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ON ALMOST KENMOTSU MANIFOLDS WITH NULLITY DISTRIBUTIONS

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Abstract. The object of this paper is to characterize the curvature conditions $R\cdot P=0$ and $P\cdot S=0$ with its characteristic vector field ξ belonging to the $(k,\mu)'$-nullity distribution and (k,μ)-nullity distribution respectively, where P is the Weyl projective curvature tensor. As a consequence of the main results we obtain several corollaries.

1. Introduction

In the present time the study of nullity distributions has become very interesting topic in Differential Geometry. Gray [7] and Tanno [12] introduced the notion of k-nullity distribution ($k\in\mathbb{R}$) in the study of Riemannian manifolds (M,g), which is defined for any $p\in M$ and $k\in\mathbb{R}$ as follows:

$$N_p(k)=\{Z\in T_pM : R(X,Y)Z=k[g(Y,Z)X-g(X,Z)Y]\}, \tag{1.1}$$

for any $X,Y\in T_pM$, where T_pM denotes the tangent vector space of M at any point $p\in M$ and R denotes the Riemannian curvature tensor of type $(1,3)$.

Next Blair, Koufogiorgos and Papantoniou [3] introduced the (k,μ)-nullity distribution which is a generalized notion of the k-nullity distribution on a contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ and defined for any $p\in M^{2n+1}$ and $k,\mu\in\mathbb{R}$ as follows:

$$\begin{aligned}N_p(k,\mu)=\{Z\in T_pM^{2n+1} : R(X,Y)Z &= k[g(Y,Z)X-g(X,Z)Y]\\ &+\mu[g(Y,Z)hX-g(X,Z)hY]\},\end{aligned} \tag{1.2}$$

where $h=\frac{1}{2}\pounds_\xi\phi$ and $\pounds$ denotes the Lie differentiation.

In [5], Dileo and Pastore introduced the notion of $(k,\mu)'$-nullity distribution, another generalized notion of the k-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$, which is defined for any $p\in M^{2n+1}$ and $k,\mu\in\mathbb{R}$ as follows:

$$N_p(k,\mu)'=\{Z\in T_pM^{2n+1} : R(X,Y)Z = k[g(Y,Z)X-g(X,Z)Y]$$

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$$+\mu[g(Y,Z)h'X - g(X,Z)h'Y]\}, \tag{1.3}$$

where $h' = h \circ \phi$.

Also, Kenmotsu [9] introduced a new type of almost contact metric manifolds named Kenmotsu manifolds nowadays. A differentiable $(2n+1)$-dimensional manifold M is said to have a (ϕ,ξ,η)-structure or an almost contact structure, if it admits a $(1,1)$ tensor field ϕ, a characteristic vector field ξ and a 1-form η satisfying [1, 2]

$$\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1, \tag{1.4}$$

where I denote the identity endomorphism. Here we include also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (1.4).

If a manifold M with a (ϕ,ξ,η)-structure admits a Riemannian metric g such that

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X and Y of T_pM^{2n+1}, then M is said to have an almost contact metric structure (ϕ,ξ,η,g). The fundamental 2-form Φ is defined by $\Phi(X,Y) = g(X,\phi Y)$ for any vector fields X, Y of T_pM^{2n+1}. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by $N_\phi = [\phi,\phi] + 2d\eta \otimes \xi$, where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ [1]. A normal almost Kenmotsu manifold is a Kenmotsu manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. Also Kenmotsu manifolds can be characterized by $(\nabla_X\phi)Y = g(\phi X,Y)\xi - \eta(Y)\phi X$ for any vector fields X, Y. It is well known [9] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$, where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by $\mathcal{D}$ and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathcal{D}$ is an integrable distribution.

A Riemannian manifold (M^{2n+1}, g) is called locally symmetric if its curvature tensor R is parallel, that is, $\nabla R = 0$, where ∇ is the Levi-Civita connection. The notion of semisymmetric manifold, a proper generalization of locally symmetric manifold, is defined by $R(X,Y)\cdot R = 0$, where $R(X,Y)$ is considered as a field of linear operators, acting on R. A complete intrinsic classification of these manifolds was given by Szabó in [11]. In a recent paper [8] Jun, De and Pathak studied Weyl semisymmetric Kenmotsu manifolds.

Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well-known Weyl projective curvature tensor P vanishes. Here P is defined by [10]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{1.5}$$

for all $X, Y, Z \in T_pM$, where R is the curvature tensor and S is the Ricci tensor of type $(0,2)$ of M. In fact, M is Weyl projectively flat if and only if the manifold is of constant curvature [17]. Thus the Weyl projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature. A Riemannian manifold is said to be Weyl projective semisymmetric if the curvature tensor P satisfies $R(X,Y)\cdot P = 0$.

In [4], Dileo and Pastore studied locally symmetric almost Kenmotsu manifolds. We refer the reader to ([4],[5],[6]) for more related results on $(k,\mu)'$-nullity distribution and (k,μ)-nullity distribution on almost Kenmotsu manifolds. In recent papers ([13],[14],[15],[16]) Wang and Liu studied almost Kenmotsu manifolds with nullity distributions. In [14], Wang and Liu studied ξ-Riemannian semisymmetric almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution and (k,μ)-nullity distribution.

Motivated by the above studies we study Weyl projective semisymmetric $(R\cdot P = 0)$ and the curvature condition $P\cdot S = 0$ in an almost Kenmotsu manifolds with nullity distributions.

The paper is organized as follows:

Section 2 focuses on almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution and ξ belonging to the (k,μ)-nullity distribution. In sections 3 and 4 we study Weyl projective semisymmetric almost Kenmotsu manifolds and almost Kenmotsu manifolds satisfying the curvature condition $P\cdot S = 0$ with characteristic vector field ξ belonging to the $(k,\mu)'$-nullity distribution and (k,μ)-nullity distribution respectively. As a consequence of the main results we obtain several corollaries.

2. Almost Kenmotsu manifolds

Let M^{2n+1} be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\pounds_\xi\phi$ and $l = R(\cdot,\xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations [4]

$$h\xi = 0,\ l\xi = 0,\ tr(h) = 0,\ tr(h\phi) = 0,\ h\phi + \phi h = 0. \tag{2.1}$$

Moreover, we have the following results [4, 5]

$$\nabla_X\xi = -\phi^2X - \phi hX (\Rightarrow \nabla_\xi\xi = 0), \tag{2.2}$$

$$\phi l\phi - l = 2(h^2 - \phi^2), \tag{2.3}$$

$$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y\phi h)X - (\nabla_X\phi h)Y, \tag{2.4}$$

for any vector fields X, Y. The $(1,1)$-type symmetric tensor field $h' = h \circ \phi$ is anticommuting with ϕ and $h'\xi = 0$. Also it is clear that

$$h = 0 \Leftrightarrow h' = 0, \;\; h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2), \tag{2.5}$$

which holds on $(k,\mu)'$-almost Kenmotsu manifold.

3. ξ belongs to the $(k,\mu)'$-nullity distribution

This section is devoted to study of almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution. Let $X \in \mathscr{D}$ be the eigen vector of h' corresponding to the eigen value λ. Then from (2.5) it is clear that $\lambda^2 = -(k+1)$, a constant. Therefore $k \le -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote by $[\lambda]'$ and $[-\lambda]'$ the corresponding eigenspaces related to the non-zero eigen value λ and $-\lambda$ of h', respectively. Before presenting our main theorems we recall some results:

Lemma 3.1 (Prop. 4.1 and Prop. 4.3 of [5]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and Spec $(h') = \{0, \lambda, -\lambda\}$, with 0 as simple eigen value and $\lambda = \sqrt{-k-1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvature are given as following:*

(a) *$K(X,\xi) = k - 2\lambda$ if $X \in [\lambda]'$ and*
$K(X,\xi) = k + 2\lambda$ if $X \in [-\lambda]'$,

(b) *$K(X,Y) = k - 2\lambda$ if $X, Y \in [\lambda]'$;*
$K(X,Y) = k + 2\lambda$ if $X, Y \in [-\lambda]'$ and $K(X,Y) = -(k+2)$ if $X \in [\lambda]'$, $Y \in [-\lambda]'$,

(c) *M^{2n+1} has constant negative scalar curvature $r = 2n(k-2n)$.*

Lemma 3.2 (Lemma 3 of [15]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. If $n > 1$, then the Ricci operator Q of M^{2n+1} is given by*

$$Q = -2n\,id + 2n(k+1)\eta \otimes \xi - 2nh'. \tag{3.1}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

Lemma 3.3 (Proposition 4.2 of [5]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $h' \neq 0$ and ξ belongs to the $(k,-2)'$-nullity distribution. Then for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor satisfies:*

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = 0,$$

$$
\begin{aligned}
R(X_{-\lambda}, Y_{-\lambda})Z_{\lambda} &= 0,\\
R(X_{\lambda}, Y_{-\lambda})Z_{\lambda} &= (k+2)g(X_{\lambda}, Z_{\lambda})Y_{-\lambda},\\
R(X_{\lambda}, Y_{-\lambda})Z_{-\lambda} &= -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_{\lambda},\\
R(X_{\lambda}, Y_{\lambda})Z_{\lambda} &= (k-2\lambda)[g(Y_{\lambda}, Z_{\lambda})X_{\lambda} - g(X_{\lambda}, Z_{\lambda})Y_{\lambda}],\\
R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= (k+2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].
\end{aligned}
$$

From (1.3) we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{3.2}$$

where $k, \mu \in \mathbb{R}$. Also we get from (3.2)

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X]. \tag{3.3}$$

Contracting Y in (3.2) we have

$$S(X,\xi) = 2nk\eta(X). \tag{3.4}$$

By applying the above results and Lemma 3.2 we obtain from (1.5)

$$P(\xi,Y)Z = (k+1)g(Y,Z)\xi - g(h'Y,Z)\xi + 2\eta(Z)h'Y - (k+1)\eta(Y)\eta(Z)\xi \tag{3.5}$$

for all vector fields Y, Z on M.

Using the above results we can present our main theorem as follows:

Theorem 3.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)(n > 1)$ be an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. If the manifold M^{2n+1} is Weyl projective semisymmetric then the manifold is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof. We suppose that the manifold M^{2n+1} is Weyl projective semisymmetric, that is, $R \cdot P = 0$. Then $(R(X,Y)\cdot P)(U,V)W = 0$ for all vector fields X, Y, U, V, W, which implies

$$R(X,Y)P(U,V)W - P(R(X,Y)U,V)W - P(U,R(X,Y)V)W - P(U,V)R(X,Y)W = 0. \tag{3.6}$$

Setting $X = U = \xi$ in (3.6) we have,

$$R(\xi,Y)P(\xi,V)W - P(R(\xi,Y)\xi,V)W - P(\xi,R(\xi,Y)V)W - P(\xi,V)R(\xi,Y)W = 0. \tag{3.7}$$

Making use of (3.3) and (3.5) we get

$$
\begin{aligned}
R(\xi,Y)P(\xi,V)W = {} & k[g(Y,P(\xi,V)W)\xi - \eta(P(\xi,V)W)Y]\\
& -2[g(h'Y,P(\xi,V)W)\xi - \eta(P(\xi,V)W)h'Y]
\end{aligned}
$$

$$
\begin{aligned}
&= k\{(k+1)g(V,W)\eta(Y)\xi - g(h'V,W)\eta(Y)\xi + 2\eta(W)g(Y,h'V)\xi \\
&\quad -(k+1)\eta(V)\eta(W)\eta(Y)\xi - (k+1)g(V,W)Y + g(h'V,W)Y \\
&\quad +(k+1)\eta(V)\eta(W)Y\} - 2\{2\eta(W)g(h'Y,h'V)\xi \\
&\quad -(k+1)g(V,W)h'Y + g(h'V,W)h'Y + (k+1)\eta(V)\eta(W)h'Y\} \qquad (3.8)
\end{aligned}
$$

for any vector fields Y, V, W on M^{2n+1}.

With the help of (3.3) and (3.5) we obtain

$$
\begin{aligned}
P(R(\xi,Y)\xi,V)W &= k\eta(Y)P(\xi,V)W - kP(Y,V)W + 2P(h'Y,V)W \\
&= k(k+1)g(V,W)\eta(Y)\xi - kg(h'V,W)\eta(Y)\xi + 2k\eta(Y)\eta(W)h'V \\
&\quad -k(k+1)\eta(Y)\eta(V)\eta(W)\xi - kP(Y,V)W + 2P(h'Y,V)W \qquad (3.9)
\end{aligned}
$$

for any vector fields Y, V, W on M^{2n+1}.

Similarly, it follows from (3.3) and (3.5) that

$$
\begin{aligned}
P(\xi,R(\xi,Y)V)W &= -k\eta(V)P(\xi,Y)W + 2\eta(V)P(\xi,h'Y)W \\
&= -k(k+1)g(Y,W)\eta(V)\xi + kg(h'Y,W)\eta(V)\xi - 2k\eta(V)\eta(W)h'Y \\
&\quad +2(k+1)g(h'Y,W)\eta(V)\xi + 2(k+1)g(Y,W)\eta(V)\xi \\
&\quad -4(k+1)\eta(V)\eta(W)Y + (k+1)(k+2)\eta(Y)\eta(V)\eta(W)\xi \qquad (3.10)
\end{aligned}
$$

for any vector fields Y, V, W on M^{2n+1}.

Again using (3.3) and (3.5) we obtain

$$
\begin{aligned}
P(\xi,V)R(\xi,Y)W &= k(k+1)g(Y,W)\eta(V)\xi - k(k+1)g(Y,V)\eta(W)\xi \\
&\quad +2(k+1)g(h'Y,V)\eta(W)\xi + kg(h'V,Y)\eta(W)\xi - 2g(h'V,h'Y)\eta(W)\xi \\
&\quad +2kg(Y,W)h'V - 2k\eta(Y)\eta(W)h'V - 4g(h'Y,W)h'V \\
&\quad -k(k+1)g(Y,W)\eta(V)\xi + k(k+1)\eta(Y)\eta(W)\eta(V)\xi \qquad (3.11)
\end{aligned}
$$

for any vector fields Y, V, W on M^{2n+1}.

Finally, using (3.8)–(3.11) we have from (3.7)

$$
\begin{aligned}
&kP(Y,V)W - 2P(h'Y,V)W + kg(h'V,Y)\eta(W)\xi + 2(k+1)g(V,W)h'Y \\
&-k(k+1)g(V,W)Y + kg(h'V,W)Y + (k^2+5k+4)\eta(V)\eta(W)Y \\
&-2g(h'Y,h'V)\eta(W)\xi - 2(k+1)^2\eta(Y)\eta(V)\eta(W)\xi \\
&-2g(h'V,W)h'Y - 2\eta(V)\eta(W)h'Y + (k^2-k-2)g(Y,W)\eta(V)\xi \\
&-(3k+2)g(h'Y,W)\eta(V)\xi + k(k+1)g(Y,V)\eta(W)\xi
\end{aligned}
$$

$$-2(k+1)g(h'Y,V)\eta(W)\xi - 2kg(Y,W)h'V + 4g(h'Y,W)h'V = 0 \tag{3.12}$$

for any vector fields Y, V, W on M^{2n+1}. Letting $Y, W \in [\lambda]'$ and $V \in [-\lambda]'$ and applying Lemma 3.3 we have

$$P(Y,V)W = (k+1-\lambda)g(Y,W)V \text{ and } P(h'Y,V)W = (\lambda+1)(k+1)g(Y,W)V. \tag{3.13}$$

By using (3.13) and noticing $Y, W \in [\lambda]'$ and $V \in [-\lambda]'$ it follows from (3.12) that

$$[k(k+1-\lambda) - 2(\lambda+1)(k+1) + 2\lambda k - 4\lambda^2]g(Y,W)V = 0. \tag{3.14}$$

Using the relationship $\lambda = \pm\sqrt{-k-1}$ in (3.14) we get

$$\lambda(\lambda+1)^2(\lambda-1) = 0. \tag{3.15}$$

If $\lambda = 0$, then $k = -1$ and consequently from (2.5) $h' = 0$, which contradicts our hypothesis $h' \neq 0$. Then it follows from (3.15) that $\lambda^2 = 1$ and hence $k = -2$. Without losing generality we may choose $\lambda = 1$. Then we can write from Lemma 3.3

$$R(X_\lambda, Y_\lambda)Z_\lambda = -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],$$
$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0$$

for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also it follows from Lemma 3.1 that $K(X,\xi) = -4$ for any $X \in [\lambda]'$ and $K(X,\xi) = 0$ for any $X \in [-\lambda]'$. Again from Lemma 3.1 we see that $K(X,Y) = -4$ for any $X, Y \in [\lambda]'$; $K(X,Y) = 0$ for any $X, Y \in [-\lambda]'$ and $K(X,Y) = 0$ for any $X \in [\lambda]', Y \in [-\lambda]'$. As is shown in [5] that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. This completes the proof of our theorem. □

Since $R \cdot R = 0$ implies $R \cdot P = 0$, we have the following:

Corollary 3.1. *A semisymmetric almost Kenmotsu manifold $M^{2n+1}(n > 1)$ with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$ is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

The above corollary have been proved by Wang and Liu [14].

Next we consider an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$ satisfying the curvature condition $P \cdot S = 0$. Then $(P(X,Y)\cdot S)(U,V) = 0$ for all vector fields X, Y, U, V, which implies

$$S(P(X,Y)U,V) + S(U,P(X,Y)V) = 0 \tag{3.16}$$

for any vector fields X, Y, U, V on M^{2n+1}.

Putting $X = U = \xi$ in (3.16) we have,

$$S(P(\xi,Y)\xi,V) + S(\xi,P(\xi,Y)V) = 0. \tag{3.17}$$

Making use of (3.4) and (3.5) the above equation implies

$$S(h'Y,V) + nk(k+1)g(Y,V) - nkg(h'Y,V) - nk(k+1)\eta(Y)\eta(V) = 0 \tag{3.18}$$

for any vector fields Y, V on M^{2n+1}.

Substituting $Y = h'Y$ in (3.18) and using (2.5) we obtain

$$(k+1)\{-S(Y,V) + nk\eta(Y)\eta(V) + nkg(h'Y,V) + nkg(Y,V)\} = 0 \tag{3.19}$$

for any vector fields Y, V on M^{2n+1}.

Again from Lemma 3.2 we have

$$S(Y,V) = -2ng(Y,V) + 2n(k+1)\eta(Y)\eta(V) - 2ng(h'Y,V) \tag{3.20}$$

for any vector fields Y, V on M^{2n+1}.

Making use of (3.20) we obtain from (3.19)

$$(k+1)(k+2)\{g(Y,V) + g(h'Y,V) - \eta(Y)\eta(V)\} = 0. \tag{3.21}$$

Letting $Y, V \in [\lambda]'$ in (3.21) implies that

$$(k+1)(k+2)(1+\lambda)g(Y,V) = 0. \tag{3.22}$$

Using the relation $\lambda = \pm\sqrt{-k-1}$ in (3.22) we have

$$\lambda^2(\lambda+1)^2(\lambda-1) = 0. \tag{3.23}$$

Suppose $\lambda = 0$, then $k = -1$ and hence it follows from (2.5) that $h' = 0$, which contradicts our hypothesis $h' \neq 0$. Then from (3.23) we have $\lambda^2 = 1$ and hence $k = -2$. Without losing the generality, we may choose $\lambda = 1$. Then we can write from Lemma 3.3

$$R(X_\lambda, Y_\lambda)Z_\lambda = -4[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda],$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0$$

for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$. Also it follows from Lemma 3.1 that $K(X,\xi) = -4$ for any $X \in [\lambda]'$ and $K(X,\xi) = 0$ for any $X \in [-\lambda]'$. Again from Lemma 3.1 we see that $K(X,Y) = -4$ for any $X,Y \in [\lambda]'$; $K(X,Y) = 0$ for any $X,Y \in [-\lambda]'$ and $K(X,Y) = 0$ for any $X \in [\lambda]', Y \in [-\lambda]'$. As is shown in [5] that the distribution $[\xi] \oplus [\lambda]'$ is integrable with totally geodesic leaves and the distribution $[-\lambda]'$ is integrable with totally umbilical leaves by $H = -(1-\lambda)\xi$, where H is the mean curvature vector field for the leaves of $[-\lambda]'$ immersed in M^{2n+1}. Here $\lambda = 1$, then two orthogonal distributions $[\xi] \oplus [\lambda]'$ and $[-\lambda]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. By the above discussions we can state the following:

Theorem 3.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)(n > 1)$ be an almost Kenmotsu manifold with ξ belonging to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. If the manifold satisfies the curvature condition $P \cdot S = 0$, then the manifold is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

4. ξ belongs to the (k,μ)-nullity distribution

In this section we deal with almost Kenmotsu manifolds of which ξ belonging to the (k,μ)-nullity distribution.

From (1.2) we obtain

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{4.1}$$

where $k,\mu \in \mathbb{R}$. Before proving our main results in this section we state the following:

Lemma 4.1 (Theorem 4.1 of [5]). *Let M be an almost Kenmotsu manifold of dimension $2n+1$. Suppose that the characteristic vector field ξ belonging to the (k,μ)-nullity distribution. Then $k = -1$, $h = 0$ and M is locally a warped product of an open interval and an almost Kähler manifold.*

In view of Lemma 4.1 it follows from (4.1) that

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{4.2}$$

$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{4.3}$$

$$S(X,\xi) = -2n\eta(X) \tag{4.4}$$

for any vector fields X, Y on M^{2n+1}.

Applying (4.3) and (4.4) in (1.5) we have the following

$$P(\xi,Y)Z = -g(Y,Z)\xi - \frac{1}{2n}S(Y,Z)\xi \tag{4.5}$$

for any vector fields Y,Z on M^{2n+1}. We can state our main theorem as follows:

Theorem 4.1. *An almost Kenmotsu manifold* $(M^{2n+1},\phi,\xi,\eta,g)$ *with* ξ *belonging to the* (k,μ)*-nullity distribution is Weyl projective semisymmetric if and only if the manifold is of constant curvature* -1.

Proof. Let M^{2n+1} be a Weyl projective semisymmetric almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution. Therefore $(R(X,Y)\cdot P)(U,V)W = 0$ for all vector fields X,Y,U,V,W, which implies

$$R(X,Y)P(U,V)W - P(R(X,Y)U,V)W - P(U,R(X,Y)V)W - P(U,V)R(X,Y)W = 0. \tag{4.6}$$

Substituting $X = U = \xi$ in (4.6) we obtain

$$R(\xi,Y)P(\xi,V)W - P(R(\xi,Y)\xi,V)W - P(\xi,R(\xi,Y)V)W - P(\xi,V)R(\xi,Y)W = 0. \tag{4.7}$$

Making use of (4.3) and (4.5) we have

$$\begin{aligned} R(\xi,Y)P(\xi,V)W &= g(V,W)\eta(Y)\xi + \frac{1}{2n}S(V,W)\eta(Y)\xi \\ &\quad - g(V,W)Y - \frac{1}{2n}S(V,W)Y \end{aligned} \tag{4.8}$$

for any vector field Y,V,W on M^{2n+1}.

Similarly using (4.3) and (4.5) we obtain

$$P(R(\xi,Y)\xi,V)W = P(Y,V)W + g(V,W)\eta(Y)\xi + \frac{1}{2n}S(V,W)\eta(Y)\xi \tag{4.9}$$

for any vector field Y,V,W on M^{2n+1}.

Again, it follows from (4.3) and (4.5) that

$$P(\xi,R(\xi,Y)V)W = -g(Y,W)\eta(V)\xi - \frac{1}{2n}S(Y,W)\eta(V)\xi \tag{4.10}$$

for any vector field Y,V,W on M^{2n+1}.

Finally, using (4.3) and (4.5) we have

$$P(\xi,V)R(\xi,Y)W = -g(V,Y)\eta(W)\xi - \frac{1}{2n}S(V,Y)\eta(W)\xi \tag{4.11}$$

for any vector field Y,V,W on M^{2n+1}.

Substituting (4.8)–(4.11) into (4.7) gives

$$P(Y,V)W = -g(V,W)Y - \frac{1}{2n}S(V,W)Y + g(Y,W)\eta(V)\xi + \frac{1}{2n}S(Y,W)\eta(V)\xi + g(V,Y)\eta(W)\xi + \frac{1}{2n}S(V,Y)\eta(W)\xi \tag{4.12}$$

for any vector field Y,V,W on M^{2n+1}.

In view of (1.5) and (4.12) we obtain

$$R(Y,V)W = -g(V,W)Y + g(Y,W)\eta(V)\xi + \frac{1}{2n}S(Y,W)\eta(V)\xi + g(V,Y)\eta(W)\xi + \frac{1}{2n}S(V,Y)\eta(W)\xi - \frac{1}{2n}S(Y,W)V. \tag{4.13}$$

Contracting Y in (4.13) it follows that

$$S(V,W) = -2ng(V,W) \tag{4.14}$$

for any vector field V,W on M^{2n+1}.

Taking account of (4.14) we have from (4.13)

$$R(Y,V)W = -[g(V,W)Y - g(Y,W)V], \tag{4.15}$$

that is, the manifold is of constant curvature -1.

Conversely, if the manifold is of constant curvature -1 then obviously Weyl projective semisymmetry follows. This completes the proof. □

Since $R\cdot R = 0$ implies $R\cdot P = 0$, we have the following:

Corollary 4.1. *An almost Kenmotsu manifold* $(M^{2n+1},\phi,\xi,\eta,g)$ *with* ξ *belonging to the* (k,μ)*-nullity distribution is semisymmetric if and only if the manifold is of constant curvature* -1.

The above corollary have been proved by Wang and Liu [14].

Let M^{2n+1} be an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution satisfying the curvature condition $P\cdot S = 0$. Then $(P(X,Y)\cdot S)(U,V) = 0$ for all vector fields X,Y,U,V, which implies

$$S(P(X,Y)U,V) + S(U,P(X,Y)V) = 0 \tag{4.16}$$

for any vector fields X,Y,U,V on M^{2n+1}.

Setting $X = U = \xi$ in (4.16) we have,

$$S(P(\xi,Y)\xi,V) + S(\xi,P(\xi,Y)V) = 0. \tag{4.17}$$

Using (4.4) and (4.5) we obtain from (4.17)

$$\eta(P(\xi,Y)V)=0. \tag{4.18}$$

In view of (4.5) and (4.18) it follows that

$$S(Y,V)=-2ng(Y,V), \tag{4.19}$$

which implies that the manifold is an Einstein manifold.

Conversely, let the manifold be an Einstein manifold of the form (4.19). Then it is obvious that $P\cdot S=0$. This leads to the following:

Theorem 4.2. *An almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (k,μ)-nullity distribution satisfies the curvature condition $P\cdot S=0$ if and only if the manifold is an Einstein one.*

References

[1] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes on Mathematics, Springer, Berlin, **509**, 1976.

[2] D. E. Blair, *Riemannian geometry on contact and symplectic manifolds*, Progr. Math., **203**, Birkhäuser, 2010.

[3] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel. J. Math. **91** (1995), 189–214.

[4] G. Dileo and A. M. Pastore, *Almost Kenmotsu manifolds and local symmetry,* Bull. Belg. Math. Soc. Simon Stevin, **14**(2007), 343–354.

[5] G. Dileo and A. M. Pastore, *Almost Kenmotsu manifolds and nullity distributions,* J. Geom. **93**(2009), 46–61.

[6] G. Dileo and A. M. Pastore, *Almost Kenmotsu manifolds with a condition of η-parallelism,* Differential Geom. Appl. **27**(2009), 671–679.

[7] A. Gray, *Spaces of constancy of curvature operators,* Proc. Amer. Math. Soc., **17**(1966), 897–902.

[8] J.-B. Jun, U.C. De and G. Pathak, *On Kenmotsu manifolds,* J. Korean Math. Soc. **42**(2005), 435–445.

[9] K. Kenmotsu, *A class of almost contact Riemannian manifolds,* Tohoku Math. J. **24**(1972), 93–103.

[10] G. Sooś, *Über die geodätischen Abbildungen von Riemannaschen Räumen auf projektiv symmetrische Riemannsche Räume,* Acta. Math. Acad. Sci. Hungar., **9**(1958), 359–361.

[11] Z. I. Szabó, *Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R=0$, the local version,* J. Diff. Geom., **17**(1982), 531–582.

[12] S. Tanno, *Some differential equations on Riemannian manifolds,* J. Math. Soc. Japan, **30**(1978), 509–531.

[13] Y. Wang and X. Liu, *Second order parallel tensors on almost Kenmotsu manifolds satisfying the nullity distributions,* Filomat **28**(2014), 839–847.

[14] Y. Wang and X. Liu, *Riemannian semisymmetric almost Kenmotsu manifolds and nullity distributions,* Ann. Polon. Math. **112**(2014), 37–46.

[15] Y. Wang and X. Liu, *On ϕ-recurrent almost Kenmotsu manifolds,* Kuwait J. Sci. **42**(2015), 65–77.

[16] Y. Wang and X. Liu, *On a type of almost Kenmotsu manifolds with harmonic curvature tensors,* Bull. Belg. Math. Soc. Simon Stevin **22**(2015), 15–24.

[17] K. Yano and S. Bochner, *Curvature and Betti numbers,* Ann. Math. Stud. **32**(1953).

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