *Let $f(n)$, $g(n)$ be real-valued functions defined on $N_{a,b}$ for which $\Delta f(n)$, $\Delta g(n)$ exist and $|\Delta f(n)| \le A$, $|\Delta g(n)| \le B$ on $N_{a,b}$. Then*

$$\left| f(n)g(n) - \frac{1}{2(b-a)}\left[g(n)\sum_{s=a}^{b-1} f(s+1) + f(n)\sum_{s=a}^{b-1} g(s+1)\right]\right| \le \frac{1}{2(b-a)}[A|g(n)| + B|f(n)|]H(n), \tag{2.1}$$

$$\left| f(n) - \frac{1}{b-a}\sum_{s=a}^{b-1} f(s+1)\right| \le \frac{A}{b-a}H(n), \tag{2.2}$$

for $n \in N_{a,b}$, where

$$H(n) = \sum_{s=a}^{b-1} |r(n,s)|, \tag{2.3}$$

in which

$$r(n,s) = \begin{cases} s-a & \text{if } s \in [a, n-1] \\ s-b & \text{if } s \in [n, b] \end{cases}. \tag{2.4}$$

for all n, $s \in N_{a,b}$ and A, B are nonneqative constants.

Theorem 2. *Let $f(n)$, $g(n)$, $\Delta f(n)$, $\Delta g(n)$, A, B be as in Theorem 1. Then*

$$\left| \frac{1}{b-a}\sum_{n=a}^{b-1} f(n)g(n) - \frac{1}{2(b-a)^2}\left[\left(\sum_{n=a}^{b-1} g(n)\right)\left(\sum_{n=a}^{b-1} f(n+1)\right) + \left(\sum_{n=a}^{b-1} f(n)\right)\left(\sum_{n=a}^{b-1} g(n+1)\right)\right]\right| \le \frac{1}{2(b-a)^2}\sum_{n=a}^{b-1}[A|g(n)| + B|f(n)|]H(n), \tag{2.5}$$

$$\left| \frac{1}{b-a}\sum_{n=a}^{b-1} f(n)g(n) - \frac{1}{(b-a)^2}\left[\left(\sum_{n=a}^{b-1} g(n)\right)\left(\sum_{n=a}^{b-1} f(n+1)\right) + \left(\sum_{n=a}^{b-1} f(n)\right)\left(\sum_{n=a}^{b-1} g(n+1)\right)\right] + \frac{1}{(b-a)^2}\left(\sum_{n=a}^{b-1} f(n+1)\right)\left(\sum_{n=a}^{b-1} g(n+1)\right)\right| \le \frac{AB}{(b-a)^3}\sum_{n=a}^{b-1}(H(n))^2, \tag{2.6}$$

where $H(n)$ is defined by (2.3).

3. Proof of Theorem 1

Using the summation by parts formula i.e.

$$\sum_{s=\alpha}^{\beta-1} u(s)\Delta v(s) = [u(\beta)v(\beta) - u(\alpha)v(\alpha)] - \sum_{s=\alpha}^{\beta-1} v(s+1)\Delta u(s),$$

where α, $\beta \in N_{a,b}$ and $u(t)$, $v(t)$ are real-valued functions defined on $N_{a,b}$, we have for $n \in N_{a,b}$,

$$\sum_{s=a}^{n-1}(s-a)\Delta f(s) = (n-a)f(n) - \sum_{s=a}^{n-1} f(s+1) \tag{3.1}$$

and

$$\sum_{s=n}^{b-1}(s-b)\Delta f(s) = -(n-b)f(n) - \sum_{s=n}^{b-1} f(s+1). \tag{3.2}$$

Adding (3.1), (3.2) and using (2.4) we get

$$\sum_{s=a}^{b-1} r(n,s)\Delta f(s) = (b-a)f(n) - \sum_{s=a}^{b-1} f(s+1)$$

i.e.

$$f(n) - \frac{1}{b-a}\sum_{s=a}^{b-1} f(s+1) = \frac{1}{b-a}\sum_{s=a}^{b-1} r(n,s)\Delta f(s). \tag{3.3}$$

Similarly, for $n \in N_{a,b}$ we have

$$g(n) - \frac{1}{b-a}\sum_{s=a}^{b-1} g(s+1) = \frac{1}{b-a}\sum_{s=a}^{b-1} r(n,s)\Delta g(s). \tag{3.4}$$

Multiplying (3.3) by $g(n)$ and (3.4) by $f(n)$, $n \in N_{a,b}$, adding the resulting identities and rewriting we get

$$\begin{aligned} f(n)g(n) &= \frac{1}{2(b-a)} g(n)\sum_{s=a}^{b-1} f(s+1) + \frac{1}{2(b-a)} f(n)\sum_{s=a}^{b-1} g(s+1) \\ &\quad + \frac{1}{2(b-a)} g(n)\sum_{s=a}^{b-1} r(n,s)\Delta f(s) + \frac{1}{2(b-a)} f(n)\sum_{s=a}^{b-1} r(n,s)\Delta g(s). \end{aligned} \tag{3.5}$$

From (3.5) and using the properties of modulus we have

$$\left| f(n)g(n) - \frac{1}{2(b-a)}\left[g(n)\sum_{s=a}^{b-1} f(s+1) + f(n)\sum_{s=a}^{b-1} g(s+1)\right]\right|$$

$$\leq \frac{1}{2(b-a)}\left[|g(n)|\sum_{s=a}^{b-1}|r(n,s)|\,|\Delta f(s)|+|f(n)|\sum_{s=a}^{b-1}|r(n,s)|\,|\Delta g(s)|\right]$$
$$\leq \frac{1}{2(b-a)}[A|g(n)|+B|f(n)|]\sum_{s=a}^{b-1}|r(n,s)|$$
$$= \frac{1}{2(b-a)}[A|g(n)|+B|f(n)|]H(n),$$

which is the requried inequality in (2.1).

The inequality (2.2) follows immediately from (3.3) and the properties of modulus. The proof is complete.

4. Proof of Theorem 2

From the hypotheses, as in the proof of Theorem 1, the identities (3.3), (3.4) and (3.5) hold. Summing both sides of (3.5) over n from a to $b-1$, and rewriting we get

$$\frac{1}{b-a}\sum_{n=a}^{b-1}f(n)g(n)$$
$$=\frac{1}{2(b-a)^2}\left[\left(\sum_{n=a}^{b-1}g(n)\right)\left(\sum_{s=a}^{b-1}f(s+1)\right)+\left(\sum_{n=a}^{b-1}f(n)\right)\left(\sum_{s=a}^{b-1}g(s+1)\right)\right]$$
$$+\frac{1}{2(b-a)^2}\left[\sum_{n=a}^{b-1}g(n)\sum_{s=a}^{b-1}r(n,s)\Delta f(s)+\sum_{n=a}^{b-1}f(n)\sum_{s=a}^{b-1}r(n,s)\Delta g(s)\right]. \tag{4.1}$$

From (4.1) and using the properties of modulus, we observe that

$$\left|\frac{1}{b-a}\sum_{n=a}^{b-1}f(n)g(n)-\frac{1}{2(b-a)^2}\left[\left(\sum_{n=a}^{b-1}g(n)\right)\left(\sum_{n=a}^{b-1}f(n+1)\right)+\left(\sum_{n=a}^{b-1}f(n)\right)\left(\sum_{n=a}^{b-1}g(n+1)\right)\right]\right|$$
$$\leq \frac{1}{2(b-a)^2}\left[\sum_{n=a}^{b-1}|g(n)|\sum_{s=a}^{b-1}|r(n,s)|\,|\Delta f(s)|+\sum_{n=a}^{b-1}|f(n)|\sum_{s=a}^{b-1}|r(n,s)|\,|\Delta g(s)|\right]$$
$$\leq \frac{1}{2(b-a)^2}\sum_{n=a}^{b-1}[A|g(n)|+B|f(n)|]H(n),$$

which proves the inequality (2.5).

Multiplying the left sides and right sides of (3.3) and (3.4) we get

$$f(n)g(n)-\frac{1}{b-a}f(n)\sum_{s=a}^{b-1}g(s+1)-\frac{1}{b-a}g(n)\sum_{s=a}^{b-1}f(s+1)$$
$$+\frac{1}{(b-a)^2}\left(\sum_{s=a}^{b-1}f(s+1)\right)\left(\sum_{s=a}^{b-1}g(s+1)\right)$$

$$= \frac{1}{(b-a)^2}\left(\sum_{s=a}^{b-1} r(n,s)\Delta f(s)\right)\left(\sum_{s=a}^{b-1} r(n,s)\Delta g(s)\right). \tag{4.2}$$

Summing both sides of (4.2) with respect to n from a to $b-1$ and rewritting we get

$$\frac{1}{b-a}\sum_{n=a}^{b-1} f(n)g(n) - \frac{1}{(b-a)^2}\left[\left(\sum_{n=a}^{b-1} f(n)\right)\left(\sum_{s=a}^{b-1} g(s+1)\right) + \left(\sum_{n=a}^{b-1} g(n)\right)\left(\sum_{s=a}^{b-1} f(s+1)\right)\right]$$
$$+\frac{1}{(b-a)^2}\left(\sum_{s=a}^{b-1} f(s+1)\right)\left(\sum_{s=a}^{b-1} g(s+1)\right)$$
$$= \frac{1}{(b-a)^3}\sum_{n=a}^{b-1}\left[\left(\sum_{s=a}^{b-1} r(n,s)\Delta f(s)\right)\left(\sum_{s=a}^{b-1} r(n,s)\Delta g(s)\right)\right]. \tag{4.3}$$

From (4.3) and following the proof of inequality (2.5) with suitable changes we get the required inequality in (2.6). The proof is complete.

References

[1] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht, 1993.

[2] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, Inequalities for Functions and Their Integrals and Derivatives, Kluwer Academic Publishers, Dordrecht, 1994.

[3] B. G. Pachpatte, *On a new Ostrowski type inequality in two independent variables*, Tamkang J. Math. **32**(2001), 45-49.

[4] B. G. Pachpatte, *A note on Ostrowski type inequalities,* Demonstratio Mathematica **35** (2002), 27-30.

[5] B. G. Pachpatte, *On Grüss type inequalities for double integrals*, J. Math. Anal. Appl. **267**(2002), 454-459.

57 Shri Niketan Colony, Near Abhninay Talkies, Aurangabad 431 001, (Maharashtra), India.