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# APPROXIMATION OF GENERALIZED RIEMANN SOLUTIONS TO COMPRESSIBLE EULER-POISSON EQUATIONS OF ISOTHERMAL FLOWS IN SPHERICALLY SYMMETRIC SPACE-TIMES

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**Abstract**. In this paper, we consider the compressible Euler-Poisson system in spherically symmetric space-times. This system, which describes the conservation of mass and momentum of physical quantity with attracting gravitational potential, can be written as a $3\times 3$ mixed-system of partial differential systems or a $2\times 2$ hyperbolic system of balance laws with $global$ source. We show that, by the equation for the conservation of mass, Euler-Poisson equations can be transformed into a standard $3\times 3$ hyperbolic system of balance laws with $local$ source. The generalized approximate solutions to the Riemann problem of Euler-Poisson equations, which is the building block of generalized Glimm scheme for solving initial-boundary value problems, are provided as the superposition of Lax's type weak solutions of the associated homogeneous conservation laws and the perturbation terms solved by the linearized hyperbolic system with coefficients depending on such Lax solutions.

## 1. Introduction

In this paper, we consider the following compressible Euler-Poisson equations of isothermal flows in spherically symmetric space-times:

$$\begin{cases} \rho_t + (\rho u)_x = -\frac{2}{x}\rho u, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\frac{2}{x}\rho u^2 - \rho\phi_x, \\ \frac{1}{x^2}(x^2\phi_x)_x = k\rho, \quad k<0, \end{cases} \tag{1.1}$$

where $\rho, u$ are, respectively, the density, velocity of gas, $\phi$ is the gravitational potential. The pressure $P = P(\rho)$ for the isothermal flow satisfies

$$P = \sigma^2\rho, \tag{1.2}$$

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where $\sigma$ is the constant sound speed. System (1.1) describing the conservation of mass, momentum for the attractive gravitational potential is a $3\times 3$ mixed system of partial differential equations. From the third equation of (1.1), it gives $\phi_x$ as an integral form of $\rho$, which follows that system (1.1) can also be written as a $2\times 2$ hyperbolic integro-differential system with global source. The global existence, uniqueness and behavior of solutions to initial-boundary value problems of (1.1) have been the important issues in physics. One always apply the generalized Glimm scheme to establish the global existence of weak solutions. Therefore, the construction of approximate generalized solutions for Riemann problem, which is the building block of generalized Glimm scheme, become crucial for the global existence result. Therefore, the goal of this article is to provide a new version of generalized Riemann solutions for the initial value problems of (1.1).

To write system (1.1) into a compact form, we multiply the system by $x^2$ and set

$$\hat{\rho} \equiv x^2\rho, \quad \hat{m} \equiv x^2\rho u, \quad \hat{Q} \equiv x^2\phi_x. \tag{1.3}$$

Then, (1.1) can be re-formulated as the following $3\times 3$ mixed system of balance laws:

$$\hat{\rho}_t + \hat{m}_x = 0, \tag{1.4a}$$

$$\hat{m}_t + \left(\frac{\hat{m}^2}{\hat{\rho}} + \sigma^2\hat{\rho}\right)_x = \frac{2\sigma^2}{x}\hat{\rho} - \frac{\hat{\rho}\hat{Q}}{x^2}, \tag{1.4b}$$

$$\hat{Q}_x = k\hat{\rho}, \tag{1.4c}$$

or equivalently

$$\begin{cases} \hat{U} + f(\hat{U})_x = g(x,\hat{U},\hat{Q}), \\ \hat{Q}_x = k\hat{\rho}, \end{cases} \tag{1.5}$$

where

$$\hat{U} \equiv \begin{bmatrix} \hat{\rho} \\ \hat{m} \end{bmatrix}, \quad f(\hat{U}) \equiv \begin{bmatrix} \hat{m} \\ \frac{\hat{m}^2}{\rho} + \sigma^2\hat{\rho} \end{bmatrix}, \quad g(x,\hat{U},\hat{Q}) \equiv \begin{bmatrix} 0 \\ \frac{2\sigma^2}{x}\hat{\rho} - \frac{\hat{\rho}\hat{Q}}{x^2} \end{bmatrix}. \tag{1.6}$$

Note that system (1.5) is also called a degenerate hyperbolic balance laws due to the missing of $\hat{Q}_t$. The initial-boundary value problem of (1.5) is given as

$$\begin{cases} \hat{U}_t + f(\hat{U})_x = g(x,\hat{U},\hat{Q}), & \\ \hat{Q}_x = k\hat{\rho}, & x > x_B, \quad t > 0, \\ \hat{U}(x,0) = \hat{U}_0(x), & x \ge x_B, \\ \hat{\rho}(x_B,t) = \hat{\rho}_B(t), & t \ge 0, \\ \hat{Q}(x_B,t) = \hat{Q}_B(t), & t \ge 0, \end{cases} \tag{1.7}$$

where $\hat{U}$, $f(\hat{U})$, $g(x,\hat{U})$, $\hat{Q}$ are in (1.6) and $x = x_B$ is the boundary. The boundary data $\hat{\rho}_B(t)$, $\hat{Q}_B(t)$ and the initial data

$$\hat{U}_0(x) = \begin{bmatrix} \hat{\rho}_0(x) \\ \hat{m}_0(x) \end{bmatrix}, \quad x \geq x_B,$$

are functions of bounded total variations.

The generalized Riemann problem of (1.5) is now given as follows. First, choose constants $\Delta x, \Delta t > 0$ satisfying Courant-Friedrichs-Lewy (C-F-L) condition

$$\frac{\Delta x}{\Delta t} > \max_{\hat{U},\hat{Q}\in\Omega} \{\lambda(\hat{U},\hat{Q})\}, \tag{1.8}$$

where $\lambda$ is an eigenvalue of $Df(\hat{U},\hat{Q})$. Next, giving constants $x_0 \gg x_B$, $t_0 > 0$ and $0 < \epsilon \ll 1$, we define the following regions (see Figures 1 and 2):

$$\begin{aligned}
D^L &= D^L\big((x_0,t_0)\big) \equiv \{(x,t); x_0 - \Delta x \leq x \leq x_0, t_0 < t < t_0 + \Delta t\},\\
D^R &= D^R\big((x_0,t_0)\big) \equiv \{(x,t); x_0 \leq x \leq x_0 + \Delta x, t_0 < t < t_0 + \Delta t\},\\
D &= D\big((x_0,t_0)\big) \equiv D^L \cup D^R,\\
D^L_\epsilon &= D^L_\epsilon\big((x_0,t_0)\big) \equiv \{(x,t); x_0 - \Delta x \leq x \leq x_0 - \epsilon\Delta x, t_0 < t < t_0 + \Delta t\},\\
D^R_\epsilon &= D^R_\epsilon\big((x_0,t_0)\big) \equiv \{(x,t); x_0 + \epsilon\Delta x \leq x \leq x_0 + \Delta x, t_0 < t < t_0 + \Delta t\},\\
D_\epsilon &= D_\epsilon\big((x_0,t_0)\big) \equiv \{(x,t); | x - x_0 | \leq \epsilon\Delta x, t_0 < t < t_0 + \Delta t\},
\end{aligned}$$

and the segments

$$\begin{aligned}
\Gamma^L &\equiv \{(x,t_0); x_0 - \Delta x \leq x < x_0\},\\
\Gamma^R &\equiv \{(x,t_0); x_0 < x < x_0 + \Delta x\},\\
\Gamma^L_\epsilon &\equiv \{(x,t_0); x_0 - \Delta x \leq x \leq x_0 - \epsilon\Delta x\},\\
\Gamma_\epsilon &\equiv \{(x,t_0); | x - x_0 | \leq \epsilon\Delta x\},\\
\Gamma_B &\equiv \{(x_B,t); t_0 \leq t < t_0 + \delta t\}.
\end{aligned}$$

Under the above definitions of regions and segments, we now give the Riemann problem of (1.5) and the modified Riemann problem of (1.5):

$$\begin{cases}
\hat{U}_t + f(\hat{U})_x = g(x,\hat{U},\hat{Q}), \qquad (x,t) \in D,\\
\hat{Q}_x = k\hat{\rho}, \qquad (x,t) \in D,\\
\hat{U}(x,t_0) = \begin{cases} \hat{U}_L, & x \in \Gamma^L,\\ \hat{U}_R, & x \in \Gamma^R, \end{cases}\\
\hat{Q}\,|_{\Gamma_B} = \hat{Q}_B(t),
\end{cases} \tag{1.9}$$

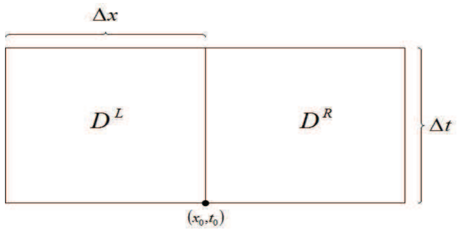


Figure 1: The Region $D$

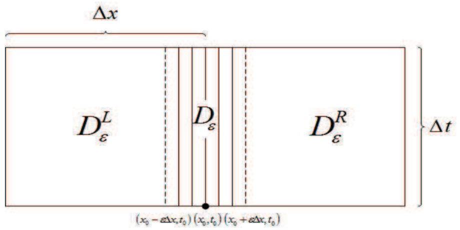


Figure 2: The Region $D_\epsilon$

$$\begin{cases} (\hat{U}_\epsilon)_t + f(\hat{U}_\epsilon)_x = g(x, \hat{U}_\epsilon, \hat{Q}_\epsilon), & (x,t) \in D, \\ (\hat{Q}_\epsilon)_x = k\hat{\rho}_\epsilon, \quad (x,t) \in D, \\ \hat{U}_\epsilon(x_0, t_0) = \begin{cases} \hat{U}_L, & x \in \Gamma_\epsilon^L, \\ \phi_\epsilon(x), & x \in \Gamma_\epsilon, \\ \hat{u}_R, & x \in \Gamma_\epsilon^R, \end{cases} \\ \hat{Q}_\epsilon \mid_{\Gamma_B} = \hat{Q}_B(t), \end{cases} \tag{1.10}$$

where $\hat{U}_L, \hat{U}_R$ are constant states given by

$$\hat{U}_L = \begin{bmatrix} \hat{\rho}_L \\ \hat{m}_L \end{bmatrix}, \quad \hat{U}_R = \begin{bmatrix} \hat{\rho}_R \\ \hat{m}_R \end{bmatrix},$$

and $\phi_\epsilon(x) = \begin{bmatrix} \phi_\epsilon^1(x) \\ \phi_\epsilon^2(x) \end{bmatrix}$ is a smooth monotone function connecting $\hat{U}_L$, $\hat{U}_R$, meaning that $\phi_\epsilon^1(x)$ (or $\phi_\epsilon^2(x)$) is a smooth monotone function connecting $\hat{\rho}_L$, $\hat{\rho}_R$ (or $\hat{m}_L, \hat{m}_R$). The reason that we invent the modified problem (1.10) is to bring in the discontinuous standing waves into the approximate generalized solutions of (1.9) so that such stationary solutions can be considered as the mollification of the smooth standing waves of (1.10) located in $D^\varepsilon$, which allows us to use the techniques in ordinary differential equations.

Here we recall some previous results in hyperbolic systems of balance laws. The entropy solutions to the Riemann problem of following strictly hyperbolic system of conservation laws

$$U_t + F(U)_x = 0 \tag{1.11}$$

were first constructed by Lax [12]. The Riemann solutions of (1.11) are self-similar and consist of constant states separated by either rarefaction waves, shocks or contact discontinuities. The global weak solutions to the Cauchy problem of (1.11) was established by Glimm [5], who applied Lax's solutions as the building blocks of a random finite difference scheme (or Glimm scheme). For the quasi-linear hyperbolic system of balance laws

$$U_t + F(x,U)_x = G(x,U), \tag{1.12}$$

the Cauchy problem was first studied by Liu [14] by using the steady state scheme. If the eigenvalues of $\partial_U F$ are nonzero and the $L^1$-norms of $G$ and $\partial_U G$ are sufficiently small, then weak solutions exist globally and tend asymptotically to the stationary solutions. When $F$ and $G$ are independent of $x$, the existence result for the nonlinear water-hammer problem was established by Luskin and Temple [15] by combining Glimm's scheme with the method of fractional steps. For the Cauchy problem of the general quasi-linear, strictly hyperbolic system of balance laws is expressed as

$$U_t + F(x,t,U)_x = G(x,t,U). \tag{1.13}$$

The local existence of entropy solutions was first established by Dafermos-Hsiao [4] under the assumption that the eigenvalues of $\partial_U F$ are nonzero and the constant solution $U \equiv 0$ is the steady state solution for all $(x,t)$. Furthermore, the global existence was also obtained under additional dissipative assumptions regarding the flux and source. In [1], the dissipative assumption in [4] was relaxed to obtain global existence results for the Cauchy problem of nozzle flow. System (1.13) has also been studied by LeFloch-Raviart [13] and Hong-LeFloch [9] using an asymptotic expansion around the classical Riemann solutions. The shock wave model of Einstein's equations, which can be written as a degenerate $4\times 4$ hyperbolic balance law (1.13), was studied by Groah-Smoller-Temple [7] by applying fractional time-step scheme.

There are several difficulties occur when we construct an approximate Riemann solutions of (1.9) and (1.10). First, due to the appearance of the source terms, it causes the breakdown of the self-similarity in the approximate solutions. Therefore, the traditional Lax method can not be applied for finding the solutions of (1.9), (1.10). The second difficulty occurs that (1.9), (1.10) are mixed-type problems, that is, the PDE systems in (1.9) and (1.10) are considered as integro-differential systems or hyperbolic systems of balance laws with global sources. The results of constructing the generalized Riemann solvers for such problems are very limited. To overcome these difficulties, we first use the equation for the conservation of mass and the one for the gravity potential to reformulate this mixed-type system into a standard form of $3 \times 3$ hyperbolic balance laws. Then, following the method of frozen-variables to our system, we give an approximate system, which is called system (A), of our original hyperbolic balance laws so that the flux of (A) only depends on $\hat{U}, \hat{Q}$. In addition, the source terms of (A) can be reduced to a much simpler form than the original ones. Next, considering the effect of source terms in (A) as a weak perturbation to the associated homogeneous conservation laws (A') of (A), it inspires us to decompose the approximate Riemann solution $\hat{U}_\epsilon$ of (1.10) as

$$\begin{cases} \hat{Q}_\epsilon = \tilde{Q}_\epsilon + \bar{Q}_\epsilon, \\ \hat{U}_\epsilon = \tilde{U}_\epsilon + \bar{U}_\epsilon, \end{cases} \tag{1.14}$$

where $\tilde{U}_\epsilon$ is the self-similar Riemann solution of (A') and $\bar{U}_\epsilon$, $\bar{Q}_\epsilon$ are some approximate solutions of the linearized system of (1.5) around $\tilde{U}_\epsilon$, $\tilde{Q}_\epsilon$. The construction of $\bar{U}_\epsilon$ is based on the following process :

(I) Operator-splitting method of the linearized system.

(II) Averaging process of discontinuous coefficients of linearized system that transforms the linearized system into an ordinary differential system with continuous coefficients.

(III) Solving $\bar{U}_\epsilon$ by the technique of variation of parameters.

The weak solution $(\tilde{U}_\epsilon, \tilde{Q}_\epsilon)$ to the Riemann problem of (A') can be solved by the standard Lax method. It consists of at most four constant states separated by the elementary waves which are rarefaction waves, shock waves and standing waves. To adopt a non-singular standing wave for solution $(\bar{U}_\epsilon, \bar{Q}_\epsilon)$, we require that the solution flow for the Riemann problem of (A') either lies entirely in the subsonic or supersonic region. By the construction of $(\tilde{U}_\epsilon, \tilde{Q}_\epsilon)$, $(\bar{U}_\epsilon, \bar{Q}_\epsilon)$, we can finally construct the approximate generalized Riemann solution $(\hat{U}_\epsilon, \hat{Q}_\epsilon)^T$ of (1.10) as given in (1.14). It is easy to see that the sequence $\{(\hat{U}_\epsilon, \hat{Q}_\epsilon)^T\}$ has bounded $L^\infty$ norm and total variations independent of $\epsilon$. So, by Helly selection principle, $\lim(\hat{U}_\epsilon, \hat{Q}_\epsilon)^T$ exists as $\epsilon$ approaches 0. However, the limit of $\hat{Q}_\epsilon$, say $\hat{Q}$, does not satisfy the third equation of (1.1) since $\hat{Q}$ is only a BV function, so $\hat{Q}_x$ is only defined in the distribution sense, which means that $\partial_x(\hat{Q}_\epsilon) = k\hat{\rho}_\epsilon$ does not hold in the weak sense as $\epsilon \to 0$. So, we need to re-construct $\hat{Q}$ as

$$\hat{Q}(x,t) = \hat{Q}_*(x,t) := Q_B(t) + \int_{x_B}^{x} k\hat{\rho}(s,t)\,ds, \quad (x,t) \in D, \tag{1.15}$$

where $\hat{\rho}$ is the first component of $\hat{U}$. It follows that

$$\hat{Q}_x = k\hat{\rho}$$

in the weak sense. Finally, we construct the approximate generalized Riemann solution of (1.9) as

$$\begin{pmatrix} \hat{U} \\ \hat{Q} \end{pmatrix}(x,t) = \lim_{\epsilon \to 0} \begin{pmatrix} \hat{U}_\epsilon(x,t) \\ \hat{Q}_*(x,t) \end{pmatrix} \tag{1.16}$$

for $(x,t) \in D\big((x_0,t_0)\big)$ where $\hat{Q}_*$ is in (1.15). By the similar calculation in [8, 9], we can show that, for all test function $\phi$,

$$|R_D(\hat{U}_\epsilon,\hat{Q}_\epsilon,\phi) - R_D(\hat{U},\hat{Q}_*,\phi)| \le \mathcal{O}(1)\{\epsilon(\Delta x)^2 + (\Delta x)^2 \cdot [osc.\{\hat{U}_\epsilon,\hat{Q}_\epsilon\}] + (\Delta x)^3\}, \tag{1.17}$$

where $osc.\{U\}$ stands for the oscillation of $U$ in $D$, and

$$\begin{aligned} R_D(U,Q,\phi) \equiv & \iint_{D((x_0,t_0))} \hat{U}\phi_t + f(\hat{U})\phi_x + \phi g(x,\hat{U},\hat{Q})\,dx\,dt \\ & + \int_{t_0}^{t_0+\Delta t} f(\hat{U}(x_0+\Delta x,t))\phi(x_0+\Delta x,t)\,dt \\ & - \int_{t_0}^{t_0+\Delta t} f(\hat{U}(x_0-\Delta x,t))\phi(x_0-\Delta x,t)\,dt \\ & + \int_{x_0-\Delta x}^{x_0+\Delta x} \hat{U}(x,t_0+\Delta x)\phi(x,t_0+\Delta t)\,dx \\ & - \int_{x_0-\Delta x}^{x_0+\Delta x} \hat{U}(x,t_0)\phi(x,t_0)\,dx, \quad \phi \in C_0^\infty(\Omega), \quad D((x_0,t_0)) \subset \Omega. \end{aligned} \tag{1.18}$$

It means that the construction of $(\hat{U},\hat{Q})$, which is the $\epsilon$-limit of $(\hat{U}_\epsilon,\hat{Q}_*)$, is reasonable under the sense of consistency of generalized Glimm scheme. Finally, We give the main theorem of this paper.

**Theorem 1.1** (Main Theorem)**.** *Consider Riemann problem* (1.9) *and modified Riemann problem* (1.10)*. There exists a unique bounded total variation function* $(\tilde{U}_\epsilon,\tilde{Q}_\epsilon)^T$ *which is the entropy solution of the associated homogeneous Riemann problem* ($g \equiv 0$) *by the Lax method. Furthermore, define*

$$\tilde{U}(x,t) \equiv \lim_{\epsilon\to 0} \tilde{U}_\epsilon(x,t), \quad \forall (x,t) \in D.$$

*Then the approximate generalized Riemann solutions of* (1.10) *and* (1.9) *are constructed, respectively, as*

$$\begin{aligned} \begin{pmatrix} \hat{U}_\epsilon \\ \hat{Q}_\epsilon \end{pmatrix}(x,t) &= \begin{pmatrix} \tilde{U}_\epsilon + \bar{U}_\epsilon \\ \tilde{Q}_\epsilon + \bar{Q}_\epsilon \end{pmatrix}(x,t), \\ \begin{pmatrix} \hat{U} \\ \hat{Q} \end{pmatrix}(x,t) &= \begin{pmatrix} \tilde{U} + \bar{U} \\ \hat{Q}_* \end{pmatrix}(x,t), \quad (x,t) \in D((x_0,t_0)), \end{aligned}$$

*where* $(\bar{U}_\epsilon,\bar{Q}_\epsilon)^T$ *is given in* (3.37)*,* $\bar{U} = \lim_{\epsilon\to 0}\bar{U}_\epsilon$ *and* $\hat{Q}_*$ *is given in* (1.15)*. Moreover,* $(\hat{U}_\epsilon,\hat{Q}_\epsilon)$ *satisfies the estimate of residual* (3.39)*.*

The outline of this paper is laid out as follows. In section 2, we re-formulate the original Euler-Poisson equations as a standard type of hyperbolic systems of balance laws. In section

3, we construct approximate generalized Riemann solutions to Riemann problem (1.9) and modified Riemann problem (1.10).

## 2. Re-formulation of Euler-Poisson equations

In this section, we transform the $2\times 2$ compressible Euler-Poisson system into as a $3\times 3$ hyperbolic system of balance laws. We begin with the equation of $\hat{Q}$ in (1.4). From the third equation of (1.4), we observe that $\hat{Q}$ is a Lipschitz continuous function when $\hat{\rho}$ is only a BV function. It means that $(\hat{Q}_x)_t$ and $(\hat{Q}_t)_x$ are defined in the sense of distributions. In addition, we have

$$(\hat{Q}_x)_t = (\hat{Q}_t)_x \tag{2.1}$$

in distributions sense. It follows by (1.4a), (1.4c), (2.1) that

$$(\hat{Q}_t)_x = (\hat{Q}_x)_t = k\hat{\rho}_t = -k\hat{m}_x,$$

which leads to

$$(\hat{Q}_t + k\hat{m})_x = 0.$$

Integrate the above equation from $x_0 - \Delta x$ to $x$, we have for any $t \in (t_0, t_0 + \triangle t)$,

$$\hat{Q}_t = -k\hat{m} + H(t), \tag{2.2}$$

where

$$H(t) \equiv \frac{dQ}{dt}(x_0 - \Delta x, t) + k\hat{m}(x_0 - \Delta x, t). \tag{2.3}$$

Next, we consider equation (1.4b). From (1.4c) we can replace $\hat{\rho}$ by $\frac{1}{k}\hat{Q}_x$ in (1.4b) and obtain

$$\hat{m}_t + (\frac{\hat{m}^2}{\hat{\rho}^2} + \sigma^2\hat{\rho})_x = \frac{1}{kx}(2\sigma^2\hat{Q}_x - \frac{1}{x}\hat{Q}\hat{Q}_x). \tag{2.4}$$

When the grid size $\Delta x$ is sufficiently small and $x > x_B$ (region $D$ is away from the boundary $x = x_B$), the function $\frac{1}{x}$ can be approximated by $\frac{1}{x_0}$ when $|x - x_0| \le \Delta x$. Therefore, equation (2.4) is approximated by

$$\hat{m}_t + \Big(\frac{\hat{m}^2}{\hat{\rho}} + \sigma^2\hat{\rho}\Big)_x = \frac{1}{kx_0}\Big(2\sigma^2\hat{Q}_x - \frac{1}{x_0}\hat{Q}\hat{Q}_x\Big) = \Big(\frac{2\sigma^2}{kx_0}\hat{Q} - \frac{1}{2kx_0^2}\hat{Q}^2\Big)_x.$$

In other words, in region $D$, equation (2.4) is approximated by the following homogeneous conservation law:

$$\hat{m}_t + \Big(\frac{\hat{m}^2}{\hat{\rho}} + \sigma^2\hat{\rho} - \frac{2\sigma^2}{kx_0}\hat{Q} + \frac{1}{2kx_0^2}\hat{Q}^2\Big)_x = 0. \tag{2.5}$$

Combining (1.4a) with (2.2), (2.5), we have the following approximate system for the compressible Euler-Poisson equations in $D$:

$$\hat{V}_t + F(\hat{V}, x_0)_x = G(t, \hat{V}), \tag{2.6}$$

where

$$\begin{cases} \hat{V} \equiv \begin{bmatrix} \hat{\rho} \\ \hat{m} \\ \hat{Q} \end{bmatrix}, \quad F(\hat{V}, x_0) \equiv \begin{bmatrix} \hat{m} \\ \frac{\hat{m}^2}{\hat{\rho}} + \sigma^2 \hat{\rho} - \frac{2\sigma^2}{k x_0} \hat{Q} + \frac{1}{2k x_0^2} \hat{Q}^2 \\ 0 \end{bmatrix}, \\ G(t, \hat{V}) \equiv \begin{bmatrix} 0 \\ 0 \\ -k\hat{m} + H(t) \end{bmatrix}. \end{cases} \tag{2.7}$$

So, the problems of giving approximate solutions of (1.9) and (1.10) are equivalent to the ones of finding the approximate solutions of

$$\begin{cases} \hat{V}_t + F(\hat{V}, x_0)_x = G(t, \hat{V}), \text{ in } D, \\ \hat{V}(x, t_0) = \begin{cases} \hat{V}_L, & x \in \Gamma^L, \\ \hat{V}_R, & x \in \Gamma^R, \end{cases} \end{cases} \tag{2.8}$$

and

$$\begin{cases} (\hat{V}_\epsilon)_t + F(\hat{V}_\epsilon, x_0)_x = G(t, \hat{V}_\epsilon), \text{ in } D, \\ \hat{V}_\epsilon(x, t_0) = \begin{cases} \hat{V}_L, & x \in \Gamma_\epsilon^L, \\ \Psi_\epsilon(x), & x \in \Gamma_\epsilon, \\ \hat{V}_R, & x \in \Gamma_\epsilon^R, \end{cases} \end{cases} \tag{2.9}$$

where $\hat{V}_L$, $\hat{V}_R$ are two constant states give by

$$\hat{V}_L \equiv \begin{bmatrix} \hat{\rho}_L \\ \hat{m}_L \\ \hat{Q}_L \end{bmatrix}, \quad \hat{V}_R \equiv \begin{bmatrix} \hat{\rho}_R \\ \hat{m}_R \\ \hat{Q}_R \end{bmatrix}.$$

Here, the smooth function $\Psi_\epsilon(x)$ can be represented as

$$\Psi_\epsilon(x) \equiv \begin{bmatrix} \phi_\epsilon(x) \\ Q_\epsilon(x) \end{bmatrix},$$

where $\phi_\epsilon(x)$ is in (1.10) and $Q_\epsilon(x)$ is a monotone function connecting $\hat{Q}_L$, $\hat{Q}_R$.

Setting $G(t,\hat{V}) \equiv 0$, we have the following associated classical Riemann problems of (2.8), (2.9):

$$\begin{cases} \tilde{V}_t + F(\tilde{V},x_0)_x = 0, \text{ in } D, \\ \tilde{V}(x,t_0) = \begin{cases} \hat{V}_L, & x \in \Gamma^L, \\ \hat{V}_R, & x \in \Gamma^R, \end{cases} \end{cases} \tag{2.10}$$

$$\begin{cases} (\tilde{V}_\epsilon)_t + F(\tilde{V}_\epsilon,x_0)_x = 0, \text{ in } D, \\ \tilde{V}_\epsilon(x,t_0) = \begin{cases} \hat{V}_L, & \text{in } \Gamma^L_\epsilon, \\ \Psi_\epsilon(x), & \text{in } \Gamma_\epsilon, \\ \hat{V}_R, & \text{in } \Gamma^R_\epsilon. \end{cases} \end{cases} \tag{2.11}$$

By the straightforward calculation, the Jacobian matrix $DF(\tilde{V},x_0)$ is given as

$$DF(\tilde{V},x_0) = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{\tilde{m}^2}{\tilde{\rho}^2} + \sigma^2 & \frac{2\tilde{m}}{\tilde{\rho}} & -\frac{2\sigma^2}{kx_0} + \frac{1}{kx_0^2}\tilde{Q} \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of $DF(\tilde{V},x_0)$ are

$$\lambda_0(\tilde{U}) = 0, \quad \lambda_1(\tilde{U}) = \frac{\tilde{m}}{\tilde{\rho}} - \sigma, \quad \lambda_2(\tilde{U}) = \frac{\tilde{m}}{\tilde{\rho}} + \sigma. \tag{2.12}$$

The right eigenvectors of $DF(\tilde{V},x_0)$ are

$$R_0(\tilde{V},x_0) = \begin{bmatrix} \frac{-\frac{2\sigma^2}{kx_0} + \frac{1}{kx_0^2}\tilde{Q}}{\lambda_1\lambda_2} \\ 0 \\ 1 \end{bmatrix}, \quad R_1(\tilde{V}) = \begin{bmatrix} 1 \\ \lambda_1 \\ 0 \end{bmatrix}, \quad R_2(\tilde{V}) = -\begin{bmatrix} 1 \\ \lambda_2 \\ 0 \end{bmatrix}. \tag{2.13}$$

Note that, the eigenvalues of $DF(\tilde{V},x_0)$ are independent of the choice of $x_0$. Following the assumption that the solution flow is either in the subsonic or supersonic region, we obtain that the systems in (2.10) and (2.11) are strictly hyperbolic. Furthermore, we have

$$\nabla\lambda_0 \cdot R_0(\tilde{V}) = 0, \quad \nabla\lambda_i \cdot R_i(\tilde{V}) > 0, \quad i = 1,2.$$

It means that the zero-characteristic field (or $\lambda_0$-field) is linear degenerate, but the first and second characteristic fields (or $\lambda_1$, $\lambda_2$-fields) are genuinely nonlinear. Therefore, by Lax method, the weak solution of (2.11) (or (2.10)) consists of rarefaction waves, shock waves from $\lambda_1$, $\lambda_2$-fields and a smooth standing wave (or standing wave discontinuity) from $\lambda_0$-field. In the next section, we will construct the wave curves of characteristic fields and use those to show the existence and uniqueness of weak solution to (2.11). Consequently, the existence and uniqueness of self-similar weak solution of (2.10) can be achieved by letting $\epsilon \to 0$.

In the end of this section, we find the linearized system of (2.6) around the weak solution of the associated homogeneous conservation laws. Let $\tilde{V}_\epsilon$ be a weak solution of (2.10). Set

$$\hat{V}_\epsilon(x,t) := \tilde{V}_\epsilon(x,t) + \bar{V}_\epsilon(x,t) \tag{2.14}$$

for some unknown $\bar{V}_\epsilon(x,t) \ll 1$. Plugging (2.14) into (2.6) and neglecting the higher order terms of $\bar{V}_\epsilon$, we have the initial value problem for the linearized system of $\bar{V}_\epsilon$:

$$\begin{cases} (\bar{V}_\epsilon)_t + (A_\epsilon(x,t)\bar{V}_\epsilon)_x = B_\epsilon(x,t)\bar{V}_\epsilon + C_\epsilon(x,t), & (x,t) \in D, \\ \bar{V}_\epsilon(x,0) = 0, \quad x \in [x_0 - \Delta x, x_0 + \Delta x], \end{cases} \tag{2.15}$$

where

$$A_\epsilon(x,t) = DF(\tilde{V}_\epsilon, x_0) = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{\tilde{m}_\epsilon^2}{\tilde{\rho}_\epsilon^2} + \sigma^2 & \frac{2\tilde{m}_\epsilon}{\tilde{\rho}_\epsilon} & -\frac{2\sigma^2}{kx_0} + \frac{1}{kx_0^2}\tilde{Q}_\epsilon \\ 0 & 0 & 0 \end{bmatrix},$$

$$B_\epsilon(x,t) = DG(\tilde{V}_\epsilon) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -k & 0 \end{bmatrix}, \quad C_\epsilon(x,t) = G(\tilde{V}_\epsilon) = (0,0,-k\tilde{m}_\epsilon + H(t))^T.$$

We notice that the entries of $A_\epsilon(x,t)$ are discontinuous functions when the shocks appear in $\tilde{U}_\epsilon$. In Section 3, we will construct the Riemann solution $\tilde{V}_\epsilon$ of (2.11) and an approximate solution $\bar{V}_\epsilon$ for (2.15) to complete the construction of $\hat{V}_\epsilon$ under (2.14).

## 3. Approximate Solutions of (2.8) and (2.9)

In this section, we first study the weak solutions of (2.10) and (2.11). The existence and uniqueness of weak solutions will be established by the Lax method. Secondly, we construct an approximate solution of (2.15) by inventing the operator splitting method and the associated ordinary differential system of (2.15).

To start, let us consider the following classical Riemann problem given in Section 2:

$$\begin{cases} (\tilde{V}_\epsilon)_t + F(\tilde{V}_\epsilon, x_0)_x = 0, \text{ in } D, \\ \tilde{V}_\epsilon(x,t_0) = \begin{cases} \hat{V}_L, & \text{in } \Gamma_\epsilon^L, \\ \Psi_\epsilon(x), & \text{in } \Gamma_\epsilon, \\ \hat{V}_R, & \text{in } \Gamma_\epsilon^R. \end{cases} \end{cases} \tag{3.1}$$

The solution of (3.1) we construct is a combination of self-similar weak solutions and a smooth standing wave which can be expressed as integral curves of the right eigenvectors of $DF(V,x_0)$.

By the results in Section 2, we know that the system in (3.1) is strictly hyperbolic in either subsonic or supersonic regions. Without loss of generality, we consider the subsonic case, which means that, for any $V_\epsilon \in \Omega$,

$$\lambda_0(\tilde{V}_\epsilon) = 0, \quad \lambda_1(\tilde{V}_\epsilon) = \frac{\tilde{m}_\epsilon}{\tilde{\rho}_\epsilon} - \sigma < 0, \quad \lambda_2(\tilde{V}_\epsilon) = \frac{\tilde{m}_\epsilon}{\tilde{\rho}_\epsilon} + \sigma > 0. \tag{3.2}$$

Furthermore, the $\lambda_0$-field is linear degenerate and $\lambda_1$, $\lambda_2$-fields are genuinely nonlinear. So, the elementary waves of $\lambda_1$, $\lambda_2$-fields consist of rarefaction waves ($C^1$ solutions) and shock waves (discontinuous solutions) satisfying the Rankine-Hugoniot condition. By assumption (3.2), we obtain that $\tilde{V}_\epsilon$ consists of at most one $\lambda_1$-wave in $D_\epsilon^L$ and at most one $\lambda_2$-wave in $D_\epsilon^R$. The $\lambda_1$-waves in $D_\epsilon^L$, including $\lambda_1$-rarefaction waves or $\lambda_1$-shock waves, have the properties as follows. The $\lambda_1$-rarefaction wave $\tilde{V}_\epsilon^1$ connecting constant state $\hat{V}_L$ on the left is a self-similar function located in $D_\epsilon^L$. That is,

$$\tilde{V}_\epsilon^1(x,t;\hat{V}_L) = \tilde{V}_\epsilon^1(\xi;\hat{V}_L) = \begin{pmatrix} \tilde{U}_\epsilon^1 \\ \tilde{Q}_\epsilon^1 \end{pmatrix} = \begin{bmatrix} \tilde{U}_\epsilon^1(\xi;\hat{V}_L) \\ \tilde{Q}_\epsilon^1(\xi;\hat{V}_L) \end{bmatrix},$$

where $\xi \equiv \frac{x-(x_0-\epsilon\triangle x)}{t-t_0}$. Plugging the above form of $\tilde{V}_\epsilon^1$ into the system in (3.1) and using the chain rule, we have

$$\left(DF(\tilde{V}_\epsilon^1, x_0) - \xi I_{3\times 3}\right)\cdot \frac{d\tilde{V}_\epsilon^1}{d\xi} = 0.$$

It follows that $\xi = \lambda_1(\tilde{V}_\epsilon^1)$ and $\tilde{V}_\epsilon^1(\xi;\hat{V}_L)$ solves

$$\begin{cases} \frac{d\tilde{V}_\epsilon}{d\xi} = R_1(\tilde{V}^\epsilon), \\ \tilde{V}_\epsilon(0) = \hat{V}_L, \end{cases} \tag{3.3}$$

where $R_1$ is the first eigenvector of $DF$. It means that the solution curve of $\tilde{V}_\epsilon^1(\xi;\hat{V}_L)$, denoted by $\mathbf{R}_1(\hat{V}_L)$, is an integral curve of $R_1$ passing through $\hat{V}_L$. In addition, by the Taylor expansion to the parameter of this curve, $\mathbf{R}_1(\tilde{V}_L)$ can be parametrized in the phase space as

$$\mathbf{R}_1(\tilde{V}_L): \quad \tilde{V}_\epsilon^1(\theta) = \hat{V}_L + \theta R_1(\hat{V}_L) + \frac{\theta^2}{2} R_1\cdot\nabla R_1(\hat{V}_L) + \mathcal{O}(\theta^3), \quad \theta > 0. \tag{3.4}$$

The $\lambda_1$-shock (or backward shock) connecting $\hat{V}_L$ on the left in $D_\epsilon^L$ satisfies the Rankine-Hugoniot condition

$$s_1[\tilde{V}_\epsilon^1] = [F(\tilde{V}_\epsilon^1)], \tag{3.5}$$

where $s_1$ is the speed of shock and

$$[\tilde{V}_\epsilon^1] \equiv \tilde{V}_\epsilon^1 - \hat{V}_L, \quad F[\tilde{V}_\epsilon^1] \equiv F(\tilde{V}_\epsilon^1) - F(\hat{V}_L).$$

Note that (3.5) is a $3\times3$ algebraic system. The third equation of (3.5) gives $\tilde{Q}_\epsilon = \hat{Q}_L$ since $s_1 \neq 0$, which means that there is no jump in $\tilde{Q}_\epsilon$. The solution curve of 1-shock starting at $\hat{V}_L$, which is denoted by $\mathbf{S}_1(\hat{V}_L)$, can be parametrized as

$$\mathbf{S}_1(\hat{V}_L): \quad \tilde{V}^1_\epsilon(\theta) = \hat{V}_L + \theta R_1(\hat{V}_L) + \frac{\theta^2}{2} R_1 \cdot \nabla R_1(\hat{V}_L) + \mathcal{O}(\theta^3), \quad \theta < 0. \tag{3.6}$$

Note that, in (3.6) the value of $\theta$ is negative due to the Lax entropy condition

$$\lambda_1(\tilde{V}^1_\epsilon(\theta)) < \lambda_1(\hat{V}_L),$$

which is equivalent to $\theta < 0$ by the results in [12].

The $\lambda_2$-elementary waves located in region $D^R_\epsilon$ can be constructed similarly as for $\lambda_1$-waves. The $\lambda_2$-rarefaction wave $\tilde{V}^2_\epsilon$ connecting some constant state $\hat{V}_2$ on the left is also a self-similar function located in $D^R_\epsilon$. That is,

$$\tilde{V}^2_\epsilon(x,t;\hat{V}_2) = \tilde{V}^2_\epsilon(\xi;\hat{V}_2) = \begin{bmatrix} \tilde{U}^1_\epsilon(\xi;\hat{V}_2) \\ \tilde{Q}^1_\epsilon(\xi;\hat{V}_2) \end{bmatrix},$$

where $\xi \equiv \frac{x-(x_0+\epsilon\triangle x)}{t-t_0}$. We have

$$\left(DF(\tilde{V}^2_\epsilon, x_0) - \xi I_{3\times3}\right)\cdot \frac{d\tilde{V}^2_\epsilon}{d\xi} = 0.$$

It follows that $\xi = \lambda_2(\tilde{V}^1_\epsilon)$ and $\tilde{V}^2_\epsilon(\xi;\hat{V}_2)$ solves the following problem

$$\begin{cases} \frac{d\tilde{V}_\epsilon}{d\xi} = R_2(\tilde{V}^\epsilon), \\ \tilde{V}_\epsilon(0) = \hat{V}_2, \end{cases} \tag{3.7}$$

where $R_2$ is the second eigenvector of $DF$. It means that the solution curve of $\tilde{V}^2_\epsilon(\xi;\hat{V}_2)$, denoted by $\mathbf{R}_1(\hat{V}_2)$, is an integral curve of $R_2$ passing through $\hat{V}_2$. Moreover, $\mathbf{R}_1(\tilde{V}_2)$ can be parametrized in the phase space as

$$\mathbf{R}_2(\tilde{V}_2): \quad \tilde{V}^2_\epsilon(\theta) = \hat{V}_2 + \theta R_2(\hat{V}_2) + \frac{\theta^2}{2} R_2 \cdot \nabla R_2(\hat{V}_2) + \mathcal{O}(\theta^3), \quad \theta > 0. \tag{3.8}$$

The $\lambda_2$-shock (or forward shock) connecting $\hat{V}_2$ on the left in $D^R_\epsilon$ satisfies the Rankine-Hugoniot condition

$$s_2[\tilde{V}^1_\epsilon] = [F(\tilde{V}^1_\epsilon)], \tag{3.9}$$

where $s_2$ is the speed of shock and

$$[\tilde{V}^2_\epsilon] \equiv \tilde{V}^2_\epsilon - \hat{V}_2, \quad F[\tilde{V}^2_\epsilon] \equiv F(\tilde{V}^2_\epsilon) - F(\hat{V}_2).$$

Note that the third equation of (3.9) gives $\tilde{Q}_\epsilon = \hat{Q}_2$ since $s_2 \neq 0$, which means that there is no jump of $\tilde{Q}_\epsilon$ in $\tilde{V}_\epsilon^2$. The solution curve of $\lambda_2$-shock starting at $\hat{V}_2$, denoted by $\mathbf{S}_2(\hat{V}_2)$, can be parametrized as

$$\mathbf{S}_2(\hat{V}_2): \quad \tilde{V}_\epsilon^2(\theta) = \hat{V}_2 + \theta R_2(\hat{V}_2) + \frac{\theta^2}{2} R_2 \cdot \nabla R_2(\hat{V}_2) + \mathcal{O}(\theta^3), \quad \theta < 0. \tag{3.10}$$

Note that, in (3.10) the value of $\theta$ is also negative due to the Lax entropy condition

$$\lambda_2(\tilde{V}_\epsilon^2(\theta)) < \lambda_2(\hat{V}_2).$$

Next, we construct the solution of (3.1) in $D_\epsilon$. We first observe that $\lambda_0(\tilde{V}_\epsilon) = 0$ and $\lambda_0$-field is linear degenerate. So it enables us to construct the smooth standing waves in $D_\epsilon$. The standing waves of (3.1) satisfy the following systems of ordinary differential equations

$$F(\tilde{V}_\epsilon, x_0)_x = 0. \tag{3.11}$$

In particular, we have

$$\begin{cases} (\tilde{m}_\epsilon)_x = 0, \\ \left(\frac{(\tilde{m}_\epsilon)^2}{(\tilde{\rho}_\epsilon)} + \sigma^2 \tilde{\rho}_\epsilon - \frac{2\sigma^2}{k x_0} \tilde{Q}_\epsilon + \frac{1}{2k x_0^2} \tilde{Q}_\epsilon^2\right)_x = 0. \end{cases} \tag{3.12}$$

The differential form of (3.12) is

$$\begin{cases} d\tilde{m}_\epsilon = 0, \\ \lambda_1 \lambda_2 \cdot d(\tilde{\rho}_\epsilon) + \left(\frac{2\sigma^2}{k x_0} - \frac{1}{k x_0^2} \tilde{Q}_\epsilon\right) \cdot d\tilde{Q}_\epsilon = 0. \end{cases} \tag{3.13}$$

From the third equation of compressible Euler equations, we observe that $d\tilde{Q}_\epsilon \neq 0$ when we rule out the case of vacuum. Therefore, we can use $\tilde{Q}_\epsilon$ to parametrize the standing wave curve passing through some constant state $\hat{V}_1$. Let $\mathbf{R}_0(\hat{V}_1)$ denote the standing wave curve connecting $\hat{V}_1$ on the left, also set

$$\theta := \tilde{Q}_\epsilon(x), \quad x \in [x_0 - \Delta x, x_0 + \Delta x].$$

Then, the standing wave $\tilde{V}_\epsilon^s(\theta)$ on $\mathbf{R}_0(\hat{V}_1)$ solves

$$\begin{cases} \frac{d\tilde{V}_\epsilon^s}{d\theta} = R_0(\tilde{V}_\epsilon^s), \\ \tilde{V}_\epsilon^s(0) = \hat{V}_1, \end{cases} \tag{3.14}$$

where $R_0$ is in (2.13). It means that $\tilde{V}_\epsilon^s(\theta)$ is an integral curve of $R_0$. In addition, the curve $\mathbf{R}_0(\hat{V}_L)$ can be parametrized as

$$\mathbf{R}_0(\hat{V}_1): \quad \tilde{V}_\epsilon^s(\theta) = \hat{V}_1 + \theta R_0(\hat{V}_1) + \frac{\theta^2}{2} R_0 \cdot \nabla R_0(\hat{V}_1) + \mathcal{O}(\theta^3), \quad |\theta| \leq \theta_* \tag{3.15}$$

for some sufficiently small $\theta_*$. Note that, since the $\lambda_0$-field is linear degenerate, the value of $\theta$ in (3.15) can be either negative or positive, which is similar to the case of contact discontinuities.

We observe that the first component of $R_0$ of $DF$ in (2.13) is nonzero under the assumption of subsonic case:

$$\lambda_1\lambda_2(\tilde{V}_\epsilon) < 0, \quad -\frac{2\sigma^2}{kx_0} + \frac{1}{kx_0^2}\tilde{Q}_\epsilon \neq 0, \quad \forall\ \tilde{V}_\epsilon \in \Omega.$$

Furthermore, we have

$$Det\Big(R_1, R_0, R_2\Big)(\hat{V}_L) = \lambda_2 - \lambda_1 = 2\sigma \neq 0, \tag{3.16}$$

where $Det(R)$ is the determinant of $R$. It means that $\{R_0(\hat{V}_L),\ R_1(\hat{V}_L),\ R_2(\hat{V}_L)\}$ form a non-singular coordinate system in a small neighborhood of $\hat{V}_L \in \Omega$.

Now, we are ready to prove the existence and uniqueness of entropy solution (weak solution satisfying Lax entropy condition) for (3.1) when $|\hat{V}_L - \hat{V}_R|$ is sufficiently small. To start, for given $\hat{V} \in \Omega$ we first define the following curves:

$$\mathbf{T}_i(\hat{V}) := \mathbf{R}_i(\hat{V})\bigcup \mathbf{S}_i(\hat{V}), \quad i = 1, 2. \tag{3.17}$$

From the idea of Lax's method, the problem of finding the entropy solution of (3.1) can be transformed into the problem of finding a sequence of paths $\{\mathbf{T}_1(\theta_1;\hat{V}_L) : \hat{V}_L \to V_1\}$, $\{\mathbf{R}_0(\theta_0;V_1) : V_1 \to V_2\}$ and $\{\mathbf{T}_2(\theta_2;V_2) : V_2 \to \hat{V}_R\}$ where $V_1$, $V_2$ are constant states needed to be decided. In addition, to decide $V_1$, $V_2$ is equivalent to decide the value of the parameter $\theta = (\theta_1,\theta_0,\theta_2)$. By previous analysis, we have

$$\hat{V}_R = \mathbf{T}_2(\theta_2;V_2) = \mathbf{T}_2(\theta_2;\mathbf{R}_0(\theta_0;V_1)) = \mathbf{T}_2(\theta_2;\mathbf{R}_0(\theta_0;\mathbf{T}_1(\theta_1;\hat{V}_L))). \tag{3.18}$$

Using (3.4), (3.6), (3.8), (3.10), (3.15) and the Taylor expansion, we finally obtain

$$\begin{aligned}\hat{V}_R &= \hat{V}_R(\theta) = V_2 + \theta_2 R_2(V_2) + O(V_2^2)\\ &= V_1 + \theta_0 R_0(V_1) + O(V_1^2) + \theta_2 R_2\big(V_1 + \theta_0 R_0(V_1) + O(V_1^2)\big) + O(V_2^2)\\ &= \hat{V}_L + \sum_{j=0}^{2}\theta_j R_j(\hat{V}_L) + O(\sum\theta_j^2).\end{aligned} \tag{3.19}$$

Define

$$\Gamma(\theta) := \hat{V}_R(\theta) - \hat{V}_L = \sum_{j=0}^{2}\theta_j R_j(\hat{V}_L) + O(\sum\theta_j^2).$$

It is easy to see that $\Gamma(0) = \hat{V}_L$, and $\frac{D\Gamma}{D\theta}(0) = [R_1, R_0, R_2](\hat{V}_L)$ which is a non-singular $3\times 3$ matrix. Then, by the implicit function theorem, we have that for any $\hat{V}_R$ sufficiently close to

$\hat{V}_L$, there exists a unique $\theta^* = (\theta_1^*, \theta_0^*, \theta_2^*)$ such that $\hat{V}_R - \hat{V}_L = \Gamma(\theta^*)$, that is, there is a unique $\theta^* = (\theta_1^*, \theta_0^*, \theta_2^*)$ such that

$$\hat{V}_R(\theta^*) = \hat{V}_L + \sum_{j=0}^{2} \theta_j^* R_j(\hat{V}_L) + O(\sum \theta_j^{*2}).$$

Consequently, constant states $\hat{V}_1$, $\hat{V}_2$ can be decided uniquely by the choice of $\theta^*$. We therefore establish the existence and uniqueness of entropy solution to (2.11). Letting $\epsilon \to 0$, we also obtain the entropy solution of (2.10). We then have the following theorems.

**Theorem 3.1.** *Consider problems* (2.10), (2.11). *Suppose* $\hat{V}_L \in \Omega$. *Then, there is a neighborhood* $\Omega_1 \subset \Omega$ *of* $\hat{V}_L$ *such that if* $\hat{V}_R \in \Omega_1$, *then* (2.11) *(or* (2.10)*) has a unique solution consisting of at most four constant states* $\hat{V}_L$, $V_1$, $V_2$ *and* $\hat{V}_R$ *separated by shocks, rarefaction waves and smooth standing wave* (*or standing wave discontinuity*).

In the remaining of this section, we study an approximate solution of the following problem in section 2:

$$\begin{cases} (\bar{V}_\epsilon)_t + (A_\epsilon(x,t)\bar{V}_\epsilon)_x = B_\epsilon(x,t)\bar{V}_\epsilon + C_\epsilon(x,t), & (x,t) \in D, \\ \bar{V}_\epsilon(x,0) = 0, & x \in [x_0 - \Delta x, x_0 + \Delta x], \end{cases} \tag{3.20}$$

where

$$A_\epsilon(x,t) = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{\tilde{m}_\epsilon^2}{\tilde{\rho}_\epsilon^2} + \sigma^2 & \frac{2\tilde{m}_\epsilon}{\tilde{\rho}_\epsilon} & -\frac{2\sigma^2}{kx_0} + \frac{1}{kx_0^2}\tilde{Q}_\epsilon \\ 0 & 0 & 0 \end{bmatrix}, B_\epsilon(x,t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -k & 0 \end{bmatrix},$$

and $C_\epsilon(x,t) = G(\tilde{V}_\epsilon) = (0, 0, -k\tilde{m}_\epsilon + H(t))^T$. We recall the characteristic method for (3.20). If $A_\epsilon(x,t)$ is a $C^1$ function, which means that there is no shock wave in $\tilde{V}_\epsilon$, then by the direct calculation, we can write system in (3.20) as

$$(\bar{V}_\epsilon)_t + A_\epsilon(x,t) \cdot (\bar{V}_\epsilon)_x = \big(B_\epsilon(x,t) - \partial_x A_\epsilon(x,t)\big) \cdot \bar{V}_\epsilon + C_\epsilon(x,t), \quad (x,t) \in D.$$

Introduce the new unknown

$$W_\epsilon = (W_\epsilon^1, W_\epsilon^2, W_\epsilon^3)^T := R_\epsilon^{-1}(\tilde{V}_\epsilon) \cdot \bar{V}_\epsilon,$$

where $R_\epsilon = [R_1, R_0, R_2]$. Then, through the diagonalization we can transform the above system into

$$(W_\epsilon)_t + \Lambda_\epsilon(x,t) \cdot (W_\epsilon)_x = \Pi_\epsilon(x,t) \cdot W_\epsilon + R_\epsilon^{-1} C_\epsilon(x,t), \quad (x,t) \in D, \tag{3.21}$$

where

$$\Lambda_\epsilon(x,t) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix},$$

$$\Pi_\epsilon(x,t) = \big((\Pi_\epsilon)_{ij}(x,t)\big) = R_\epsilon^{-1}\cdot(B_\epsilon - \partial_x A_\epsilon)R_\epsilon + \Lambda_\epsilon(R_\epsilon^{-1})_x R_\epsilon + (R_\epsilon^{-1})_t R_\epsilon. \tag{3.22}$$

Also, the initial data of $W_\epsilon$ is

$$W_\epsilon(x,0) = R_\epsilon^{-1}(x,0)\cdot \bar{V}_\epsilon(x,0) = 0.$$

Therefore, initial value problem (3.20) is equivalent to

$$\begin{cases} (W_\epsilon)_t + \Lambda_\epsilon(x,t)\cdot(W_\epsilon)_x = \Pi_\epsilon(x,t)\cdot W_\epsilon + R_\epsilon^{-1}C_\epsilon(x,t), & (x,t)\in D, \\ W_\epsilon(x,0) = 0, \quad x\in[x_0-\Delta x, x_0+\Delta x], \end{cases} \tag{3.23}$$

where $\Pi_\epsilon(x,t)$ is in (3.22). For the convenience, we denote the $\lambda_0$-field as $\lambda_3$-field. Then, according to the characteristic method to (3.23), we have that along the $\lambda_i$-field

$$\frac{dx_i}{dt} = \lambda_i(x_i(t),t), \quad i = 1,2,3,$$

the unknown $W_\epsilon^i(x_i(t),t)$ satisfies

$$\frac{dW_\epsilon^i}{dt} = \sum_{j=1}^{3}(\Pi_\epsilon)_{ij}\cdot W_\epsilon^j(x_i(t),t) + \Upsilon_\epsilon^i(x_i(t),t), \tag{3.24}$$

where $\Upsilon_\epsilon(x,t) = (\Upsilon_\epsilon^1(x,t), \Upsilon_\epsilon^2(x,t), \Upsilon_\epsilon^3(x,t)) := R_\epsilon^{-1}C_\epsilon(x,t)$.

We can see there are several difficulties for solving problem (3.23). Firstly, in the case that $\tilde{V}_\epsilon$ consists of shock waves, it means $\partial_x A_\epsilon$ and $\Pi_\epsilon$ are not defined in the classical sense. So, equations (3.24) can not be solved by traditional techniques in ODEs. Secondly, the information of $W_\epsilon$ is missing in some sub-region of $D$ due to the fact that the characteristic curves starting from $[x_0-\Delta x, x_0+\Delta x]$ cannot cover the whole domain of $D$. The third difficulty occurs due to the lack of information of $W_\epsilon^j(x_i(t),t)$, $j\neq i$, in the left hand side of (3.24). To overcome the above difficulties, we first use the averaging process for $A_\epsilon(x,t)$:

$$A_\epsilon^*(x) := \frac{1}{\Delta x}\int_{t_0}^{t_0+\Delta t} A_\epsilon(x,s)\,ds. \tag{3.25}$$

Similarly, we can obtain $B_\epsilon^*(x)$ and $C_\epsilon^*(x)$. Then, we solve an approximate problem of (3.20):

$$\begin{cases} (\bar{V}_\epsilon)_t + A_\epsilon^*(x)\cdot(\bar{V}_\epsilon)_x = (B_\epsilon^*(x) - \partial_x A_\epsilon^*(x))\cdot\bar{V}_\epsilon + C_\epsilon^*(x), & (x,t)\in D, \\ \bar{V}_\epsilon(x,0) = 0, \quad x\in[x_0-\Delta x, x_0+\Delta x]. \end{cases} \tag{3.26}$$

However, we still have to face the second and third difficulties addressed above. For these difficulties, we invent the idea of operator-splitting method to (3.26) and observe that the following problem only admits zero solution:

$$\begin{cases} (\bar{V}_\epsilon)_t + A_\epsilon^*(x)\cdot(\bar{V}_\epsilon)_x = 0, & (x,t)\in D, \\ \bar{V}_\epsilon(x,0) = 0, \quad x\in[x_0-\Delta x, x_0+\Delta x]. \end{cases} \tag{3.27}$$

Therefore, for any fixed $x \in [x_0 - \Delta x, x_0 + \Delta x]$, an approximate solution of (3.20) is obtained by solving the following ODEs problem:

$$\begin{cases} (\bar{V}_\epsilon)_t = \Theta^*_\epsilon(x) \cdot \bar{V}_\epsilon + C^*_\epsilon(x), & (x,t) \in D, \\ \bar{V}_\epsilon(x,0) = 0, & x \in [x_0 - \Delta x, x_0 + \Delta x], \end{cases} \tag{3.28}$$

where

$$\Theta^*_\epsilon(x) = B^*_\epsilon(x) - \partial_x A^*_\epsilon(x) = \begin{bmatrix} 0 & 0 & 0 \\ -\partial_x\left(\frac{\tilde{m}^2_\epsilon}{\tilde{\rho}^2_\epsilon}\right)^*, & \partial_x\left(\frac{2\tilde{m}_\epsilon}{\tilde{\rho}_\epsilon}\right)^* & \frac{1}{kx_0^2}\partial_x(\tilde{Q}_\epsilon)^* \\ 0 & -k & 0 \end{bmatrix}.$$

In view of the first equation and the initial data of (3.28), we obtain

$$\bar{\rho}_\epsilon(x,t) = 0 \quad \forall (x,t) \in D.$$

Plugging this into (3.28), we obtain the following problem of $2 \times 2$ system:

$$\begin{cases} (\bar{m}_\epsilon)_t = \left(\partial_x\left(\frac{2\tilde{m}_\epsilon}{\tilde{\rho}_\epsilon}\right)^*\right)\bar{m}_\epsilon + \left(\frac{1}{kx_0^2}\partial_x(\tilde{Q}_\epsilon)^*\right)\bar{Q}_\epsilon, \\ (\bar{Q}_\epsilon)_t = -k\bar{m}_\epsilon + H^*, (x,t) \in D, \\ \bar{m}^\epsilon(x,t_0) = 0,\ \bar{Q}^\epsilon(x,t_0) = 0,\ x \in [x_0 - \Delta x, x_0 + \Delta x], \end{cases} \tag{3.29}$$

where

$$H^* := \int_{t_0}^{t_0+\Delta t} H(s)\,ds, \tag{3.30}$$

and $H(t)$ is in (2.3). Note that, in (3.29) functions $(\frac{2\tilde{m}_\epsilon}{\tilde{\rho}_\epsilon})^*$ and $(\tilde{Q}_\epsilon)^*$ stand for the average of $\frac{2\tilde{m}_\epsilon}{\tilde{\rho}_\epsilon}$ and $\tilde{Q}_\epsilon$ over $[x_0 - \Delta x, x_0 + \Delta x]$ which are Lipschitz continuous function of $x$. The derivatives $\partial_x(\frac{2\tilde{m}_\epsilon}{\tilde{\rho}_\epsilon})^*$ and $\partial_x((\tilde{Q}_\epsilon)^*)$ are well-defined almost everywhere in $D$. Therefore, problem (3.29) is solvable for any $x \in [x_0 - \Delta x, x_0 + \Delta x]$.

We study the solution of (3.29) in each sub-region of $D$. In $D^\epsilon_L \cap D^\epsilon_R$, we have that $\tilde{Q}_\epsilon$ is a constant function, which implies that $\partial_x(\tilde{Q}_\epsilon)^* = 0$. Therefore, by the first equation and the initial data of (3.29), we have

$$\bar{m}_\epsilon(x,t) = 0, \quad \forall\ (x,t) \in D^\epsilon_L \bigcap D^\epsilon_R. \tag{3.31}$$

Using (3.31) for (3.29), we have

$$\bar{Q}_\epsilon(x,t) = (t - t_0)H^*, \quad \forall\ (x,t) \in D^\epsilon_L \bigcap D^\epsilon_R. \tag{3.32}$$

Next, we construct the solution of (3.29) in $D_\epsilon$. We observe that $\tilde{m}_\epsilon$ is a constant function in $D_\epsilon$. So, we assume that there is a constant $m_1$ such that

$$\tilde{m}_\epsilon = m_1, \quad \forall\ (x,t) \in D_\epsilon.$$

By the third equation of compressible Euler-Poisson equations, we have $\partial_x(\tilde{Q}_\epsilon) = \tilde{\rho}_\epsilon$ in $D_\epsilon$. Therefore, (3.29) becomes

$$\begin{cases} (\bar{m}_\epsilon)_t = \left(2m_1\partial_x\left(\frac{1}{\tilde{\rho}_\epsilon}\right)^*\right)\bar{m}_\epsilon + \frac{\tilde{\rho}_\epsilon}{x_0^2}\bar{Q}_\epsilon, (x,t) \in D, \\ (\bar{Q}_\epsilon)_t = -k\bar{m}_\epsilon + H^*, (x,t) \in D, \\ \bar{m}^\epsilon(x,t_0) = 0, \quad \bar{Q}^\epsilon(x,t_0) = 0, \quad x \in [x_0 - \epsilon\Delta x, x_0 + \epsilon\Delta x], \end{cases} \tag{3.33}$$

Define

$$\bar{Y}_\epsilon := (\bar{m}_\epsilon, \bar{Q}_\epsilon)^T, \quad q_\epsilon(x) := 2m_1\partial_x\left(\frac{1}{\tilde{\rho}_\epsilon}\right)^*, \quad \bar{H} := [0, H^*]^T. \tag{3.34}$$

Then, for any fixed $x \in [x_0 - \epsilon\Delta x, x_0 + \epsilon\Delta x]$, problem (3.33) can be written as the following initial value problem of ODEs:

$$(\bar{Y}_\epsilon)_t = Z_\epsilon(x)\cdot\bar{Y}_\epsilon + \bar{H}, \quad \bar{Y}_\epsilon(t_0) = 0, \tag{3.35}$$

where

$$Z_\epsilon(x) = \begin{bmatrix} q_\epsilon(x) & \frac{\tilde{\rho}_\epsilon}{2x_0} \\ -k & 0 \end{bmatrix}.$$

Let $e^{Z_\epsilon(x)t}$ denote the fundamental matrix of $(\bar{Y}_\epsilon)_t = Z_\epsilon(x)\cdot\bar{Y}_\epsilon$. Then, by the variation of parameters, we obtain that, for any $(x,t) \in D_\epsilon$,

$$\bar{Y}_\epsilon(x,t) = (\bar{Y}_\epsilon^1(x,t), \bar{Y}_\epsilon^2(x,t))^T = e^{Z_\epsilon(x)t}\int_{t_0}^t e^{-Z_\epsilon(x)\mu}\cdot\bar{H}d\mu = \int_{t_0}^t e^{-Z_\epsilon(x)(t-\mu)}\cdot\bar{H}d\mu. \tag{3.36}$$

Therefore, by our construction of approximate solution in each sub-region of $D$, we have the following expression of $\bar{V}_\epsilon$ in $D$:

$$\bar{V}_\epsilon(x,t) = (\bar{U}_\epsilon(x,t), \bar{Q}_\epsilon(x,t))^T = \begin{cases} \begin{bmatrix} 0 \\ 0 \\ (t-t_0)H^* \end{bmatrix}, & (x,t) \in D_\epsilon^L \bigcap D_\epsilon^R, \\ \begin{bmatrix} 0 \\ \bar{Y}_\epsilon(x,t) \end{bmatrix}, & (x,t) \in D_\epsilon, \end{cases} \tag{3.37}$$

where $\bar{Y}_\epsilon$ is in (3.36). For any $(x,t) \in D$, we define

$$\hat{U}_\epsilon(x,t) = \tilde{U}_\epsilon(x,t) + \bar{U}_\epsilon(x,t), \quad \hat{Q}_\epsilon(x,t) = \tilde{Q}_\epsilon(x,t) + \bar{Q}_\epsilon(x,t), \tag{3.38}$$

where $(\tilde{U}_\epsilon, \tilde{Q}_\epsilon)$ is the weak solution of (3.1), and $(\bar{U}_\epsilon, \bar{Q}_\epsilon)$ is given in (3.37). Then, we have the following theorem for the estimate of residual defined in (1.18).

**Theorem 3.2.** *Let $\phi \in C_0^\infty(\bar{D})$ be the test function and $R(\cdot, \bar{D}, \phi)$ be the residual defined in (1.18). Then*

$$\begin{aligned} R((\hat{U}_\epsilon, \hat{Q}_\epsilon)^T, \bar{D}, \phi) = &\int_{x_0-\Delta x}^{x_0+\Delta x} (\hat{U}_\epsilon \phi)(x, t_0+\Delta t)\, dx - \int_{x_0-\Delta x}^{x_0+\Delta x} \hat{U}_\epsilon(x, t_0^+)\phi(x, t_0)\, dx \\ &+ \int_{t_0}^{t_0+\Delta t} [f(\hat{U}_\epsilon)\phi](x_0+\Delta x, t)\, dt - \int_{t_0}^{t_0+\Delta t} [f(\hat{U}_\epsilon)\phi](x_0-\Delta x, t)\, dt \\ &+ O(1)\left(\epsilon(\Delta x)^2 + (\Delta x)^3 + (\Delta x)^2 \underset{\bar{D}}{\text{osc.}}\{\widetilde{U}\}\right) \|\phi\|_\infty, \end{aligned} \tag{3.39}$$

*where $(\hat{U}_\epsilon, \hat{Q}_\epsilon)^T$ is given in* (3.38).

**Proof.** The proof is based on the similar estimates of residuals in [8, 9] and the facts that $\bar{V}_\epsilon(x, t)$ is a continuous function in $D$ and $|\bar{V}_\epsilon(x, t)| \le C\Delta x$ for some bounded positive constant $C$ and all $(x, t) \in D$. Here, we only give the sketch of the proof. First, we calculate the main term of the residual and omit the higher order terms of $\hat{U}_\epsilon$ in the integral. We obtain

$$\begin{aligned} &\iint_D \hat{U}_\epsilon \phi_t + f(\hat{U}_\epsilon)\phi_x + \phi g(\hat{U}_\epsilon, \hat{Q}_\epsilon)\ dx\, dt \\ &\quad \cong \iint_D (\tilde{U}_\epsilon + \bar{U}_\epsilon)\phi_t + (f(\tilde{U}_\epsilon) + Df(\tilde{U}_\epsilon)\cdot \bar{U}_\epsilon)\phi_x + \phi(g(\tilde{U}_\epsilon) + Dg(\tilde{U}_\epsilon)\cdot \bar{U}_\epsilon)\ dx\, dt \\ &\quad = \iint_D \bar{U}_\epsilon \phi_t + A_\epsilon \cdot \bar{U}_\epsilon \phi_x + (B_\epsilon \cdot \bar{U}_\epsilon + C_\epsilon)\phi\ dx\, dt + I_{\partial D}(\tilde{U}_\epsilon) \\ &\quad = -\iint_D \{(\bar{U}_\epsilon)_t + (A_\epsilon \bar{U}_\epsilon)_x - B_\epsilon \cdot \bar{U}_\epsilon - C_\epsilon\}\phi\ dx\, dt + I_{\partial D}(\tilde{U}_\epsilon, \bar{U}_\epsilon) \\ &\quad \equiv J_\epsilon(D) + I_{\partial D}(\tilde{U}_\epsilon, \bar{U}_\epsilon), \end{aligned}$$

where $I_{\partial D}(\tilde{U}_\epsilon)$, $I_{\partial D}(\tilde{U}_\epsilon, \bar{U}_\epsilon)$ are the boundary integral terms related to $\tilde{U}_\epsilon$, $\bar{U}_\epsilon$. Next, we define

$$I_\epsilon \equiv \partial_t \bar{U}_\epsilon + \partial_x(A_\epsilon \bar{U}_\epsilon) - B_\epsilon \cdot \bar{U}_\epsilon - C_\epsilon.$$

Then

$$\begin{aligned} I_\epsilon &= \partial_t \bar{U}_\epsilon + A_\epsilon \partial_x \bar{U}_\epsilon - (B_\epsilon - \partial_x A_\epsilon)\bar{U}_\epsilon - C_\epsilon \\ &= \{\partial_t \bar{U}_\epsilon + \partial_x A_\epsilon^* \bar{U}_\epsilon - B_\epsilon^* \bar{U}_\epsilon - C_\epsilon^*\} + A_\epsilon \partial_x \bar{U}_\epsilon - (B_\epsilon - B_\epsilon^*)\bar{U}_\epsilon \\ &\quad + (\partial_x A_\epsilon - \partial_x A_\epsilon^*)\bar{U}_\epsilon - (C_\epsilon - C_\epsilon^*) \\ &= A_\epsilon \partial_x \bar{U}_\epsilon - (B_\epsilon - B_\epsilon^*)\bar{U}_\epsilon + (\partial_x A_\epsilon - \partial_x A_\epsilon^*)\bar{U}_\epsilon - (C_\epsilon - C_\epsilon^*), \end{aligned}$$

where $A_\epsilon^* = A(\tilde{U}_\epsilon^*)$ is the average of $A(\tilde{U}_\epsilon)$ described in (3.25). Similarly for $B_\epsilon^*$ and $C_\epsilon^*$. In the above equalities, we used the fact that $\bar{U}_\epsilon$ is the solution of (3.28). It follows that

$$\begin{aligned} J_\epsilon(D) &= \iint_D A_\epsilon \phi \partial_x \bar{U}_\epsilon\ dx\, dt - \iint_D (B_\epsilon - B_\epsilon^*)\phi \bar{U}_\epsilon\ dx\, dt \\ &\quad + \iint_D (\partial_x A_\epsilon - \partial_x A_\epsilon^*)\bar{U}_\epsilon\ dx\, dt - \iint_D (C_\epsilon - C_\epsilon^*)\phi\ dx\, dt \\ &\equiv \sum_{k=1}^{4} I_\epsilon^k. \end{aligned}$$

Next, we compute each $I_\epsilon^k$. By the fact that $|\bar{U}_\epsilon| = \mathcal{O}(1)\,\triangle t$, we can easily obtain

$$I_\epsilon^2 = \mathcal{O}(1)(\Delta x)^3, \quad I_\epsilon^4 = \mathcal{O}(1)(\Delta x)^2 \cdot \underset{D}{osc}(\tilde{U}_\epsilon).$$

Furthermore, we have

$$\iint_D (\partial_x A_\epsilon - \partial_x A_\epsilon^*)\bar{U}_\epsilon \, dx\, dt \le K\|\bar{U}_\epsilon\|_\infty \cdot \iint_D |\partial_x A_\epsilon - \partial_x A_\epsilon^*| \, dx\, dt \le K_3(\triangle x)^2 \underset{D}{osc}(\tilde{U}_\epsilon),$$

which implies that

$$I_\epsilon^3 = \mathcal{O}(1)(\Delta x)^2 \cdot \underset{D}{osc}(\tilde{U}_\epsilon).$$

Finally, in view of $\bar{U}_\epsilon$ in (3.37) we have $|\partial_x \bar{U}_\epsilon| < K'$ almost everywhere in $D_\epsilon$ for some bounded constant $K'$. It follows that there is a positive constant $K$ such that

$$\iint A_\epsilon \phi \partial_x \bar{U}_\epsilon \, dx\, dt \le \|\phi\|_\infty \|A_\epsilon\|_{L^\infty(D)} \int_{t_0}^{t_0+\triangle t} \int_{\Gamma_\epsilon} |\partial_x \bar{U}_\epsilon| \, dx\, dt \le K\epsilon(\triangle x)^2.$$

We complete the proof of the theorem. □

We notice that the estimate of $R((\hat{U}_\epsilon, \hat{Q}_\epsilon)^T, \bar{D}, \phi)$ turns out to be the important property of proving the consistency of the generalized Glimm scheme to the initial-boundary value problem of Euler-Poisson equations.

We already constructed $\hat{V}_\epsilon$ for (2.9). We have $\hat{V}_\epsilon = \tilde{V}_\epsilon + \bar{V}_\epsilon^*$ where $\tilde{V}_\epsilon$ is the entropy solution of Riemann problem to the associated homogeneous conservation laws and $\bar{V}_\epsilon$ in (3.37) is an approximate solution of (3.20) . For our approximate solution of (2.8), we define

$$\hat{\rho}(x,t) := \lim_{\epsilon\to 0} \hat{\rho}_\epsilon(x,t), \; \hat{m}(x,t) := \lim_{\epsilon\to 0} \hat{m}_\epsilon(x,t), \quad (x,t) \in D.$$

Then, $(\hat{\rho}, \hat{m})$ can be constructed as an approximate solution of (2.8). But $\hat{Q}(x,t) := \lim_{\epsilon\to 0} \hat{Q}_\epsilon(x,t)$ fails to be an approximate solution of (2.8) due to the facts that $\hat{\rho}$, $\hat{Q}$ both are functions of bounded total variation, which violates the equation $\hat{Q}_x = k\hat{\rho}$ in the compressible Euler-Poisson equations. To overcome this problem, we instead construct $\hat{Q}$ as

$$\hat{Q}(x,t) = \hat{Q}_*(x,t) := \lim_{\epsilon\to 0} \left( Q_B(t) + k \int_{x_0-\Delta x}^{x} \hat{\rho}_\epsilon(s,t)\,ds \right), \quad (x,t) \in D. \tag{3.40}$$

Then, function $\hat{Q}_*$ in (3.40) is indeed a Lipschitz continuous function satisfying $\partial_x \hat{Q}_* = k\hat{\rho}$ in the weak sense. Therefore, $(\hat{U}, \hat{Q}_*(x,t))$ can be used as an approximate solution of (2.8). Finally, by (3.40), Theorems 3.1 and 3.2, we establish the results in the main theorem of this paper.

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