



LEVITIN-POLYAK WELL-POSEDNESS OF COMPLETELY GENERALIZED MIXED VARIATIONAL INEQUALITIES IN REFLEXIVE BANACH SPACES

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Abstract. Let X be a real reflexive Banach space. In this paper, we first introduce the concept of Levitin-Polyak well-posedness of a completely generalized mixed variational inequality in X , and establish some characterizations of its Levitin-Polyak well-posedness. Under suitable conditions, we prove that the Levitin-Polyak well-posedness of a completely generalized mixed variational inequality is equivalent both to the Levitin-Polyak well-posedness of a corresponding inclusion problem and to the Levitin-Polyak well-posedness of a corresponding fixed point problem. We also derive some conditions under which a completely generalized mixed variational inequality in X is Levitin-Polyak well-posed. Our results improve, extend and develop the early and recent ones in the literature.

1. Introduction

In 1966, Tykhonov [1] first introduced the concept of Tykhonov well-posedness of a minimization problem. More precisely, this concept consists of the existence and uniqueness of minimizers, and the convergence of every minimizing sequence to the unique minimizer. In many practical situations, there are more than one minimizer for a minimization problem. In a natural way, the concept of Tykhonov well-posedness in the generalized sense was introduced, which means the existence of minimizers and the convergence of some subsequence of every minimizing sequence to a minimizer. Without question, the concept of well-posedness is motivated by the numerical methods producing optimizing sequences. Because it plays an important role in the study of optimization problems, various concepts of

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well-posedness have been introduced and studied widely for minimization problems in past decades. A great deal of effort has gone into those concepts; see, e.g., [2]-[4] and the references therein.

The Tykhonov well-posedness of a constrained minimization problem requires that every minimizing sequence should lie in the constraint set. It is well known that in many practical situations, the minimizing sequence produced by a numerical optimization method usually fails to be feasible but gets closer and closer to the constraint set. Such a sequence is called a generalized minimizing sequence for constrained minimization problems. Taking into account this case, Levitin and Polyak [5] strengthened the concept of Tykhonov well-posedness by requiring the existence and uniqueness of minimizers, and the convergence of every generalized minimizing sequence to the unique minimizer, which is called Levitin and Polyak (for short, LP) well-posedness. There have been a large number of results involving Tykhonov well-posedness, LP well-posedness and their generalizations for minimization problems. For details, we refer the readers to [1, 2, 3, 5, 6, 7].

In 2008, Fang, Huang and Yao [11] considered and studied the well-posedness of a mixed variational inequality in a real Hilbert space H , which includes as a special case the classical variational inequality, and derived some results for the well-posedness of such a mixed variational inequality, the corresponding inclusion problem and the corresponding fixed-point problem. Subsequently, Ceng and Yao [9] extended the concept of well-posedness to a generalized mixed variational inequality in H , which includes as a special case the mixed variational inequality, and gave some characterizations of its well-posedness. Under suitable conditions, the authors [9] proved that the well-posedness of the generalized mixed variational inequality is equivalent both to the well-posedness of the corresponding inclusion problem and to the corresponding fixed-point problem, and derived some conditions under which the generalized mixed variational inequality is well-posed. Recently, some authors made the further extension and development on the concept of well-posedness; see, e.g., [20]–[24] and the references therein.

On the other hand, Hu and Huang [14] considered the Levitin-Polyak well-posedness of a general variational inequality in \mathbf{R}^n . They derived some characterizations of the Levitin-Polyak well-posedness by considering the size of Levitin-Polyak approximating solution sets of general variational inequalities. They also proved that the Levitin-Polyak well-posedness of a general variational inequality is closely related to the Levitin-Polyak well-posedness of a minimization problem and a fixed point problem. Finally, they proved that under suitable conditions, the Levitin-Polyak well-posedness of a general variational inequality is equivalent to the uniqueness and existence of its solutions.

Let X be a real reflexive Banach space. In 2012, Li and Xia [19] extended the notion of Levitin-Polyak well-posedness to a generalized mixed variational inequality in X , and gave

some characterizations of its Levitin-Polyak well-posedness. Under suitable conditions, they proved that the Levitin-Polyak well-posedness of a generalized mixed variational inequality is closely related to the Levitin-Polyak well-posedness of a corresponding inclusion problem and a corresponding fixed point problem, and derived some conditions under which a generalized mixed variational inequality is Levitin-Polyak well-posed. However, there is no result for the Levitin-Polyak well-posedness of a completely generalized mixed variational inequality. Therefore, it is worth studying implementable results for the Levitin-Polyak well-posedness of a completely generalized mixed variational inequality.

Motivated and inspired by the research work going on this field, we extend the notion of Levitin-Polyak well-posedness to a completely generalized mixed variational inequality in a real reflexive Banach space X , and give some characterizations of its Levitin-Polyak well-posedness. Under suitable conditions, we prove the Levitin-Polyak well-posedness of a completely generalized mixed variational inequality is closely related to the Levitin-Polyak well-posedness of a corresponding inclusion problem and a corresponding fixed point problem. Finally, we derive some conditions under which a completely generalized mixed variational inequality is Levitin-Polyak well-posed. Our results improve, extend and develop the early and recent ones announced by some others, e.g., Ceng and Yao [9] and Li and Xia [19].

2. Preliminaries

Let X be a real reflexive Banach space with its dual X^* and K be a nonempty, closed and convex subset of X . Let $g : X \rightarrow X$ be a single-valued mapping, $N : X^* \times X^* \rightarrow X^*$ be a single-valued mapping, and $F, E : X \rightarrow 2^{X^*}$ be two set-valued mappings. Let $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ be such that for each $x \in X$, $\phi(\cdot, x)$ is a proper, convex and lower semicontinuous functional. For each $x \in X$, we denote by $\text{dom}\phi(\cdot, x)$ the domain of $\phi(\cdot, x)$, i.e.,

$$\text{dom}\phi(\cdot, x) = \{y \in X : \phi(y, x) < +\infty\}.$$

In this paper we always assume that $g(X) \cap \text{dom}\phi(\cdot, x) \cap K \neq \emptyset$ for each $x \in X$. Consider the following completely generalized mixed variational inequality associated with $(N(F, E), g, \phi, K)$:

$$\begin{aligned} \text{CGMVI}(N(F, E), g, \phi, K) : \text{ find } x \in K \text{ such that } g(x) \in K \text{ and} \\ \langle N(u, v), g(x) - y \rangle + \phi(g(x), x) - \phi(y, x) \leq 0, \quad \forall y \in K, \\ \text{for some } u \in F(x) \text{ and } v \in E(x). \end{aligned}$$

It is easy to see that the $\text{CGMVI}(N(F, E), g, \phi, K)$ is equivalent to the following inclusion problem associated with $N(F, E) + \partial(\phi + \delta_K) \circ g$:

$$\begin{aligned} \text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K) : \text{ find } x \in K \text{ such that } g(x) \in K \text{ and} \\ 0 \in N(F(x), E(x)) + \partial(\phi(\cdot, x) + \delta_K)(g(x)), \end{aligned}$$

where δ_K denotes the indicator function associated with K (i.e., $\delta_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise) and for a fixed $x \in X$, $\partial(\phi(\cdot, x) + \delta_K)(g(x))$ denotes the subdifferential of the convex function $\phi(\cdot, x) + \delta_K$ at $g(x)$. It is clear that $\text{CGMVI}(N(F, E), g, \phi, K)$ includes as a special case the following generalized mixed variational inequality associated with (F, ϕ, K) (considered in [19]):

$$\begin{aligned} \text{GMVI}(F, \phi, K) : \text{ find } x \in K \text{ such that for some } u \in F(x), \\ \langle u, x - y \rangle + \phi(x) - \phi(y) \leq 0, \quad \forall y \in K. \end{aligned}$$

Moreover, $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$ obviously includes as a special case the following inclusion problem associated with $F + \partial(\phi + \delta_K)$ (considered in [19]):

$$\text{IP}(F + \partial(\phi + \delta_K), K) : \text{ find } x \in K \text{ such that } 0 \in F(x) + \partial(\phi + \delta_K)(x).$$

It is not hard to find that there holds the following:

Proposition 2.1. *Let K be a nonempty, closed and convex subset of X . Let $g : X \rightarrow X$ be a single-valued mapping, $N : X^* \times X^* \rightarrow X^*$ be a single-valued mapping, and $F, E : X \rightarrow 2^{X^*}$ be two nonempty set-valued mappings. Let $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ be such that for each $x \in X$, $\phi(\cdot, x)$ is a proper, convex and lower semicontinuous functional. Then the following conclusions are equivalent:*

- (i) x solves $\text{CGMVI}(N(F, E), g, \phi, K)$;
- (ii) x solves $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$.

Definition 2.1. Let A, B be nonempty subsets of X . The Hausdorff metric $\mathcal{H}(\cdot, \cdot)$ between A and B is defined by

$$\mathcal{H}(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a \in A} d(a, B)$ with $d(a, B) = \inf_{b \in B} \|a - b\|$.

Lemma 2.1 (Nadler's Theorem [7]). *Let $(X, \|\cdot\|)$ be a normed vector space and $\mathcal{H}(\cdot, \cdot)$ be the Hausdorff metric on the collection $CB(X)$ of all nonempty, closed and bounded subsets of X , induced by a metric d in terms of $d(u, v) = \|u - v\|$, which is defined by $\mathcal{H}(U, V) = \max\{e(U, V), e(V, U)\}$ for U and V in $CB(X)$ where $e(U, V) = \sup_{x \in U} d(x, V)$ with $d(x, V) = \inf_{y \in V} \|x - y\|$. If U and V lie in $CB(X)$, then for any $\epsilon > 0$ and any $u \in U$, there exists $v \in V$ such that $\|u - v\| \leq (1 + \epsilon)\mathcal{H}(U, V)$. In particular, whenever U and V are compact subsets in X , one has $\|u - v\| \leq \mathcal{H}(U, V)$.*

Definition 2.2. A single-valued mapping $N : X^* \times X^* \rightarrow X^*$ is called mixed Lipschitz continuous with respect to the first and second arguments if there exists a pair of constants $\zeta, \xi > 0$ such that

$$\|N(u_1, v_1) - N(u_2, v_2)\| \leq \zeta \|u_1 - u_2\| + \xi \|v_1 - v_2\|, \quad \forall (u_i, v_i) \in X^* \times X^*, i = 1, 2.$$

Definition 2.3. Let $F : X \rightarrow CB(X^*)$ be a set-valued mapping. Then,

- (i) F is called \mathcal{H} -continuous at a point $x_0 \in X$, if for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$ with $\|x - x_0\| < \delta$, one has $\mathcal{H}(F(x), F(x_0)) < \epsilon$. If F is \mathcal{H} -continuous at each point $x \in X$, then F is called \mathcal{H} -continuous.
- (ii) F is called \mathcal{H} -uniformly continuous [9], if for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $\|x - y\| < \delta$, one has $\mathcal{H}(F(x), F(y)) < \epsilon$.

Definition 2.4. Let X and Y be two topological spaces and $x \in X$. A set-valued mapping $F : X \rightarrow 2^Y$ is said to be upper semicontinuous (in short, u.s.c) at x , if for any neighborhood V of $F(x)$, there exists a neighborhood U of x such that $F(y) \subset V, \forall y \in U$. If F is u.s.c at each point of X , we say that F is u.s.c on X .

Definition 2.5 ([15]). Let A be a nonempty subset of X . The measure of noncompactness μ of the set A is defined by

$$\mu(A) = \inf\{\epsilon > 0 : A \subset \bigcup_{i=1}^n A_i, \text{diam} A_i < \epsilon, i = 1, 2, \dots, n\},$$

where diam means the diameter of a set.

3. Levitin-Polyak Well-Posedness of $\text{CGMVI}(N(F, E), g, \phi, K)$

In this section, we extend the concept of Levitin-Polyak well-posedness to the completely generalized mixed variational inequality and establish its metric characterizations. In the sequel, we always denote by \rightarrow and \rightharpoonup the strong convergence and weak convergence, respectively. Let $\alpha \geq 0$ be a given number, and let X, K, N, F, E, g, ϕ be defined as in the previous section.

Definition 3.1. A sequence $\{x_n\} \subset K$ is called a LP α -approximating sequence for $\text{CGMVI}(N(F, E), g, \phi, K)$, if there exists $w_n \in X$ with $w_n \rightarrow 0$ and $0 < \epsilon_n \rightarrow 0$ such that $g(x_n) + w_n \in K$ for all $n \geq 1$, and there exist $u_n \in F(x_n)$ and $v_n \in E(x_n)$ such that

$$g(x_n) \in \text{dom}\phi(\cdot, x_n), \quad \langle N(u_n, v_n), g(x_n) - y \rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \leq \frac{\alpha}{2} \|g(x_n) - y\|^2 + \epsilon_n$$

for all $y \in K$ and $n \geq 1$. If $\alpha_1 > \alpha_2 \geq 0$, then every LP α_2 -approximating sequence is LP α_1 -approximating. When $\alpha = 0$, we say that $\{x_n\}$ is a LP approximating sequence for $\text{CGMVI}(N(F, E), g, \phi, K)$.

Definition 3.2. We say that $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly (resp. weakly) LP α -well-posed if $\text{CGMVI}(N(F, E), g, \phi, K)$ has a unique solution and every LP α -approximating sequence converges strongly (resp. weakly) to the unique solution. In the sequel, strong (resp. weak) LP 0-well-posedness is always known as strong (resp. weak) LP well-posedness. If $\alpha_1 > \alpha_2 \geq 0$, then strong (resp. weak) LP α_1 -well-posedness implies strong (resp. weak) LP α_2 -well-posedness.

Definition 3.3. We say that $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly (resp. weakly) LP α -well-posed in the generalized sense if $\text{CGMVI}(N(F, E), g, \phi, K)$ has nonempty solution set S and every LP α -approximating sequence has a subsequence which converges strongly (resp. weakly) to some point of S . In the sequel, strong (resp. weak) LP 0-well-posedness in the generalized sense is always known as strong (resp. weak) LP well-posedness in the generalized sense. If $\alpha_1 > \alpha_2 \geq 0$, then strong (resp. weak) LP α_1 -well-posedness in the generalized sense implies strong (resp. weak) LP α_2 -well-posedness in the generalized sense.

Remark 3.1. When $\text{CGMVI}(N(F, E), g, \phi, K) = \text{GMVI}(F, \phi, K)$, Definitions 3.1, 3.2 and 3.3 reduce to the Definitions 3.1, 3.2 and 3.3 in [19], respectively. When X is a real Hilbert space, $K = X$ and $w_n \equiv 0$, Definitions 3.2 and 3.3 in [19] reduce to the Definitions 3.2 and 3.3 in [11], respectively. When $X = \mathbf{R}^n$, $\alpha = 0$, $\phi = \delta_K$ and F is single-valued, Definitions 3.2 and 3.3 in [19] reduce to the Definitions 3.3 and 3.4 in [14], respectively.

To derive the metric characterizations of LP α -well-posedness, we consider the following LP α -approximating solution set of $\text{CGMVI}(N(F, E), g, \phi, K)$:

$$\begin{aligned} \Omega_\alpha(\epsilon) = \{x \in X : g(x) \in \text{dom}\phi(\cdot, x), d(g(x), K) \leq \epsilon, \text{ and} \\ \text{there exists } u \in F(x) \text{ and } v \in E(x) \text{ such that } \forall y \in K, \\ \langle N(u, v), g(x) - y \rangle + \phi(g(x), x) - \phi(y, x) \leq \frac{\alpha}{2} \|g(x) - y\|^2 + \epsilon\}, \quad \forall \epsilon \geq 0. \end{aligned}$$

Theorem 3.1. Let K be a nonempty, closed and convex subset of X . Let $F, E : X \rightarrow 2^{X^*}$ be nonempty compact-valued mappings which both are \mathcal{H} -continuous. Let $g : X \rightarrow X$ be a continuous mapping and $N : X^* \times X^* \rightarrow X^*$ be mixed Lipschitz continuous with respect to the first and second arguments. Assume that $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ satisfies the conditions:

- (C1) ϕ is proper, convex and lower semicontinuous in the first argument;
- (C2) ϕ is upper semicontinuous in the second argument;
- (C3) $\liminf_{n \rightarrow \infty} \inf_{y \in \text{dom}\phi(\cdot, x_n)} (\phi(y, x_n) - \phi(y, \bar{x})) \geq 0, \quad \forall \{x_n\} \subset K : x_n \rightarrow \bar{x} (n \rightarrow \infty)$.

Then $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP α -well-posed if and only if

$$\Omega_\alpha(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0 \text{ and } \text{diam}(\Omega_\alpha(\epsilon)) \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (3.1)$$

Proof. Suppose that $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP α -well-posed and $x^* \in K$ is the unique solution of $\text{CGMVI}(N(F, E), g, \phi, K)$. It is obvious that $x^* \in \Omega_\alpha(\epsilon)$. If $\text{diam}(\Omega_\alpha(\epsilon)) \not\rightarrow 0$ as $\epsilon \rightarrow 0$, then there exist constant $l > 0$ and sequences $\{\epsilon_n\} \subset \mathbf{R}^+$ with $\epsilon_n \rightarrow 0$, and $\{x_n^{(1)}\}, \{x_n^{(2)}\}$ with $x_n^{(1)}, x_n^{(2)} \in \Omega_\alpha(\epsilon_n)$ such that

$$\|x_n^{(1)} - x_n^{(2)}\| > l, \quad \forall n \geq 1. \quad (3.2)$$

Since $x_n^{(1)}, x_n^{(2)} \in \Omega_\alpha(\epsilon_n)$, for $x_n^{(1)}$ we have

$$d(g(x_n^{(1)}), K) \leq \epsilon_n < \epsilon_n + \frac{1}{n},$$

and there exists $u_n \in F(x_n^{(1)})$ and $v_n \in E(x_n^{(1)})$ such that

$$\langle N(u_n, v_n), g(x_n^{(1)}) - y \rangle + \phi(g(x_n^{(1)}), x_n^{(1)}) - \phi(y, x_n^{(1)}) \leq \frac{\alpha}{2} \|g(x_n^{(1)}) - y\|^2 + \epsilon_n, \quad \forall y \in K.$$

Since K is closed and convex, then there exists $\bar{x}_n^{(1)} \in K$ such that $\|g(x_n^{(1)}) - \bar{x}_n^{(1)}\| < \epsilon_n + \frac{1}{n}$. Putting $w_n = \bar{x}_n^{(1)} - g(x_n^{(1)})$, we have $w_n + g(x_n^{(1)}) = \bar{x}_n^{(1)} \in K$ and $\|w_n\| = \|g(x_n^{(1)}) - \bar{x}_n^{(1)}\| \rightarrow 0$. This implies that $w_n \rightarrow 0$. Thus, $\{x_n^{(1)}\}$ is a LP approximating sequence for $\text{CGMVI}(N(F, E), g, \phi, K)$. By the similar argument, we obtain that $\{x_n^{(2)}\}$ is a LP approximating sequence for $\text{CGMVI}(N(F, E), g, \phi, K)$. So they have to converge strongly to the unique solution of $\text{CGMVI}(N(F, E), g, \phi, K)$, a contradiction to (3.2).

Conversely, suppose that the conclusion (3.1) holds. Let $\{x_n\} \subset K$ be a LP α -approximating sequence for $\text{CGMVI}(N(F, E), g, \phi, K)$. Then there exists $w_n \in X$ with $w_n \rightarrow 0$ and $0 < \epsilon'_n \rightarrow 0$ such that $g(x_n) + w_n \in K$ for all $n \geq 1$, and there exist $u_n \in F(x_n)$ and $v_n \in E(x_n)$ such that

$$g(x_n) \in \text{dom}\phi(\cdot, x_n), \quad \langle N(u_n, v_n), g(x_n) - y \rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \leq \frac{\alpha}{2} \|g(x_n) - y\|^2 + \epsilon'_n \quad (3.3)$$

for all $y \in K$ and $n \geq 1$. Since $g(x_n) + w_n \in K$, then there exists $k_n \in K$ such that $g(x_n) + w_n = k_n$. It is easy to see that $d(g(x_n), K) \leq \|g(x_n) - k_n\| = \|w_n\| \rightarrow 0$. Set $\epsilon_n = \max\{\epsilon'_n, \|w_n\|\}$, it follows that $x_n \in \Omega_\alpha(\epsilon_n)$. From (3.1), we deduce that $\{x_n\}$ is a Cauchy sequence and so it converges strongly to a point $\bar{x} \in K$. Since $g(x_n) + w_n \in K$, $w_n \rightarrow 0$ and g is continuous, we know that $g(\bar{x}) \in K$. So, it follows from (3.3) and conditions (C1)–(C3) that

$$\liminf_{n \rightarrow \infty} (\phi(g(x_n), x_n) - \phi(g(x_n), \bar{x})) \geq \liminf_{n \rightarrow \infty} \inf_{y \in \text{dom}\phi(\cdot, x_n)} (\phi(y, x_n) - \phi(y, \bar{x})) \geq 0,$$

and

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \langle N(u_n, v_n), g(x_n) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \\ & \leq \liminf_{n \rightarrow \infty} \langle N(u_n, v_n), g(x_n) - y \rangle + \liminf_{n \rightarrow \infty} \phi(g(x_n), \bar{x}) + \liminf_{n \rightarrow \infty} (\phi(g(x_n), x_n) - \phi(g(x_n), \bar{x})) \\ & \quad + \liminf_{n \rightarrow \infty} (-\phi(y, x_n)) \\ & \leq \liminf_{n \rightarrow \infty} \{ \langle N(u_n, v_n), g(x_n) - y \rangle + \phi(g(x_n), \bar{x}) + \phi(g(x_n), x_n) - \phi(g(x_n), \bar{x}) - \phi(y, x_n) \} \\ & = \liminf_{n \rightarrow \infty} \{ \langle N(u_n, v_n), g(x_n) - y \rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \} \\ & \leq \liminf_{n \rightarrow \infty} \left\{ \frac{\alpha}{2} \|g(x_n) - y\|^2 + \epsilon'_n \right\} \\ & = \frac{\alpha}{2} \|g(\bar{x}) - y\|^2. \end{aligned} \quad (3.4)$$

Since F and E are nonempty compact-valued mappings, in terms of Lemma 2.1, for each $u_n \in F(x_n)$ and $v_n \in E(x_n)$ there exist $\bar{u}_n \in F(\bar{x})$ and $\bar{v}_n \in E(\bar{x})$ such that $\|u_n - \bar{u}_n\| \leq \mathcal{H}(F(x_n), F(\bar{x}))$

and $\|v_n - \bar{v}_n\| \leq \mathcal{H}(E(x_n), E(\bar{x}))$. Since F and E both are \mathcal{H} -continuous, one deduces that $\|u_n - \bar{u}_n\| \leq \mathcal{H}(F(x_n), F(\bar{x})) \rightarrow 0$ and $\|v_n - \bar{v}_n\| \leq \mathcal{H}(E(x_n), E(\bar{x})) \rightarrow 0$ as $n \rightarrow \infty$. Since $F(\bar{x})$ and $E(\bar{x})$ both are compact, without loss of generality we may assume that $\bar{u}_n \rightarrow \bar{u} \in F(\bar{x})$ and $\bar{v}_n \rightarrow \bar{v} \in E(\bar{x})$ as $n \rightarrow \infty$. Thus, we conclude that

$$\|u_n - \bar{u}\| \leq \|u_n - \bar{u}_n\| + \|\bar{u}_n - \bar{u}\| \leq \mathcal{H}(F(x_n), F(\bar{x})) + \|\bar{u}_n - \bar{u}\| \rightarrow 0,$$

and

$$\|v_n - \bar{v}\| \leq \|v_n - \bar{v}_n\| + \|\bar{v}_n - \bar{v}\| \leq \mathcal{H}(E(x_n), E(\bar{x})) + \|\bar{v}_n - \bar{v}\| \rightarrow 0.$$

This implies that $u_n \rightarrow \bar{u}$ and $v_n \rightarrow \bar{v}$ as $n \rightarrow \infty$. Note that N is mixed Lipschitz continuous with respect to the first and second arguments. Hence there exists a pair of constants $\zeta, \xi > 0$ such that

$$\|N(u_n, v_n) - N(\bar{u}, \bar{v})\| \leq \zeta \|u_n - \bar{u}\| + \xi \|v_n - \bar{v}\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which together with (3.4), immediately yields

$$\langle N(\bar{u}, \bar{v}), g(\bar{x}) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \leq \frac{\alpha}{2} \|g(\bar{x}) - y\|^2, \quad \forall y \in K.$$

For any $y \in K$, put $y_t = g(\bar{x}) + t(y - g(\bar{x}))$ for all $t \in [0, 1]$. Since K is a nonempty, closed and convex subset, this implies that $y_t \in K$. Then

$$\langle N(\bar{u}, \bar{v}), g(\bar{x}) - y_t \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y_t, \bar{x}) \leq \frac{\alpha}{2} \|g(\bar{x}) - y_t\|^2.$$

Since ϕ is convex in the first argument, we have

$$\langle N(\bar{u}, \bar{v}), g(\bar{x}) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \leq \frac{\alpha t}{2} \|g(\bar{x}) - y\|^2, \quad \forall y \in K. \quad (3.5)$$

Letting $t \rightarrow 0^+$, from (3.5) we get

$$\langle N(\bar{u}, \bar{v}), g(\bar{x}) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \leq 0, \quad \forall y \in K.$$

Therefore, \bar{x} solves $\text{CGMVI}(N(F, E), g, \phi, K)$.

To complete the proof, we need only to prove that $\text{CGMVI}(N(F, E), g, \phi, K)$ has a unique solution. Assume by contradiction that $\text{CGMVI}(N(F, E), g, \phi, K)$ has two distinct solutions x_1 and x_2 in K . Then it is easy to see that $x_1, x_2 \in \Omega_\alpha(\epsilon)$ for all $\epsilon > 0$ and

$$0 < \|x_1 - x_2\| \leq \text{diam}(\Omega_\alpha(\epsilon)) \rightarrow 0,$$

a contradiction to (3.1). The proof is complete. \square

Theorem 3.2. *Let K be a nonempty, closed and convex subset of X . Let $F, E : X \rightarrow 2^{X^*}$ be nonempty compact-valued mappings which both are upper semicontinuous. Let $g : X \rightarrow X$ be a continuous mapping and $N : X^* \times X^* \rightarrow X^*$ be mixed Lipschitz continuous with respect to the first and second arguments. Assume that $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ satisfies the conditions:*

(C1) ϕ is proper, convex and lower semicontinuous in the first argument;

(C2) ϕ is upper semicontinuous in the second argument;

(C3) $\liminf_{n \rightarrow \infty} \inf_{y \in \text{dom}\phi(\cdot, x_n)} (\phi(y, x_n) - \phi(y, \bar{x})) \geq 0, \quad \forall \{x_n\} \subset K : x_n \rightarrow \bar{x} (n \rightarrow \infty)$.

Then $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP α -well-posed in the generalized sense if and only if

$$\Omega_\alpha(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0 \text{ and } \mu(\Omega_\alpha(\epsilon)) \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (3.6)$$

Proof. Suppose that $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP α -well-posed in the generalized sense. Let S be the solution set of $\text{CGMVI}(N(F, E), g, \phi, K)$. Then S is nonempty and compact. Indeed, let $\{x_n\}$ be any sequence in S . Then $\{x_n\}$ is a LP α -approximating sequence for $\text{CGMVI}(N(F, E), g, \phi, K)$. Since $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP α -well-posed in the generalized sense, $\{x_n\}$ has a subsequence which converges strongly to some point of S . Thus S is compact. It is obvious that $\emptyset \neq S \subset \Omega_\alpha(\epsilon)$ for all $\epsilon > 0$. Now we show that

$$\mu(\Omega_\alpha(\epsilon)) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Observe that for every $\epsilon > 0$,

$$\mathcal{H}(\Omega_\alpha(\epsilon), S) = \max\{e(\Omega_\alpha(\epsilon), S), e(S, \Omega_\alpha(\epsilon))\} = e(\Omega_\alpha(\epsilon), S).$$

Taking into account the compactness of S , we get

$$\mu(\Omega_\alpha(\epsilon)) \leq 2\mathcal{H}(\Omega_\alpha(\epsilon), S) + \mu(S) = 2e(\Omega_\alpha(\epsilon), S).$$

To prove (3.6), it is sufficient to show that

$$e(\Omega_\alpha(\epsilon), S) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Indeed, if $e(\Omega_\alpha(\epsilon), S) \not\rightarrow 0$ as $\epsilon \rightarrow 0$, then there exist $l > 0$ and $\{\epsilon_n\} \subset \mathbf{R}^+$ with $\epsilon_n \rightarrow 0$, and $x_n \in \Omega_\alpha(\epsilon_n)$ such that

$$x_n \notin S + B(0, l), \quad \forall n \geq 1, \quad (3.7)$$

where $B(0, l)$ is the closed ball centered at 0 with radius l . By the definition of $\Omega_\alpha(\epsilon_n)$, we know $d(g(x_n), K) \leq \epsilon_n < \epsilon_n + \frac{1}{n}$, and there exists $u_n \in F(x_n)$ and $v_n \in E(x_n)$ such that

$$\langle N(u_n, v_n), g(x_n) - y \rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \leq \frac{\alpha}{2} \|g(x_n) - y\|^2 + \epsilon_n, \quad \forall y \in K.$$

Thus, there exists $k_n \in K$ such that $\|k_n - g(x_n)\| < \epsilon_n + \frac{1}{n}$. Let $w_n = k_n - g(x_n)$, then we have $w_n + g(x_n) \in K$ with $w_n \rightarrow 0$. So $\{x_n\}$ is a LP α -approximating sequence for $\text{CGMVI}(N(F, E), g, \phi, K)$. Since $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP α -well-posed in the generalized sense, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges strongly to some point of S . This contradicts (3.7) and so

$$e(\Omega_\alpha(\epsilon), S) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Conversely, assume that (3.6) holds. We first show that $\Omega_\alpha(\epsilon)$ is closed for all $\epsilon > 0$. Let $\{x_n\} \subset \Omega_\alpha(\epsilon)$ with $x_n \rightarrow \bar{x}$. Then there exists $u_n \in F(x_n)$ and $v_n \in E(x_n)$ such that $d(g(x_n), K) \leq \epsilon$ and

$$\langle N(u_n, v_n), g(x_n) - y \rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \leq \frac{\alpha}{2} \|g(x_n) - y\|^2 + \epsilon, \quad \forall y \in K, n \geq 1. \quad (3.8)$$

Since F and E are nonempty compact-valued mappings which both are upper semicontinuous, there exist subsequences $\{u_{n_i}\} \subset \{u_n\}$ and $\{v_{n_i}\} \subset \{v_n\}$ such that $u_{n_i} \rightarrow \bar{u}$ and $v_{n_i} \rightarrow \bar{v}$ for some $\bar{u} \in F(\bar{x})$ and $\bar{v} \in E(\bar{x})$. Note that g is continuous and N is mixed Lipschitz continuous with respect to the first and second arguments. So it follows that $g(x_{n_i}) \rightarrow g(\bar{x})$ and $N(u_{n_i}, v_{n_i}) \rightarrow N(\bar{u}, \bar{v})$ as $i \rightarrow \infty$. Therefore, utilizing the similar argument process to that in the proof of Theorem 3.1, we obtain from (3.8) and conditions (C1)–(C3) that

$$\langle N(\bar{u}, \bar{v}), g(\bar{x}) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \leq \frac{\alpha}{2} \|g(\bar{x}) - y\|^2 + \epsilon, \quad \forall y \in K.$$

It is easy to see that $d(\bar{x}, K) \leq \epsilon$. This shows that $\bar{x} \in \Omega_\alpha(\epsilon)$ and so $\Omega_\alpha(\epsilon)$ is nonempty closed for all $\epsilon > 0$. Observe that

$$S = \bigcap_{\epsilon > 0} \Omega_\alpha(\epsilon).$$

Since $\mu(\Omega_\alpha(\epsilon)) \rightarrow 0$, the theorem in page 412 of [15] can be applied and one concludes that S is nonempty and compact with

$$e(\Omega_\alpha(\epsilon), S) = \mathcal{H}(\Omega_\alpha(\epsilon), S) \rightarrow 0.$$

Let $\{\hat{x}_n\} \subset K$ be a LP α -approximating sequence for $\text{CGMVI}(N(F, E), g, \phi, K)$. Then there exists $w_n \in X$ with $w_n \rightarrow 0$ such that $g(\hat{x}_n) + w_n \in K$, and there exist $\hat{u}_n \in F(\hat{x}_n)$, $\hat{v}_n \in E(\hat{x}_n)$ and $0 < \epsilon'_n \rightarrow 0$ such that

$$\langle N(\hat{u}_n, \hat{v}_n), g(\hat{x}_n) - y \rangle + \phi(g(\hat{x}_n), \hat{x}_n) - \phi(y, \hat{x}_n) \leq \frac{\alpha}{2} \|g(\hat{x}_n) - y\|^2 + \epsilon'_n, \quad \forall y \in K, n \geq 1.$$

Since $g(\hat{x}_n) + w_n \in K$, then there exists $k_n \in K$ such that $g(\hat{x}_n) + w_n = k_n$. It follows that

$$d(g(\hat{x}_n), K) \leq \|g(\hat{x}_n) - k_n\| = \|w_n\| \rightarrow 0.$$

Set $\epsilon_n = \max\{\|w_n\|, \epsilon'_n\}$, we get $\hat{x}_n \in \Omega_\alpha(\epsilon_n)$. From (3.6) and the definition of $\Omega_\alpha(\epsilon_n)$, we have

$$d(\hat{x}_n, S) \leq e(\Omega_\alpha(\epsilon_n), S) \rightarrow 0.$$

Since S is compact, there exists $p_n \in S$ such that

$$\|p_n - \hat{x}_n\| = d(\hat{x}_n, S) \rightarrow 0.$$

Again from the compactness of S , there exists a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ which converges strongly to $\bar{p} \in S$. Hence the corresponding subsequence $\{\hat{x}_{n_i}\}$ of $\{\hat{x}_n\}$ converges strongly to $\bar{p} \in S$. Therefore, $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP α -well-posed in the generalized sense. The proof is complete. \square

Remark 3.2. Our Theorems 3.1 and 3.2 improve, extend and develop Theorems 3.1 and 3.2 in [19] to a great extent because our $\text{CGMVI}(N(F, E), g, \phi, K)$ is more general than the $\text{GMVI}(F, \phi, K)$ in [19].

4. Links with Levitin-Polyak Well-Posedness of Inclusion Problems

In this section, we shall show that the Levitin-Polyak well-posedness of a completely generalized mixed variational inequality is closely related to the Levitin-Polyak well-posedness of an inclusion problem. Let $g : X \rightarrow X$ be a single-valued mapping and $A : X \times X \rightarrow 2^{X^*}$ be a set-valued mapping. The inclusion problem associated with (A, g, K) is defined by

$$\text{IP}(A, g, K) : \text{ find } x \in K \text{ such that } 0 \in A(x, g(x)).$$

Definition 4.1. A sequence $\{x_n\} \subset K$ is called a LP approximating sequence for $\text{IP}(A, g, K)$ if there exists $w_n \in X$ with $w_n \rightarrow 0$ such that $g(x_n) + w_n \in K$ and $d(0, A(x_n, g(x_n))) \rightarrow 0$ as $n \rightarrow \infty$, or equivalently, there exists $y_n \in A(x_n, g(x_n))$ such that $\|y_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Definition 4.2. We say that $\text{IP}(A, g, K)$ is strongly (resp. weakly) LP well-posed if it has a unique solution and every LP approximating sequence converges strongly (resp. weakly) to the unique solution of $\text{IP}(A, g, K)$. $\text{IP}(A, g, K)$ is said to be strongly (resp. weakly) LP well-posed in the generalized sense if the solution set S of $\text{IP}(A, g, K)$ is nonempty and every LP approximating sequence has a subsequence which converges strongly (resp. weakly) to a point of S .

Remark 4.1. When $g = I$, Definitions 4.1 and 4.2 coincide with the Definitions 4.1 and 4.2 in [19], respectively. When X is a real Hilbert space, $g = I$, $K = X$ and $w_n \equiv 0$, Definitions 4.1 and 4.2 coincide with the Definitions 4.1 and 4.2 in [11], respectively.

Theorem 4.1. *Let K be a nonempty, compact and convex subset of X . Let $F, E : X \rightarrow 2^{X^*}$ be nonempty compact-valued mappings which both are \mathcal{H} -continuous. Let $g : X \rightarrow X$ be a continuous mapping and $N : X^* \times X^* \rightarrow X^*$ be mixed Lipschitz continuous with respect to the first and second arguments. Assume that $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ satisfies the conditions:*

(C1) ϕ is proper, convex and lower semicontinuous in the first argument;

(C2) ϕ is upper semicontinuous in the second argument;

(C3) $\liminf_{n \rightarrow \infty} \inf_{y \in \text{dom}\phi(\cdot, x_n)} (\phi(y, x_n) - \phi(y, \bar{x})) \geq 0, \quad \forall \{x_n\} \subset K : x_n \rightarrow \bar{x} (n \rightarrow \infty)$.

If CGMVI($N(F, E), g, \phi, K$) is strongly LP well-posed, then $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$ is strongly LP well-posed.

Proof. Suppose that CGMVI($N(F, E), g, \phi, K$) is strongly LP well-posed and x^* is the unique solution of CGMVI($N(F, E), g, \phi, K$). By Proposition 2.1, we obtain that x^* is also the unique solution of $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$. Let $\{x_n\}$ be a LP approximating sequence for $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$. Then, there exists $w_n \in X$ with $w_n \rightarrow 0$ such that $g(x_n) + w_n \in K$, and there exists $y_n \in N(F(x_n), E(x_n)) + \partial(\phi(\cdot, x_n) + \delta_K)(g(x_n))$ such that $\|y_n\| \rightarrow 0$ as $n \rightarrow \infty$. It is easy to see that there exist $u_n \in F(x_n)$ and $v_n \in E(x_n)$ such that

$$\phi(y, x_n) - \phi(g(x_n), x_n) \geq \langle y_n - N(u_n, v_n), y - g(x_n) \rangle, \quad \forall y \in K. \quad (4.1)$$

Let $\{x_{n_i}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_i} \rightarrow \bar{x}$ as $i \rightarrow \infty$. Clearly $\bar{x} \in K$. Since $\{g(x_{n_i})\} \subset K$ and g is continuous, we obtain from $g(x_{n_i}) + w_{n_i} \in K$ and $w_{n_i} \rightarrow 0$ that $g(\bar{x}) \in K$. So, it follows from (4.1), $\|y_{n_i}\| \rightarrow 0$ and conditions (C1)–(C3) that

$$\liminf_{i \rightarrow \infty} (\phi(g(x_{n_i}), x_{n_i}) - \phi(g(x_{n_i}), \bar{x})) \geq \liminf_{i \rightarrow \infty} \inf_{y \in \text{dom}\phi(\cdot, x_{n_i})} (\phi(y, x_{n_i}) - \phi(y, \bar{x})) \geq 0,$$

and

$$\begin{aligned} & \liminf_{i \rightarrow \infty} \langle N(u_{n_i}, v_{n_i}), g(x_{n_i}) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \\ & \leq \liminf_{i \rightarrow \infty} \langle N(u_{n_i}, v_{n_i}) - y_{n_i}, g(x_{n_i}) - y \rangle + \liminf_{i \rightarrow \infty} \phi(g(x_{n_i}), \bar{x}) \\ & \quad + \liminf_{i \rightarrow \infty} (\phi(g(x_{n_i}), x_{n_i}) - \phi(g(x_{n_i}), \bar{x})) + \liminf_{i \rightarrow \infty} (-\phi(y, x_{n_i})) \end{aligned} \quad (4.2)$$

$$\begin{aligned} & \leq \liminf_{i \rightarrow \infty} \{ \langle N(u_{n_i}, v_{n_i}) - y_{n_i}, g(x_{n_i}) - y \rangle + \phi(g(x_{n_i}), \bar{x}) \\ & \quad + \phi(g(x_{n_i}), x_{n_i}) - \phi(g(x_{n_i}), \bar{x}) - \phi(y, x_{n_i}) \} \end{aligned} \quad (4.3)$$

$$\begin{aligned} & = \liminf_{i \rightarrow \infty} \{ \langle N(u_{n_i}, v_{n_i}) - y_{n_i}, g(x_{n_i}) - y \rangle + \phi(g(x_{n_i}), x_{n_i}) - \phi(y, x_{n_i}) \} \\ & \leq 0. \end{aligned} \quad (4.4)$$

Since F and E are nonempty compact-valued mappings, in terms of Lemma 2.1, for each $u_n \in F(x_n)$ and $v_n \in E(x_n)$ there exist $\bar{u}_n \in F(\bar{x})$ and $\bar{v}_n \in E(\bar{x})$ such that $\|u_n - \bar{u}_n\| \leq \mathcal{H}(F(x_n), F(\bar{x}))$ and $\|v_n - \bar{v}_n\| \leq \mathcal{H}(E(x_n), E(\bar{x}))$. Since F and E both are \mathcal{H} -continuous, one deduces that $\|u_{n_i} - \bar{u}_{n_i}\| \leq \mathcal{H}(F(x_{n_i}), F(\bar{x})) \rightarrow 0$ and $\|v_{n_i} - \bar{v}_{n_i}\| \leq \mathcal{H}(E(x_{n_i}), E(\bar{x})) \rightarrow 0$ as $i \rightarrow \infty$. Since $F(\bar{x})$

and $E(\bar{x})$ both are compact, without loss of generality we may assume that $\bar{u}_{n_i} \rightarrow \bar{u} \in F(\bar{x})$ and $\bar{v}_{n_i} \rightarrow \bar{v} \in E(\bar{x})$ as $i \rightarrow \infty$. Thus, we conclude that

$$\|u_{n_i} - \bar{u}\| \leq \|u_{n_i} - \bar{u}_{n_i}\| + \|\bar{u}_{n_i} - \bar{u}\| \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

and

$$\|v_{n_i} - \bar{v}\| \leq \|v_{n_i} - \bar{v}_{n_i}\| + \|\bar{v}_{n_i} - \bar{v}\| \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

That is, $u_{n_i} \rightarrow \bar{u}$ and $v_{n_i} \rightarrow \bar{v}$ as $i \rightarrow \infty$. Note that N is mixed Lipschitz continuous with respect to the first and second arguments. Hence there exists a pair of constants $\zeta, \xi > 0$ such that

$$\|N(u_{n_i}, v_{n_i}) - N(\bar{u}, \bar{v})\| \leq \zeta \|u_{n_i} - \bar{u}\| + \xi \|v_{n_i} - \bar{v}\| \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

So, it immediately follows from (4.4) that

$$\langle N(\bar{u}, \bar{v}), g(\bar{x}) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \leq 0, \quad \forall y \in K.$$

Thus, \bar{x} solves $\text{CGMVI}(N(F, E), g, \phi, K)$. Since $\text{CGMVI}(N(F, E), g, \phi, K)$ has a unique solution x^* , we get $\bar{x} = x^*$. This shows that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Therefore, $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$ is strongly LP well-posed. The proof is complete. \square

Theorem 4.2. *Let K be a nonempty, closed and convex subset of X . Let $F, E : X \rightarrow 2^{X^*}$ be nonempty compact-valued mappings which both are \mathcal{H} -uniformly continuous. Let $g : X \rightarrow X$ be a homeomorphic mapping and $N : X^* \times X^* \rightarrow X^*$ be mixed Lipschitz continuous with respect to the first and second arguments. Assume that $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ satisfies the conditions:*

- (C1) ϕ is proper, convex and subdifferentiable in the first argument, and for each $y \in X$ the subdifferential of $\phi(\cdot, y)$ at $x \in X$ is denoted by $\partial_1 \phi(x, y)$;
- (C2) for each $y \in X$, $\partial_1 \phi(\cdot, y) : X \rightarrow 2^{X^*}$ is a nonempty compact-valued mapping, which is \mathcal{H} -Lipschitz continuous with respect to g , that is, for some $\lambda > 0$, one has

$$\mathcal{H}(\partial_1 \phi(g(x), y), \partial_1 \phi(g(x), z)) \leq \lambda \|g(y) - g(z)\|, \quad \forall x, y, z \in X.$$

If $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$ is strongly LP well-posed, then $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP well-posed.

Proof. Let $\{x_n\} \subset K$ be a LP approximating sequence for $\text{CGMVI}(N(F, E), g, \phi, K)$. Then there exist $w_n \in X$ with $w_n \rightarrow 0$ and $0 < \epsilon_n \rightarrow 0$ such that $g(x_n) + w_n \in K$, and there exist $u_n \in F(x_n)$ and $v_n \in E(x_n)$ satisfying

$$\phi(g(x_n), x_n) \geq \phi(y, x_n) + \langle N(u_n, v_n), y - g(x_n) \rangle + \epsilon_n, \quad \forall y \in K, n \geq 1. \quad (4.5)$$

Since $g(x_n) + w_n \in K$, there exists $k_n \in K$ such that $g(x_n) + w_n = k_n$.

Define $\tilde{\phi}_n : X \rightarrow \mathbf{R} \cup \{+\infty\}$ as follows:

$$\tilde{\phi}_n(y) = \phi(y, x_n) + \langle N(u_n, v_n), y - g(x_n) \rangle, \quad \forall y \in K, n \geq 1.$$

Since ϕ is proper, convex and subdifferentiable in the first argument, then we have that $\tilde{\phi}_n$ is proper, convex and subdifferentiable for all $n \geq 1$. It follows from Proposition 2.2.6 of [16] that $\tilde{\phi}_n$ is Lipschitz continuous. Since $\|w_n\| = \|k_n - g(x_n)\| \rightarrow 0$, then there exists $0 < \delta_n \rightarrow 0$ such that

$$\tilde{\phi}_n(k_n) - \tilde{\phi}_n(g(x_n)) \leq \delta_n. \quad (4.6)$$

By (4.5) and (4.6), we have

$$\tilde{\phi}_n(k_n) \leq \tilde{\phi}_n(y) + \delta_n + \epsilon_n, \quad \forall y \in K.$$

By Ekeland's theorem [17], there exists $\tilde{x}_n \in K$ such that

$$\|g(\tilde{x}_n) - k_n\| \leq \sqrt{\delta_n + \epsilon_n},$$

and

$$\tilde{\phi}_n(g(\tilde{x}_n)) \leq \tilde{\phi}_n(y) + \sqrt{\delta_n + \epsilon_n} \|y - g(\tilde{x}_n)\|, \quad \forall y \in K.$$

Thus, $g(\tilde{x}_n)$ minimizes the function $\tilde{\phi}_n(\cdot) + \sqrt{\delta_n + \epsilon_n} \|\cdot - g(\tilde{x}_n)\|$. It follows that $0^* \in \partial(\tilde{\phi}_n(\cdot) + \sqrt{\delta_n + \epsilon_n} \|\cdot - g(\tilde{x}_n)\|)(g(\tilde{x}_n))$. That is,

$$0^* \in \partial\tilde{\phi}_n(g(\tilde{x}_n)) + \sqrt{\delta_n + \epsilon_n} B_{X^*}.$$

So there exists

$$x_n^* \in \partial\tilde{\phi}_n(g(\tilde{x}_n)) = \partial\phi_1(g(\tilde{x}_n), x_n) + N(u_n, v_n) \quad (4.7)$$

such that

$$\|x_n^*\| \leq \sqrt{\delta_n + \epsilon_n}.$$

Since $\|g(x_n) - k_n\| \rightarrow 0$ and $\|g(\tilde{x}_n) - k_n\| \rightarrow 0$, we know that $\|g(\tilde{x}_n) - g(x_n)\| \leq \|g(\tilde{x}_n) - k_n\| + \|g(x_n) - k_n\| \rightarrow 0$. Since F and E are nonempty compact-valued mappings which both are \mathcal{H} -uniformly continuous, by Lemma 2.1 we deduce that for each $u_n \in F(x_n)$ and $v_n \in E(x_n)$ there exist $\tilde{u}_n \in F(\tilde{x}_n)$ and $\tilde{v}_n \in E(\tilde{x}_n)$ such that $\|u_n - \tilde{u}_n\| \leq \mathcal{H}(F(x_n), F(\tilde{x}_n)) \rightarrow 0$ and $\|v_n - \tilde{v}_n\| \leq \mathcal{H}(E(x_n), E(\tilde{x}_n)) \rightarrow 0$.

On the other hand, putting $z_n = x_n^* - N(u_n, v_n)$, from (4.7) we get $z_n \in \partial\phi_1(g(\tilde{x}_n), x_n)$. Since for each $y \in X$, $\partial_1\phi(\cdot, y) : X \rightarrow 2^{X^*}$ is a nonempty compact-valued mapping, which is \mathcal{H} -Lipschitz continuous with respect to g , we obtain that for each $z_n \in \partial\phi_1(g(\tilde{x}_n), x_n)$, there exists $\tilde{z}_n \in \partial\phi_1(g(\tilde{x}_n), \tilde{x}_n)$ such that for some $\lambda > 0$,

$$\|z_n - \tilde{z}_n\| \leq \mathcal{H}(\partial\phi_1(g(\tilde{x}_n), x_n), \partial\phi_1(g(\tilde{x}_n), \tilde{x}_n)) \leq \lambda \|g(x_n) - g(\tilde{x}_n)\| \rightarrow 0. \quad (4.8)$$

Putting $e_n = z_n - \tilde{z}_n$, we deduce from (4.7) that

$$\begin{aligned}
 x_n^* + N(\tilde{u}_n, \tilde{v}_n) - N(u_n, v_n) - e_n + \partial\delta_K(g(\tilde{x}_n)) \\
 &= N(\tilde{u}_n, \tilde{v}_n) + z_n - e_n + \partial\delta_K(g(\tilde{x}_n)) \\
 &= N(\tilde{u}_n, \tilde{v}_n) + \tilde{z}_n + \partial\delta_K(g(\tilde{x}_n)) \\
 &\subset N(\tilde{u}_n, \tilde{v}_n) + \partial\phi_1(g(\tilde{x}_n), \tilde{x}_n) + \partial\delta_K(g(\tilde{x}_n)) \\
 &\subset N(F(\tilde{x}_n), E(\tilde{x}_n) + \partial(\phi(\cdot, \tilde{x}_n) + \delta_K)(g(\tilde{x}_n))).
 \end{aligned}$$

From $0^* \in \partial\delta_K(g(\tilde{x}_n))$, we have

$$x_n^* + N(\tilde{u}_n, \tilde{v}_n) - N(u_n, v_n) - e_n \in N(F(\tilde{x}_n), E(\tilde{x}_n) + \partial(\phi(\cdot, \tilde{x}_n) + \delta_K)(g(\tilde{x}_n))).$$

Since N be mixed Lipschitz continuous with respect to the first and second arguments, there exists a pair of constants $\zeta, \xi > 0$ such that

$$\|N(u_n, v_n) - N(\tilde{u}_n, \tilde{v}_n)\| \leq \zeta\|u_n - \tilde{u}_n\| + \xi\|v_n - \tilde{v}_n\| \rightarrow 0. \quad (4.9)$$

Combining (4.7)–(4.9), we get

$$\|x_n^* + N(\tilde{u}_n, \tilde{v}_n) - N(u_n, v_n) - e_n\| \leq \|x_n^*\| + \|N(\tilde{u}_n, \tilde{v}_n) - N(u_n, v_n)\| + \|e_n\| \rightarrow 0.$$

This shows that $\{\tilde{x}_n\}$ is a LP approximating sequence for $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$. Let x^* be the unique solution of $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$. By Proposition 2.1, x^* is also the unique solution of $\text{CGMVI}(N(F, E), g, \phi, K)$. If $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$ is strongly LP well-posed, then $\tilde{x}_n \rightarrow x^*$. Noticing that $\|g(x_n) - g(x^*)\| \leq \|g(x_n) - g(\tilde{x}_n)\| + \|g(\tilde{x}_n) - g(x^*)\|$, we know that $g(x_n) \rightarrow g(x^*)$ as $n \rightarrow \infty$. Since $g : X \rightarrow X$ is a homeomorphic mapping, we immediately obtain that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Thus, $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP well-posed. \square

Theorem 4.3. *Let K be a nonempty, closed and convex subset of X . Let $F, E : X \rightarrow 2^{X^*}$ be nonempty compact-valued mappings. Let $g : X \rightarrow X$ be a continuous mapping and $N : X^* \times X^* \rightarrow X^*$ be a continuous mapping. Assume that $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous in the first argument. If $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly (resp. weakly) LP 1-well-posed in the generalized sense, then $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$ is strongly (resp. weakly) LP well-posed in the generalized sense.*

Proof. Let $\{x_n\}$ be a LP approximating sequence for $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$. Then there exists $w_n \in X$ with $w_n \rightarrow 0$ such that $g(x_n) + w_n \in K$, and there exists $y_n \in N(F(x_n), E(x_n)) + \partial(\phi(\cdot, x_n) + \delta_K)(g(x_n))$ such that $\|y_n\| \rightarrow 0$ as $n \rightarrow \infty$. It is easy to see that there exist $u_n \in F(x_n)$ and $v_n \in E(x_n)$ such that

$$\phi(y, x_n) - \phi(g(x_n), x_n) \geq \langle y_n - N(u_n, v_n), y - g(x_n) \rangle, \quad \forall y \in K.$$

Thus,

$$\begin{aligned} & \langle N(u_n, v_n), g(x_n) - y \rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \\ & \leq \langle y_n, g(x_n) - y \rangle \leq \frac{1}{2} \|g(x_n) - y\|^2 + \frac{1}{2} \|y_n\|^2, \quad \forall y \in K, n \geq 1. \end{aligned}$$

This together with $\|y_n\| \rightarrow 0$, implies that $\{x_n\}$ be a LP 1-approximating sequence for $\text{CGMVI}(N(F, E), g, \phi, K)$. Since $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly (resp. weakly) LP 1-well-posed in the generalized sense, $\{x_n\}$ has a subsequence which converges strongly (resp. weakly) to some solution x^* of $\text{CGMVI}(N(F, E), g, \phi, K)$. By Proposition 2.1, x^* is also a solution of $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$ and so $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$ is strongly (resp. weakly) LP well-posed in the generalized sense. The proof is complete. \square

Theorem 4.4. *Let K be a nonempty, closed and convex subset of X . Let $F, E : X \rightarrow 2^{X^*}$ be nonempty compact-valued mappings which both are \mathcal{H} -uniformly continuous. Let $g : X \rightarrow X$ be a homeomorphic mapping and $N : X^* \times X^* \rightarrow X^*$ be mixed Lipschitz continuous with respect to the first and second arguments. Assume that $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ satisfies the conditions:*

- (C1) *ϕ is proper, convex and subdifferentiable in the first argument, and for each $y \in X$ the subdifferential of $\phi(\cdot, y)$ at $x \in X$ is denoted by $\partial_1 \phi(x, y)$;*
- (C2) *for each $y \in X$, $\partial_1 \phi(\cdot, y) : X \rightarrow 2^{X^*}$ is a nonempty compact-valued mapping, which is \mathcal{H} -Lipschitz continuous with respect to g , that is, for some $\lambda > 0$, one has*

$$\mathcal{H}(\partial_1 \phi(g(x), y), \partial_1 \phi(g(x), z)) \leq \lambda \|g(y) - g(z)\|, \quad \forall x, y, z \in X.$$

If $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$ is strongly LP well-posed in the generalized sense, then $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP well-posed in the generalized sense.

Proof. The conclusion follows from the similar arguments to those in the proof of Theorem 4.2. \square

Remark 4.2. Our Theorems 4.1–4.4 improve, extend and develop Theorems 4.1–4.4 in [19] to a great extent because our $\text{CGMVI}(N(F, E), g, \phi, K)$ and $\text{IP}(N(F, E) + \partial(\phi + \delta_K) \circ g, K)$ are more general than the $\text{GMVI}(F, \phi, K)$ and $\text{IP}(F + \partial(\phi + \delta_K), K)$ in [19], respectively.

5. Links with Levitin-Polyak Well-Posedness of fixed point problems

In this section, we shall investigate the relations between the Levitin-Polyak well-posedness of completely generalized mixed variational inequalities and the Levitin-Polyak well-posedness of the corresponding fixed point problems. Let $g : X \rightarrow X$ be a single-valued mapping and

$T : X \times X \rightarrow 2^{X^*}$ be a set-valued mapping. The fixed point problem associated with (T, g, K) is defined by

$$\text{FP}(T, g, K) : \text{find } x \in K \text{ such that } x \in T(x, g(x)).$$

Let $U = \{x \in X : \|x\| = 1\}$ be the unit sphere. A Banach space X is said to be (i) strictly convex if for any $x, y \in U$, $x \neq y \Rightarrow \|\frac{x+y}{2}\| < 1$; (ii) smooth if the limit $\lim_{t \rightarrow 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for all $x, y \in U$. The modulus of convexity of X is defined by

$$\delta_X(\epsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in U, \|x+y\| \geq \epsilon\},$$

and the modulus of smoothness of X is defined by

$$\rho_X(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in U, \|y\| \leq \tau\}.$$

In this section, we suppose that $q > 1$ and $s > 1$ are fixed numbers.

Definition 5.1. A Banach space X is said to be

- (i) q -uniformly convex if there exists a constant $c > 0$ such that $\delta_X(\epsilon) \geq c\epsilon^q$ for all $\epsilon \in (0, 2)$;
- (ii) q -uniformly smooth if there exists a constant $k > 0$ such that $\rho_X(\tau) \leq k\tau^q$.

The generalized duality mapping $J_q : X \rightarrow 2^{X^*}$ is defined by

$$J_q(x) = \{j_q(x) \in X^* : \langle j_q(x), x \rangle = \|x\|^q, \|j_q(x)\| = \|x\|^{q-1}\}.$$

In particular, $J = J_2$ is called the normalized duality mapping. It is well known that J_q has the following properties: (a) J_q is bounded; (b) if X is smooth, then J_q is single-valued; (c) if X is strictly convex, then J_q is one-to-one and strictly monotone.

Lemma 5.1 (see [18]). *Let X be a q -uniformly smooth Banach space. Then there exists a constant $L_q > 0$ such that*

$$\|J_q(x) - J_q(y)\| \leq L_q \|x - y\|^{q-1}, \quad \forall x, y \in X.$$

Lemma 5.2 (see [18]). *Let X be a q -uniformly convex Banach space. Then there exists a constant $K_q > 0$ such that*

$$\langle J_q(x) - J_q(y), x - y \rangle \geq K_q \|x - y\|^q, \quad \forall x, y \in X.$$

Lemma 5.3 (see [10]). *Let X be a q -uniformly convex Banach space and $M : X \rightarrow 2^{X^*}$ be a maximal monotone operator. Then for every $\lambda > 0$, $(J_q + \lambda M)^{-1}$ is well-defined and single-valued.*

We denote $\Pi_\lambda^{\phi(\cdot, x)} = (J_q + \lambda \partial \phi(\cdot, x))^{-1}$ for each $x \in X$. By the definition of $\Pi_\lambda^{\phi(\cdot, x)}$ and Lemma 5.3, it is easy to prove the following proposition.

Proposition 5.1. *Let X be a q -uniformly convex Banach space, and K be a nonempty, closed and convex subset of X . Let $g : X \rightarrow X$ be a single-valued mapping, $N : X^* \times X^* \rightarrow X^*$ be a single-valued mapping, and $F, E : X \rightarrow 2^{X^*}$ be two nonempty set-valued mappings. Let $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ be such that for each $x \in X$, $\phi(\cdot, x)$ is a proper, convex and lower semicontinuous functional. Then the following conclusions are equivalent:*

- (i) x solves CGMVI($N(F, E)$, g , ϕ , K);
- (ii) x solves the fixed-point problem $\text{FP}(I - g + \Pi_\lambda^{\phi + \delta_k}(J_q \circ g - \lambda N(F, E)), K)$:

$$\text{find } x \in K \text{ such that } g(x) \in \Pi_\lambda^{\phi(\cdot, x) + \delta_k}(J_q(g(x)) - \lambda N(F(x), E(x))).$$

Definition 5.2. A sequence $\{x_n\} \subset K$ is called a LP approximating sequence for $\text{FP}(T, g, K)$ if there exists $w_n \in X$ with $w_n \rightarrow 0$ such that $g(x_n) + w_n \in K$, and there exists $y_n \in T(x_n, g(x_n))$ such that $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Definition 5.3. We say that $\text{FP}(T, g, K)$ is strongly (resp. weakly) LP well-posed if it has a unique solution and every LP approximating sequence for $\text{FP}(T, g, K)$ converges strongly (resp. weakly) to the unique solution. $\text{FP}(T, g, K)$ is said to be strongly (resp. weakly) LP well-posed in the generalized sense if the solution set S of $\text{FP}(T, g, K)$ is nonempty and every LP approximating sequence has a subsequence which converges strongly (resp. weakly) to a point of S .

Remark 5.1. When $g = I$, Definitions 5.2 and 5.3 coincide with the Definitions 5.2 and 5.3 in [19], respectively.

Theorem 5.1. *Let X be a s -uniformly convex and q -uniformly smooth Banach space and K be a nonempty, compact and convex subset of X . Let $F, E : X \rightarrow 2^{X^*}$ be nonempty compact-valued mappings which both are \mathcal{H} -continuous. Let $g : X \rightarrow X$ be a continuous mapping and $N : X^* \times X^* \rightarrow X^*$ be mixed Lipschitz continuous with respect to the first and second arguments. Assume that $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ satisfies the conditions:*

- (C1) ϕ is proper, convex and lower semicontinuous in the first argument;
- (C2) ϕ is upper semicontinuous in the second argument;
- (C3) $\liminf_{n \rightarrow \infty} \inf_{y \in \text{dom} \phi(\cdot, x_n)} (\phi(y, x_n) - \phi(y, \bar{x})) \geq 0, \quad \forall \{x_n\} \subset K : x_n \rightarrow \bar{x} (n \rightarrow \infty)$.

If CGMVI($N(F, E)$, g , ϕ , K) is strongly LP well-posed, then $\text{FP}(I - g + \Pi_\lambda^{\phi + \delta_k}(J_q \circ g - \lambda N(F, E)), K)$ is strongly LP well-posed, where $\lambda > 0$ is a constant.

Proof. We suppose that $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP well-posed and x^* is the unique solution of $\text{CGMVI}(N(F, E), g, \phi, K)$. By Proposition 5.1, x^* is the unique solution of $\text{FP}(I - g + \Pi_\lambda^{\phi + \delta_k}(J_q \circ g - \lambda N(F, E)), K)$. Let $\{x_n\}$ be a LP approximating sequence for $\text{FP}(I - g + \Pi_\lambda^{\phi + \delta_k}(J_q \circ g - \lambda N(F, E)), K)$, then there exists $w_n \in X$ with $w_n \rightarrow 0$ such that $g(x_n) + w_n \in K$, and there exists $y_n \in (I - g)(x_n) + \Pi_\lambda^{\phi(\cdot, x_n) + \delta_k}(J_q(g(x_n)) - \lambda N(F(x_n), E(x_n)))$ such that $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$. By the definition of $\Pi_\lambda^{\phi(\cdot, x_n) + \delta_k}$, there exist $u_n \in F(x_n)$ and $v_n \in E(x_n)$ such that

$$\frac{J_q(g(x_n)) - J_q(y_n - x_n + g(x_n))}{\lambda} - N(u_n, v_n) \in \partial(\phi(\cdot, x_n) + \delta_k)(y_n - x_n + g(x_n)).$$

It is easy to see that $\{y_n - x_n + g(x_n)\} \subset K$ and

$$\begin{aligned} & \left\langle \frac{J_q(g(x_n)) - J_q(y_n - x_n + g(x_n))}{\lambda} - N(u_n, v_n), y - (y_n - x_n + g(x_n)) \right\rangle \\ & \leq \phi(y, x_n) - \phi(y_n - x_n + g(x_n), x_n), \quad \forall y \in K, n \geq 1. \end{aligned} \quad (5.1)$$

Putting $z_n = \frac{J_q(g(x_n)) - J_q(y_n - x_n + g(x_n))}{\lambda}$, by Lemma 5.1 we deduce from $\|x_n - y_n\| \rightarrow 0$ that as $n \rightarrow \infty$,

$$\|z_n\| = \left\| \frac{J_q(g(x_n)) - J_q(y_n - x_n + g(x_n))}{\lambda} \right\| \leq \frac{L_q \|y_n - x_n\|^{q-1}}{\lambda} \rightarrow 0. \quad (5.2)$$

Let $\{x_{n_i}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_i} \rightarrow \bar{x}$ as $i \rightarrow \infty$. Clearly $\bar{x} \in K$. Since $\{y_n - x_n + g(x_n)\} \subset K$ and g is continuous, we know that $g(\bar{x}) \in K$. So, it follows from (5.1), (5.2) and conditions (C1)–(C3) that

$$\begin{aligned} & \liminf_{i \rightarrow \infty} (\phi(y_{n_i} - x_{n_i} + g(x_{n_i}), x_{n_i}) - \phi(y_{n_i} - x_{n_i} + g(x_{n_i}), \bar{x})) \\ & \geq \liminf_{i \rightarrow \infty} \inf_{y \in \text{dom}\phi(\cdot, x_{n_i})} (\phi(y, x_{n_i}) - \phi(y, \bar{x})) \geq 0, \end{aligned}$$

and

$$\begin{aligned} & \liminf_{i \rightarrow \infty} \langle N(u_{n_i}, v_{n_i}), y_{n_i} - x_{n_i} + g(x_{n_i}) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \\ & \leq \liminf_{i \rightarrow \infty} \langle N(u_{n_i}, v_{n_i}) - z_{n_i}, y_{n_i} - x_{n_i} + g(x_{n_i}) - y \rangle + \liminf_{i \rightarrow \infty} \phi(y_{n_i} - x_{n_i} + g(x_{n_i}), \bar{x}) \\ & \quad + \liminf_{i \rightarrow \infty} (\phi(y_{n_i} - x_{n_i} + g(x_{n_i}), x_{n_i}) - \phi(y_{n_i} - x_{n_i} + g(x_{n_i}), \bar{x})) + \liminf_{i \rightarrow \infty} (-\phi(y, x_{n_i})) \\ & \leq \liminf_{i \rightarrow \infty} \{ \langle N(u_{n_i}, v_{n_i}) - z_{n_i}, y_{n_i} - x_{n_i} + g(x_{n_i}) - y \rangle + \phi(y_{n_i} - x_{n_i} + g(x_{n_i}), \bar{x}) \\ & \quad + \phi(y_{n_i} - x_{n_i} + g(x_{n_i}), x_{n_i}) - \phi(y_{n_i} - x_{n_i} + g(x_{n_i}), \bar{x}) - \phi(y, x_{n_i}) \} \\ & = \liminf_{i \rightarrow \infty} \{ \langle N(u_{n_i}, v_{n_i}) - z_{n_i}, y_{n_i} - x_{n_i} + g(x_{n_i}) - y \rangle + \phi(y_{n_i} - x_{n_i} + g(x_{n_i}), x_{n_i}) - \phi(y, x_{n_i}) \} \\ & \leq 0. \end{aligned} \quad (5.3)$$

Since F and E are nonempty compact-valued mappings, in terms of Lemma 2.1, for each $u_n \in F(x_n)$ and $v_n \in E(x_n)$ there exist $\bar{u}_n \in F(\bar{x})$ and $\bar{v}_n \in E(\bar{x})$ such that $\|u_n - \bar{u}_n\| \leq \mathcal{H}(F(x_n), F(\bar{x}))$ and $\|v_n - \bar{v}_n\| \leq \mathcal{H}(E(x_n), E(\bar{x}))$. Since F and E both are \mathcal{H} -continuous, one deduces that $\|u_{n_i} - \bar{u}_{n_i}\| \leq \mathcal{H}(F(x_{n_i}), F(\bar{x})) \rightarrow 0$ and $\|v_{n_i} - \bar{v}_{n_i}\| \leq \mathcal{H}(E(x_{n_i}), E(\bar{x})) \rightarrow 0$ as $i \rightarrow \infty$. Since

$F(\bar{x})$ and $E(\bar{x})$ both are compact, without loss of generality we may assume that $\bar{u}_{n_i} \rightarrow \bar{u} \in F(\bar{x})$ and $\bar{v}_{n_i} \rightarrow \bar{v} \in E(\bar{x})$ as $i \rightarrow \infty$. Thus, we conclude that

$$\|u_{n_i} - \bar{u}\| \leq \|u_{n_i} - \bar{u}_{n_i}\| + \|\bar{u}_{n_i} - \bar{u}\| \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

and

$$\|v_{n_i} - \bar{v}\| \leq \|v_{n_i} - \bar{v}_{n_i}\| + \|\bar{v}_{n_i} - \bar{v}\| \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

That is, $u_{n_i} \rightarrow \bar{u}$ and $v_{n_i} \rightarrow \bar{v}$ as $i \rightarrow \infty$. Note that N is mixed Lipschitz continuous with respect to the first and second arguments. Hence there exists a pair of constants $\zeta, \xi > 0$ such that

$$\|N(u_{n_i}, v_{n_i}) - N(\bar{u}, \bar{v})\| \leq \zeta \|u_{n_i} - \bar{u}\| + \xi \|v_{n_i} - \bar{v}\| \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

This together with $u_{n_i} \rightarrow \bar{u}$ and $v_{n_i} \rightarrow \bar{v}$, implies that $N(u_{n_i}, v_{n_i}) \rightarrow N(\bar{u}, \bar{v})$ as $i \rightarrow \infty$. So, it immediately follows from (5.3) that

$$\langle N(\bar{u}, \bar{v}), g(\bar{x}) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \leq 0, \quad \forall y \in K.$$

Thus, \bar{x} solves CGMVI($N(F, E), g, \phi, K$). Since CGMVI($N(F, E), g, \phi, K$) has a unique solution x^* , we get $\bar{x} = x^*$. This shows that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Therefore, FP($I - g + \Pi_\lambda^{\phi+\delta_k}(J_q \circ g - \lambda N(F, E)), K$) is strongly LP well-posed. The proof is complete. \square

Theorem 5.2. *Let X be a q -uniformly convex Banach space and K be a nonempty, closed and convex subset of X . Let $F, E : X \rightarrow 2^{X^*}$ be nonempty compact-valued mappings which both are \mathcal{H} -uniformly continuous. Let $g : X \rightarrow X$ be a homeomorphic mapping and $N : X^* \times X^* \rightarrow X^*$ be mixed Lipschitz continuous with respect to the first and second arguments. Assume that $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ satisfies the conditions:*

(C1) *for each $x \in X$, $\phi(\cdot, x)$ is a proper, convex and subdifferentiable functional;*

(C2) $\|\Pi_\lambda^{\phi(\cdot, x)+\delta_K}(v^*) - \Pi_\lambda^{\phi(\cdot, y)+\delta_K}(v^*)\| \leq \beta \|g(x) - g(y)\|, \forall x, y \in X, v^* \in X^*$ *for some $\beta > 0$.*

If FP($I - g + \Pi_\lambda^{\phi+\delta_k}(J_q \circ g - \lambda N(F, E)), K$) is strongly LP well-posed, then CGMVI($N(F, E), g, \phi, K$) is strongly LP well-posed.

Proof. Let $\{x_n\} \subset K$ be a LP approximating sequence for CGMVI($N(F, E), g, \phi, K$). Then there exist $w_n \in X$ with $w_n \rightarrow 0$ and $0 < \epsilon_n \rightarrow 0$ such that $g(x_n) + w_n \in K$, and there exist $u_n \in F(x_n)$ and $v_n \in E(x_n)$ satisfying

$$\phi(g(x_n), x_n) \geq \phi(y, x_n) + \langle N(u_n, v_n), y - g(x_n) \rangle + \epsilon_n, \quad \forall y \in K, n \geq 1. \quad (5.4)$$

Since $g(x_n) + w_n \in K$, there exists $k_n \in K$ such that $g(x_n) + w_n = k_n$. Define $\tilde{\phi}_n : X \rightarrow \mathbf{R} \cup \{+\infty\}$ as follows:

$$\tilde{\phi}_n(y) = \phi(y, x_n) + \langle N(u_n, v_n), y - g(x_n) \rangle, \quad \forall y \in K, n \geq 1.$$

Since ϕ is proper, convex and subdifferentiable in the first argument, then we have that $\tilde{\phi}_n$ is proper, convex and subdifferentiable for all $n \geq 1$. It follows from Proposition 2.2.6 of [16] that $\tilde{\phi}_n$ is Lipschitz continuous. Since $\|w_n\| = \|k_n - g(x_n)\| \rightarrow 0$, then there exists $0 < \delta_n \rightarrow 0$ such that

$$\tilde{\phi}_n(k_n) - \tilde{\phi}_n(g(x_n)) \leq \delta_n. \quad (5.5)$$

It follows from (5.4) and (5.5) that

$$\tilde{\phi}_n(k_n) \leq \tilde{\phi}_n(y) + \delta_n + \epsilon_n, \quad \forall y \in K.$$

By Ekeland's theorem [17], there exists $\tilde{x}_n \in K$ such that

$$\|g(\tilde{x}_n) - k_n\| \leq \sqrt{\delta_n + \epsilon_n},$$

and

$$\tilde{\phi}_n(g(\tilde{x}_n)) \leq \tilde{\phi}_n(y) + \sqrt{\delta_n + \epsilon_n} \|y - g(\tilde{x}_n)\|, \quad \forall y \in K.$$

Thus, $g(\tilde{x}_n)$ minimizes the function $\tilde{\phi}_n(\cdot) + \sqrt{\delta_n + \epsilon_n} \|\cdot - g(\tilde{x}_n)\|$. It follows that $0^* \in \partial(\tilde{\phi}_n(\cdot) + \sqrt{\delta_n + \epsilon_n} \|\cdot - g(\tilde{x}_n)\|)(g(\tilde{x}_n))$. That is,

$$0^* \in \partial\tilde{\phi}_n(g(\tilde{x}_n)) + \sqrt{\delta_n + \epsilon_n} B_{X^*}.$$

So there exists

$$x_n^* \in \partial\tilde{\phi}_n(g(\tilde{x}_n)) = \partial\phi_1(g(\tilde{x}_n), x_n) + N(u_n, v_n) \quad (5.6)$$

such that

$$\|x_n^*\| \leq \sqrt{\delta_n + \epsilon_n}.$$

It follows from (5.6) that

$$g(\tilde{x}_n) = \Pi_{\lambda}^{\phi(\cdot, x_n) + \delta_K}(J_q(g(\tilde{x}_n)) + \lambda x_n^* - \lambda N(u_n, v_n)).$$

Since $\|g(x_n) - k_n\| \rightarrow 0$ and $\|g(\tilde{x}_n) - k_n\| \rightarrow 0$, we have $\|g(x_n) - g(\tilde{x}_n)\| \leq \|g(x_n) - k_n\| + \|g(\tilde{x}_n) - k_n\| \rightarrow 0$. Since $F, E : X \rightarrow 2^{X^*}$ are nonempty compact-valued mappings which both are \mathcal{H} -uniformly continuous, in terms of Lemma 2.1 we know that for each $u_n \in F(x_n)$ and $v_n \in E(x_n)$ there exist $\tilde{u}_n \in F(\tilde{x}_n)$ and $\tilde{v}_n \in E(\tilde{x}_n)$ such that $\|u_n - \tilde{u}_n\| \leq \mathcal{H}(F(x_n), F(\tilde{x}_n)) \rightarrow 0$ and $\|v_n - \tilde{v}_n\| \leq \mathcal{H}(E(x_n), E(\tilde{x}_n)) \rightarrow 0$ as $n \rightarrow \infty$. Note that N is mixed Lipschitz continuous with respect to the first and second arguments. Hence, it follows that there exists a pair of constants $\zeta, \xi > 0$ such that

$$\|N(u_n, v_n) - N(\tilde{u}_n, \tilde{v}_n)\| \leq \zeta \|u_n - \tilde{u}_n\| + \xi \|v_n - \tilde{v}_n\| \rightarrow 0,$$

as $n \rightarrow \infty$. Since

$$\Pi_{\lambda}^{\phi(\cdot, \tilde{x}_n) + \delta_K}(J_q(g(\tilde{x}_n)) - \lambda N(\tilde{u}_n, \tilde{v}_n)) \in \Pi_{\lambda}^{\phi(\cdot, \tilde{x}_n) + \delta_K}(J_q(g(\tilde{x}_n)) - \lambda N(F(\tilde{x}_n), E(\tilde{x}_n))),$$

and as $n \rightarrow \infty$,

$$\begin{aligned}
& \|g(\tilde{x}_n) - \Pi_\lambda^{\phi(\cdot, \tilde{x}_n) + \delta_K} (J_q(g(\tilde{x}_n)) - \lambda N(\tilde{u}_n, \tilde{v}_n))\| \\
&= \|\Pi_\lambda^{\phi(\cdot, x_n) + \delta_K} (J_q(g(\tilde{x}_n)) + \lambda x_n^* - \lambda N(u_n, v_n)) - \Pi_\lambda^{\phi(\cdot, \tilde{x}_n) + \delta_K} (J_q(g(\tilde{x}_n)) - \lambda N(\tilde{u}_n, \tilde{v}_n))\| \\
&\leq \|\Pi_\lambda^{\phi(\cdot, x_n) + \delta_K} (J_q(g(\tilde{x}_n)) + \lambda x_n^* - \lambda N(u_n, v_n)) - \Pi_\lambda^{\phi(\cdot, \tilde{x}_n) + \delta_K} (J_q(g(\tilde{x}_n)) + \lambda x_n^* - \lambda N(u_n, v_n))\| \\
&\quad + \|\Pi_\lambda^{\phi(\cdot, \tilde{x}_n) + \delta_K} (J_q(g(\tilde{x}_n)) + \lambda x_n^* - \lambda N(u_n, v_n)) - \Pi_\lambda^{\phi(\cdot, \tilde{x}_n) + \delta_K} (J_q(g(\tilde{x}_n)) - \lambda N(\tilde{u}_n, \tilde{v}_n))\| \\
&\leq \beta \|g(x_n) - g(\tilde{x}_n)\| + \|(J_q(g(\tilde{x}_n)) + \lambda x_n^* - \lambda N(u_n, v_n)) - (J_q(g(\tilde{x}_n)) - \lambda N(\tilde{u}_n, \tilde{v}_n))\| \\
&= \beta \|g(x_n) - g(\tilde{x}_n)\| + \|\lambda x_n^* + \lambda(N(\tilde{u}_n, \tilde{v}_n) - N(u_n, v_n))\| \\
&\leq \beta \|g(x_n) - g(\tilde{x}_n)\| + \|\lambda x_n^*\| + \|\lambda(N(\tilde{u}_n, \tilde{v}_n) - N(u_n, v_n))\| \rightarrow 0,
\end{aligned} \tag{5.7}$$

this shows that

$$\lim_{n \rightarrow \infty} \|\tilde{x}_n - [(I - g)(\tilde{x}_n) + \Pi_\lambda^{\phi(\cdot, \tilde{x}_n) + \delta_K} (J_q(g(\tilde{x}_n)) - \lambda N(\tilde{u}_n, \tilde{v}_n))]\| = 0.$$

Thus, $\{\tilde{x}_n\}$ is a LP approximating sequence for $\text{FP}(I - g + \Pi_\lambda^{\phi + \delta_K} (J_q \circ g - \lambda N(F, E)), K)$. Let x^* be the unique solution of $\text{CGMVI}(N(F, E), g, \phi, K)$. By Proposition 5.1, we obtain that x^* is also the unique solution of $\text{FP}(I - g + \Pi_\lambda^{\phi + \delta_K} (J_q \circ g - \lambda N(F, E)), K)$. If $\text{FP}(I - g + \Pi_\lambda^{\phi + \delta_K} (J_q \circ g - \lambda N(F, E)), K)$ is strongly LP well-posed, then $\tilde{x}_n \rightarrow x^*$. It follows that

$$\|g(x_n) - g(x^*)\| \leq \|g(x_n) - g(\tilde{x}_n)\| + \|g(\tilde{x}_n) - g(x^*)\| \rightarrow 0.$$

Since g is a homeomorphic mapping, we immediately obtain that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Thus, $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP well-posed. \square

Theorem 5.3. *Let X be a s -uniformly convex and q -uniformly smooth Banach space and K be a nonempty, compact and convex subset of X . Let $F, E : X \rightarrow 2^{X^*}$ be nonempty compact-valued mappings such that $F(K) \cup E(K)$ is bounded. Let $g : X \rightarrow X$ be a continuous mapping and $N : X^* \times X^* \rightarrow X^*$ be mixed Lipschitz continuous with respect to the first and second arguments. Assume that $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ satisfies the conditions:*

(C1) ϕ is proper, convex and lower semicontinuous in the first argument;

(C2) $\lim_{n \rightarrow \infty} |\phi(e_n + g(x_n), x_n) - \phi(g(x_n), x_n)| = 0$, $\forall \{x_n\} \subset K : \{e_n + g(x_n)\} \subset K$ with $e_n \rightarrow 0$.
If $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly (resp. weakly) LP $\frac{1}{\lambda}$ -well-posed in the generalized sense, then $\text{FP}(I - g + \Pi_\lambda^{\phi + \delta_K} (J_q \circ g - \lambda N(F, E)), K)$ is strongly (resp. weakly) LP well-posed in the generalized sense.

Proof. Let $\{x_n\} \subset K$ be a LP approximating sequence for $\text{FP}(I - g + \Pi_\lambda^{\phi + \delta_K} (J_q \circ g - \lambda N(F, E)), K)$. Then there exists $w_n \in X$ with $w_n \rightarrow 0$ such that $g(x_n) + w_n \in K$, and there exists $y_n \in (I - g)(x_n) + \Pi_\lambda^{\phi(\cdot, x_n) + \delta_K} (J_q(g(x_n)) - \lambda N(F(x_n), E(x_n)))$ such that $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$. By the

definition of $\Pi_\lambda^{\phi(\cdot, x_n) + \delta_k}$, we know that $\{y_n - x_n + g(x_n)\} \subset K$ and there exist $u_n \in F(x_n)$ and $v_n \in E(x_n)$ such that

$$\begin{aligned} & \left\langle \frac{J_q(g(x_n)) - J_q(y_n - x_n + g(x_n))}{\lambda} - N(u_n, v_n), y - (y_n - x_n + g(x_n)) \right\rangle \\ & \leq \phi(y, x_n) - \phi(y_n - x_n + g(x_n), x_n), \quad \forall y \in K, n \geq 1. \end{aligned} \quad (5.8)$$

It follows from (5.8) that for any $y \in K$,

$$\begin{aligned} & \langle N(u_n, v_n), g(x_n) - y \rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \\ & \leq \langle N(u_n, v_n), x_n - y_n \rangle + \phi(g(x_n), x_n) - \phi(y_n - x_n + g(x_n), x_n) \\ & \quad + \left\langle \frac{J_q(g(x_n)) - J_q(y_n - x_n + g(x_n))}{\lambda}, g(x_n) - y \right\rangle + \left\langle \frac{J_q(g(x_n)) - J_q(y_n - x_n + g(x_n))}{\lambda}, y_n - x_n \right\rangle \\ & \leq \|N(u_n, v_n)\| \|x_n - y_n\| + |\phi(g(x_n), x_n) - \phi(y_n - x_n + g(x_n), x_n)| \\ & \quad + \frac{\|J_q(g(x_n)) - J_q(y_n - x_n + g(x_n))\|}{\lambda} \|g(x_n) - y\| + \frac{\|J_q(g(x_n)) - J_q(y_n - x_n + g(x_n))\|}{\lambda} \|y_n - x_n\| \\ & \leq \|N(u_n, v_n)\| \|x_n - y_n\| + |\phi(g(x_n), x_n) - \phi(y_n - x_n + g(x_n), x_n)| \\ & \quad + \frac{1}{2\lambda} \|J_q(g(x_n)) - J_q(y_n - x_n + g(x_n))\|^2 + \frac{1}{2\lambda} \|g(x_n) - y\|^2 \\ & \quad + \frac{\|J_q(g(x_n)) - J_q(y_n - x_n + g(x_n))\|}{\lambda} \|y_n - x_n\| \\ & = \frac{1}{2\lambda} \|g(x_n) - y\|^2 + [\|N(u_n, v_n)\| + \frac{1}{\lambda} \|J_q(g(x_n)) - J_q(y_n - x_n + g(x_n))\|] \|y_n - x_n\| \\ & \quad + |\phi(g(x_n), x_n) - \phi(y_n - x_n + g(x_n), x_n)| + \frac{1}{2\lambda} \|J_q(g(x_n)) - J_q(y_n - x_n + g(x_n))\|^2. \end{aligned} \quad (5.9)$$

Since N is mixed Lipschitz continuous with respect to the first and second arguments, there exists a pair of constants $\zeta, \xi > 0$ such that

$$\|N(x_1, y_1) - N(x_2, y_2)\| \leq \zeta \|x_1 - x_2\| + \xi \|y_1 - y_2\|, \quad \forall (x_i, y_i) \in X \times X, i = 1, 2,$$

which together with $(u_n, v_n) \in F(K) \times E(K)$ and the boundedness of $F(K) \cup E(K)$, implies that $\{N(u_n, v_n)\}$ is bounded. Putting $e_n = y_n - x_n$, from condition (C2) we get

$$\lim_{n \rightarrow \infty} |\phi(y_n - x_n + g(x_n), x_n) - \phi(g(x_n), x_n)| = \lim_{n \rightarrow \infty} |\phi(e_n + g(x_n), x_n) - \phi(g(x_n), x_n)| = 0.$$

Again, by Lemma 5.1 we have

$$\|J_q(y_n - x_n + g(x_n)) - J_q(g(x_n))\| \leq L_q \|y_n - x_n\|^{q-1} \rightarrow 0.$$

It follows from (5.9) that $\{x_n\}$ is a $\text{LP } \frac{1}{\lambda}$ -approximating sequence for $\text{CGMVI}(N(F, E), g, \phi, K)$. If $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly $\text{LP } \frac{1}{\lambda}$ -well-posed in the generalized sense, then $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \rightarrow x^*$ as $i \rightarrow \infty$, where x^* is a solution of $\text{CGMVI}(N(F, E), g, \phi, K)$. By Proposition 5.1, we obtain that x^* is also a solution of $\text{FP}(I - g + \Pi_\lambda^{\phi + \delta_k}(J_q \circ g - \lambda N(F, E)))$,

K). Thus, $\text{FP}(I - g + \Pi_{\lambda}^{\phi+\delta_k}(J_q \circ g - \lambda N(F, E)), K)$ is strongly LP well-posed in the generalized sense.

If $\text{CGMVI}(N(F, E), g, \phi, K)$ is weakly LP $\frac{1}{\lambda}$ -well-posed in the generalized sense, then $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \rightarrow x^*$ as $i \rightarrow \infty$, where x^* is a solution of $\text{CGMVI}(N(F, E), g, \phi, K)$. By Proposition 5.1, we obtain that x^* is also a solution of $\text{FP}(I - g + \Pi_{\lambda}^{\phi+\delta_k}(J_q \circ g - \lambda N(F, E)), K)$. Thus, $\text{FP}(I - g + \Pi_{\lambda}^{\phi+\delta_k}(J_q \circ g - \lambda N(F, E)), K)$ is weakly LP well-posed in the generalized sense. \square

Theorem 5.4. *Let X be a q -uniformly convex Banach space and K be a nonempty, closed and convex subset of X . Let $F, E : X \rightarrow 2^{X^*}$ be nonempty compact-valued mappings which both are \mathcal{H} -uniformly continuous. Let $g : X \rightarrow X$ be a homeomorphic mapping and $N : X^* \times X^* \rightarrow X^*$ be mixed Lipschitz continuous with respect to the first and second arguments. Assume that $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ satisfies the conditions:*

- (C1) *for each $x \in X$, $\phi(\cdot, x)$ is a proper, convex and subdifferentiable functional;*
 (C2) $\|\Pi_{\lambda}^{\phi(\cdot, x)+\delta_k}(v^*) - \Pi_{\lambda}^{\phi(\cdot, y)+\delta_k}(v^*)\| \leq \beta \|g(x) - g(y)\|, \forall x, y \in X, v^* \in X^*$ *for some $\beta > 0$.*

If $\text{FP}(I - g + \Pi_{\lambda}^{\phi+\delta_k}(J_q \circ g - \lambda N(F, E)), K)$ is strongly LP well-posed in the generalized sense, then $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP well-posed in the generalized sense, where $\lambda > 0$ is a constant.

Proof. The conclusion follows from the similar arguments to those in the proof of Theorem 5.2. \square

Remark 5.2. Our Theorems 5.1–5.4 improve, extend and develop Theorems 5.1–5.4 in [19] to a great extent because our $\text{CGMVI}(N(F, E), g, \phi, K)$ and $\text{FP}(I - g + \Pi_{\lambda}^{\phi+\delta_k}(J_q \circ g - \lambda N(F, E)), K)$ are more general than the $\text{GMVI}(F, \phi, K)$ and $\text{FP}(\Pi_{\lambda}^{\phi+\delta_k}(J_q - \lambda F, K)$ in [19], respectively.

6. Conditions for Levitin-Polyak Well-Posedness

In this section, we shall derive some conditions under which a completely generalized mixed variational inequality in Banach spaces is Levitin-Polyak well-posed.

Theorem 6.1. *Let K be a nonempty, compact and convex subset of X . Let $F, E : X \rightarrow 2^{X^*}$ be nonempty compact-valued mappings which both are \mathcal{H} -continuous. Let $g : X \rightarrow X$ be a continuous mapping and $N : X^* \times X^* \rightarrow X^*$ be mixed Lipschitz continuous with respect to the first and second arguments. Assume that $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ satisfies the conditions:*

- (C1) *ϕ is proper, convex and lower semicontinuous in the first argument;*
 (C2) *ϕ is upper semicontinuous in the second argument;*

(C3) $\liminf_{n \rightarrow \infty} \inf_{y \in \text{dom}\phi(\cdot, x_n)} (\phi(y, x_n) - \phi(y, \bar{x})) \geq 0, \quad \forall \{x_n\} \subset K: x_n \rightarrow \bar{x} (n \rightarrow \infty).$

Then $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP well-posed if and only if it has a unique solution.

Proof. The necessity is obvious. For the sufficiency, suppose that $\text{CGMVI}(N(F, E), g, \phi, K)$ has a unique solution x^* . If $\text{CGMVI}(N(F, E), g, \phi, K)$ is not strongly LP well-posed, then there exists a LP approximating sequence $\{x_n\} \subset K$ for $\text{CGMVI}(N(F, E), g, \phi, K)$ such that $\{x_n\}$ is not strongly convergent to x^* . Thus, there exist $w_n \in X$ with $w_n \rightarrow 0$ and $0 < \epsilon_n \rightarrow 0$ such that $g(x_n) + w_n \in K$, and there exist $u_n \in F(x_n)$ and $v_n \in E(x_n)$ such that

$$\langle N(u_n, v_n), g(x_n) - y \rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \leq \epsilon_n, \quad \forall y \in K, n \geq 1. \quad (6.1)$$

Let $\{x_{n_i}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_i} \rightarrow \bar{x}$ as $i \rightarrow \infty$. Clearly $\bar{x} \in K$. Since $\{g(x_{n_i}) + w_{n_i}\} \subset K$ and g is continuous, we know that $g(\bar{x}) \in K$. So, it follows from (6.1), $\epsilon_n \rightarrow 0$ and conditions (C1)–(C3) that

$$\liminf_{i \rightarrow \infty} (\phi(g(x_{n_i}), x_{n_i}) - \phi(g(x_{n_i}), \bar{x})) \geq \liminf_{i \rightarrow \infty} \inf_{y \in \text{dom}\phi(\cdot, x_{n_i})} (\phi(y, x_{n_i}) - \phi(y, \bar{x})) \geq 0,$$

and

$$\begin{aligned} & \liminf_{i \rightarrow \infty} \langle N(u_{n_i}, v_{n_i}), g(x_{n_i}) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \\ & \leq \liminf_{i \rightarrow \infty} \langle N(u_{n_i}, v_{n_i}), g(x_{n_i}) - y \rangle + \liminf_{i \rightarrow \infty} \phi(g(x_{n_i}), \bar{x}) \\ & \quad + \liminf_{i \rightarrow \infty} (\phi(g(x_{n_i}), x_{n_i}) - \phi(g(x_{n_i}), \bar{x})) + \liminf_{i \rightarrow \infty} (-\phi(y, x_{n_i})) \\ & \leq \liminf_{i \rightarrow \infty} \{ \langle N(u_{n_i}, v_{n_i}), g(x_{n_i}) - y \rangle + \phi(g(x_{n_i}), \bar{x}) + \phi(g(x_{n_i}), x_{n_i}) - \phi(g(x_{n_i}), \bar{x}) - \phi(y, x_{n_i}) \} \\ & = \liminf_{i \rightarrow \infty} \{ \langle N(u_{n_i}, v_{n_i}) - y_{n_i}, g(x_{n_i}) - y \rangle + \phi(g(x_{n_i}), x_{n_i}) - \phi(y, x_{n_i}) \} \\ & \leq \liminf_{i \rightarrow \infty} \epsilon_{n_i} = 0. \end{aligned} \quad (6.2)$$

Since F and E are nonempty compact-valued mappings, in terms of Lemma 2.1, for each $u_n \in F(x_n)$ and $v_n \in E(x_n)$ there exist $\bar{u}_n \in F(\bar{x})$ and $\bar{v}_n \in E(\bar{x})$ such that $\|u_n - \bar{u}_n\| \leq \mathcal{H}(F(x_n), F(\bar{x}))$ and $\|v_n - \bar{v}_n\| \leq \mathcal{H}(E(x_n), E(\bar{x}))$. Since F and E both are \mathcal{H} -continuous, one deduces that $\|u_{n_i} - \bar{u}_{n_i}\| \leq \mathcal{H}(F(x_{n_i}), F(\bar{x})) \rightarrow 0$ and $\|v_{n_i} - \bar{v}_{n_i}\| \leq \mathcal{H}(E(x_{n_i}), E(\bar{x})) \rightarrow 0$ as $i \rightarrow \infty$. Since $F(\bar{x})$ and $E(\bar{x})$ both are compact, without loss of generality we may assume that $\bar{u}_{n_i} \rightarrow \bar{u} \in F(\bar{x})$ and $\bar{v}_{n_i} \rightarrow \bar{v} \in E(\bar{x})$ as $i \rightarrow \infty$. Thus, we conclude that

$$\|u_{n_i} - \bar{u}\| \leq \|u_{n_i} - \bar{u}_{n_i}\| + \|\bar{u}_{n_i} - \bar{u}\| \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

and

$$\|v_{n_i} - \bar{v}\| \leq \|v_{n_i} - \bar{v}_{n_i}\| + \|\bar{v}_{n_i} - \bar{v}\| \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

That is, $u_{n_i} \rightarrow \bar{u}$ and $v_{n_i} \rightarrow \bar{v}$ as $i \rightarrow \infty$. Note that N is mixed Lipschitz continuous with respect to the first and second arguments. Hence there exists a pair of constants $\zeta, \xi > 0$ such that

$$\|N(u_{n_i}, v_{n_i}) - N(\bar{u}, \bar{v})\| \leq \zeta \|u_{n_i} - \bar{u}\| + \xi \|v_{n_i} - \bar{v}\| \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

This together with $u_{n_i} \rightarrow \bar{u}$ and $v_{n_i} \rightarrow \bar{v}$, implies that $N(u_{n_i}, v_{n_i}) \rightarrow N(\bar{u}, \bar{v})$ as $i \rightarrow \infty$. So, it immediately follows from (6.2) that

$$\langle N(\bar{u}, \bar{v}), g(\bar{x}) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \leq 0, \quad \forall y \in K.$$

Thus, \bar{x} solves $\text{CGMVI}(N(F, E), g, \phi, K)$. Since $\text{CGMVI}(N(F, E), g, \phi, K)$ has a unique solution x^* , we get $\bar{x} = x^*$. So, $x_n \rightarrow x^*$ as $n \rightarrow \infty$. This leads to a contradiction. Therefore, $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP well-posed. The proof is complete. \square

Let $K \subset X$ and $g : X \rightarrow X$ be a mapping. Now, for any $\delta_0 \geq 0$, we denote $M_g(\delta_0) = \{x \in X : d(g(x), K) \leq \delta_0\}$. We have the following result.

Theorem 6.2. *Let K be a nonempty, closed and convex subset of X . Let $F, E : X \rightarrow 2^{X^*}$ be nonempty compact-valued mappings which both are upper semicontinuous. Let $g : X \rightarrow X$ be a continuous mapping and $N : X^* \times X^* \rightarrow X^*$ be mixed Lipschitz continuous with respect to the first and second arguments. Assume that $\phi : X \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ satisfies the conditions:*

(C1) ϕ is proper, convex and lower semicontinuous in the first argument;

(C2) ϕ is upper semicontinuous in the second argument;

(C3) $\liminf_{n \rightarrow \infty} \inf_{y \in \text{dom}\phi(\cdot, x_n)} (\phi(y, x_n) - \phi(y, \bar{x})) \geq 0, \quad \forall \{x_n\} \subset K : x_n \rightarrow \bar{x} (n \rightarrow \infty)$.

If there exists some $\delta_0 > 0$ such that $M(\delta_0)$ is compact, then $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP α -well-posed in the generalized sense.

Proof. Let $\{x_n\} \subset K$ be a LP α -approximating sequence for $\text{CGMVI}(N(F, E), g, \phi, K)$. Then there exist $0 < \epsilon'_n \rightarrow 0$ and $w_n \in X$ with $w_n \rightarrow 0$ such that

$$g(x_n) + w_n \in K,$$

and there exist $u_n \in F(x_n)$ and $v_n \in E(x_n)$ satisfying

$$\langle N(u_n, v_n), g(x_n) - y \rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \leq \frac{\alpha}{2} \|g(x_n) - y\|^2 + \epsilon'_n, \quad \forall y \in K, n \geq 1. \quad (6.3)$$

Since $g(x_n) + w_n \in K$, then there exists $k_n \in K$ such that $g(x_n) + w_n = k_n$. Thus,

$$d(g(x_n), K) \leq \|g(x_n) - k_n\| = \|w_n\| \rightarrow 0.$$

Setting $\epsilon_n = \max\{\epsilon'_n, \|w_n\|\}$, we can get $d(g(x_n), K) \leq \epsilon_n$. Without loss of generality, we may assume that $\{x_n\} \subset M(\delta_0)$ for n sufficiently large. By the compactness of $M(\delta_0)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $\bar{x} \in M_g(\delta_0)$ such that $x_{n_i} \rightarrow \bar{x}$ as $i \rightarrow \infty$. It is easy to see $\bar{x} \in K$. Furthermore, by the u.s.c. of F and E at \bar{x} and compactness of $F(\bar{x})$ and $E(\bar{x})$, there exist a subsequence $\{(u_{n_i}, v_{n_i})\}$ of $\{(u_n, v_n)\}$ some $(\bar{u}, \bar{v}) \in F(\bar{x}) \times E(\bar{x})$ such that $u_{n_i} \rightarrow \bar{u}$ and $v_{n_i} \rightarrow \bar{v}$. Since N is mixed Lipschitz continuous with respect to the first and second arguments, there exists a pair of constants $\zeta, \xi > 0$ such that as $i \rightarrow \infty$,

$$\|N(u_{n_i}, v_{n_i}) - N(\bar{u}, \bar{v})\| \leq \zeta \|u_{n_i} - \bar{u}\| + \xi \|v_{n_i} - \bar{v}\| \rightarrow 0.$$

Since ϕ satisfies conditions (C1)–(C3), from (6.3) we deduce that for all $y \in K$,

$$\begin{aligned}
 & \langle N(\bar{u}, \bar{v}), g(\bar{x}) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \\
 & \leq \liminf_{i \rightarrow \infty} \langle N(u_{n_i}, v_{n_i}), g(x_{n_i}) - y \rangle + \liminf_{i \rightarrow \infty} \phi(g(x_{n_i}), \bar{x}) \\
 & \quad + \liminf_{i \rightarrow \infty} (\phi(g(x_{n_i}), x_{n_i}) - \phi(g(x_{n_i}), \bar{x})) + \liminf_{i \rightarrow \infty} (-\phi(y, x_{n_i})) \\
 & \leq \liminf_{i \rightarrow \infty} \{ \langle N(u_{n_i}, v_{n_i}), g(x_{n_i}) - y \rangle + \phi(g(x_{n_i}), \bar{x}) + \phi(g(x_{n_i}), x_{n_i}) - \phi(g(x_{n_i}), \bar{x}) - \phi(y, x_{n_i}) \} \\
 & = \liminf_{i \rightarrow \infty} \{ \langle N(u_{n_i}, v_{n_i}), g(x_{n_i}) - y \rangle + \phi(g(x_{n_i}), x_{n_i}) - \phi(y, x_{n_i}) \} \\
 & \leq \liminf_{i \rightarrow \infty} \left\{ \frac{\alpha}{2} \|g(x_{n_i}) - y\|^2 + \epsilon'_{n_i} \right\} \\
 & = \frac{\alpha}{2} \|g(\bar{x}) - y\|^2,
 \end{aligned}$$

That is,

$$\langle N(\bar{u}, \bar{v}), g(\bar{x}) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \leq \frac{\alpha}{2} \|g(\bar{x}) - y\|^2, \quad \forall y \in K. \quad (6.4)$$

For any $y \in K$, put $y_t = g(\bar{x}) + t(y - g(\bar{x}))$ for all $t \in (0, 1)$, it is easy to see $y_t \in K$. Now, utilizing (6.4), one has

$$\langle N(\bar{u}, \bar{v}), g(\bar{x}) - y_t \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y_t, \bar{x}) \leq \frac{\alpha}{2} \|g(\bar{x}) - y_t\|^2.$$

By the convexity of ϕ in the first argument, we conclude that for each $t \in (0, 1)$, one has

$$\langle N(\bar{u}, \bar{v}), g(\bar{x}) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \leq \frac{\alpha t}{2} \|g(\bar{x}) - y\|^2, \quad \forall y \in K.$$

Letting $t \rightarrow 0^+$ in the last inequality, we have

$$\langle N(\bar{u}, \bar{v}), g(\bar{x}) - y \rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \leq 0, \quad \forall y \in K.$$

This shows that \bar{x} solves $\text{CGMVI}(N(F, E), g, \phi, K)$. Thus, $\text{CGMVI}(N(F, E), g, \phi, K)$ is strongly LP α -well-posed in the generalized sense. The proof is complete. \square

Remark 6.1. Our Theorems 6.1–6.2 improve, extend and develop Theorems 6.1–6.2 in [19] to a great extent because our $\text{CGMVI}(N(F, E), g, \phi, K)$ is more general than the $\text{GMVI}(F, \phi, K)$ in [19].

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