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ON THE APPROXIMATION OF FUNCTION BELONGING TO WEIGHTED $(L^p, \xi(t))$ CLASS BY ALMOST MATRIX SUMMABILITY METHOD OF ITS FOURIER SERIES

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Abstract. In this paper, the degree of approximation of function belonging to weighted $W(L^p, \xi(t))$ class by almost matrix summability of its Fourier series has been determined. The main theorem improves all the previously known theorems in this line of work.

1. Introduction

The concept of convergence and divergence of a series are at common places of analysis. In 1948, Lorentz [6], for first time, defined almost convergence of a bounded sequence. It is easy to verify that a convergent sequence is almost convergent and the limits are the same [17]. The idea of almost convergence led to the formulation of almost matrix summability method. Bernstein [16], Alexits [5], Sahney [3], Chandra [15] and several others have determined the degree of approximation of a function $f \in \text{Lip}\ \alpha$ by $(C, 1)$, (C, δ), (N, p_n) and $(\overline{N}, p_n)$ means of its Fourier series. Working in the same direction Sahney & Rao [13] and Khan [7] have studied the degree of approximation of function belonging to Lip α and $\text{Lip}(\alpha, p)$ by (N, p_n) means an $\text{d}(N, p, q)$ mean respectively. After the definition of almost summabilities methods, Qureshi [8, 9, 10] determined the degree of approximation of certian functions by almost (N, p_n) and almost (N, p, q) means. But till now no work seems to have been done to obtain the degree of approximation of function belonging to weighted $W(L^p, \xi(t))$ class by almost matrix summability method of its Fourier series. It is important to note that almost matrix summability method includes as special cases the methods of almost $(C, 1)$, (C, δ), (N, p_n), $(\overline{N}, p_n)$ and (N, p, q). The weighted $W(L^p, \xi(t))$ class is the generalization of Lip α, $\text{Lip}(\alpha, p)$ and $\text{Lip}(\xi(t), p)$. In an attempt to make an advance study in this direction, in this paper, one theorem on degree of approximation of function of $W(L^p, \xi(t))$ class by almost matrix summability means of its Fourier series has been established.

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2. Definitions and Notations

If a function f is 2π-periodic and Lebesgue integrable in $[-\pi, \pi]$ then its Fourier series is given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{2.1}$$

A function $f \in \text{Lip } \alpha$ if

$$f(x+t) - f(x) = O(|t|^{\alpha}) \quad \text{for } \ 0 < \alpha \leq 1.$$

$f \in \text{Lip}(\alpha, p)$ if

$$\left\{\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right\}^{1/p} = o(t^{\alpha}), \quad 0 < \alpha \leq 1, \ p \geq 1.$$

(def. 5.38 of Mcfadden [12]).

Given a positive increasing function $\xi(t)$ and an integer $p > 1$, we find (Siddiqui [1]) that

$$f \in \text{Lip}(\xi(t), p) \quad \text{if} \quad \left\{\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right\}^{1/p} = O(\xi(t))$$

so that $f \in W(L^p, \xi(t))$ (Lal [18])

$$\left\{\int_0^{2\pi} |f(x+t) - f(x)|^p \sin^{\beta p} x dx\right\}^{1/p} = O(\xi(t)), \quad \beta \geq 0.$$

If $\beta = 0$, then $W(L^p, \xi(t))$ class coincides with the class $\text{Lip}(\xi(t), p)$.

We define the norm by

$$\|f\|_p = \left\{\int_0^{2\pi} |f(x)|^p dx\right\}^{1/p}, \quad p \geq 1$$

The degree of approximation of f is given by (Zygmund [2])

$$E_n(f) = \min_{T_n} \|f - T_n\|_p$$

where $T_n(x)$ is some n^{th} degree trigonometric polynomial.

Lorentz [6] has defined:

A sequence $\{S_n\}$ is said to be almost convergent to a limit S if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=m}^{n+m} S_k = S, \quad \text{uniformly with respect to } m.$$

Let $T = (a_{n,k})$ be an infinite triangular matrix satisfying the Silverman-Toeplitz [13] condition of regularity i.e.

$$\sum_{k=0}^{n} a_{n,k} \to 1, \quad \text{as } n \to \infty,$$
$$a_{n,k} = 0, \quad \text{for } k > n$$
$$\text{and} \qquad \sum_{k=0}^{n} |a_{n,k}| \leq M, \quad \text{a finite constant.}$$

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series such that

$$S_k = \sum_{v=0}^{k} u_v.$$

A series $\sum_{n=0}^{\infty} u_n$ with the sequence of partial sums $\{S_n\}$ is said to be almost matrix summable to S provided

$$t_{n,m} = \sum_{k=0}^{n} a_{n,k} S_{k,m} = \sum_{k=0}^{n} a_{n,n-k} S_{n-k,m} \to S$$

uniformly with respect to m where

$$S_{n-k,m} = \frac{1}{n-k+1} \sum_{v=m}^{n-k+m} S_v$$

and $(a_{n,k})$ is an infinite regular triangular matrix such that the elements $a_{n,k}$ is non-negative and non-decreasing with k, $A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k}$, $A_{n,n} = 1\ \forall n$.

Seven important particular cases of matrix means are

1. $(C, 1)$ mean, when $a_{n,k} = \dfrac{1}{n+1}\ \forall k$
2. Harmonic means, when $a_{n,k} = \dfrac{1}{(n-k+1)\log n}$
3. (C, δ) means when $a_{n,k} = \dfrac{\binom{n-k+\delta+1}{\delta-1}}{\binom{n+\delta}{\delta}}$,
4. (H, p) means, when $a_{n,k} = \dfrac{1}{\log^{p-1}(n+1)} \displaystyle\prod_{q=0}^{p-1} \log^q(k+1)$
5. Nörlund means [12] when $a_{n,k} = \dfrac{p_{n-k}}{P_n}$ where $P_n = \displaystyle\sum_{k=0}^{n} p_k$,
6. Riesz means $(\overline{N}, p_n)$ when $a_{n,k} = \dfrac{p_k}{P_n}$
7. Generalized Nörlund means (N, p, q) [4] when $a_{n,k} = \dfrac{p_{n-k} q_k}{R_n}$ where $R_n = \displaystyle\sum_{k=0}^{n} p_k q_{n-k}$.

We shall use the following notations:

$$\Phi(t) = f(x+t) - f(x-t) - 2f(x), \quad A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k}$$

$$\tau = \left[\frac{1}{t}\right] = \text{Integral part of } \frac{1}{t}$$

$$M_{n,m}(t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\sin(2m+n-k+1)t/2 \sin(n-k+1)t/2}{(n-k+1)\sin^2(t/2)}. \tag{2.2}$$

3. Main Theorem

Quite good amount of works are known on approximation of function belonging to Lipschitz class by almost Nörlund summability means. The purpose of this paper is to determine the degree of approximation of function $f \in W(L^p, \xi(t))$ by almost matrix summability means. In fact, in this paper, we shall prove the following theorem:

Theorem. *Let $T = (a_{n,k})$ is an infinite regular triangular matrix such that $(a_{n,k})$ is non-negative and non-decreasing with k, $A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k}$, $A_{n,n} = 1\ \forall n$. If $f(x)$ is 2π-periodic function belonging to the class $W(L^p, \xi(t))$ then its degree of approximation by $t_{n,m} = \sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \sum_{\nu=m}^{n-k+m} S_\nu$ i.e., almost matrix means of its Fourier series* (2.1) *is given by*

$$\|t_{n,m}(x) - f(x)\|_p = O\{\xi(1/n) n^{\beta+\frac{1}{p}}\} \tag{3.1}$$

provided $\xi(t)$ satisfies the following conditions

$$\left\{\int_0^{1/n} \left(\frac{t|\phi(t)|}{\xi(t)}\right)^p \sin^{\beta p} t dt\right\}^{\frac{1}{p}} = O\Big(\frac{1}{n}\Big) \tag{3.2}$$

$$\left\{\int_{1/n}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} = O(n^\delta) \tag{3.3}$$

where δ is an arbitrary number such that $q(1-\delta)-1 > 0$, conditions (3.2) *and* (3.3) *hold uniformly in x, and*

$$\frac{1}{p} + \frac{1}{q} = 1$$

4. Lemma

For the proof of our theorem following lemmas are required.

Lemma 4.1. *Let $M_{n,m}(t)$ be given by* (2.2). *Then*

$$M_{n,m}(t) = O(n) \quad \text{for} \quad 0 < t \leq \frac{1}{n}.$$

Proof.

$$
\begin{aligned}
|M_{n,m}(t)| &= \left| \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\sin(2m+n-k+1)(t/2)\sin(n-k+1)(t/2)}{(n-k+1)\sin^2(t/2)} \right| \\
&\le \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{|\sin^2(n-k+1)(t/2)|}{(n-k+1)|\sin^2(t/2)|} \\
&\le \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{(n-k+1)^2|\sin^2(t/2)|}{(n-k+1)|\sin^2(t/2)|} \\
&\le \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k}(n-k+1) \\
&\le \left(\frac{n}{\pi}\right) A_{n,n} \\
&= O(n).
\end{aligned}
$$

Lemma 4.2. *Lal* [19]*: If $(a_{n,k})$ is non-negative and non-increasing with k, then for $0 \le a \le b \le \infty$, $0 \le t \le \pi$, and for any n,*

$$
\left| \sum_{k=a}^{b} a_{n,n-k} e^{i(n-k)t} \right| = O(A_{n,\tau}),
$$

where $\tau =$ Integral part of $\frac{1}{t} = [\frac{1}{t}]$.

Lemma 4.3. *Let $M_{n,m}(t)$ be given by* (2.2)*. Then*

$$
M_{n,m}(t) = O\left(\frac{A_{n,\tau}}{t}\right) \quad \text{for} \quad \frac{1}{n} \le t \le \pi \quad \text{where } \tau = [\frac{1}{t}] = \text{integral part of } \frac{1}{t}.
$$

Proof.

$$
\begin{aligned}
|M_{n,m}(t)| &= \frac{1}{2\pi} \left| \sum_{k=0}^{n} \frac{a_{n,n-k}\sin(2m+n-k+1)t/2\sin(n-k+1)t/2}{(n-k+1)|\sin^2(t/2)|} \right| \\
&\le \frac{1}{2\pi t} \sum_{k=0}^{n} a_{n,n-k}\sin(2m+n-k+1)t/2 \\
&\le \frac{1}{2\pi t} \text{ imaginary part of } \left| \sum_{k=0}^{n} a_{n,n-k} e^{i(2m+n-k+1)t/2} \right| \\
&\le \frac{1}{\pi t} \left| \sum_{k=0}^{n} a_{n,n-k} e^{i(n-k)t} \right| \\
&= O\left(\frac{A_{n,\tau}}{t}\right) \quad \text{by lemma (4.2)}
\end{aligned}
$$

5. Proof of Theorem

It is well known that vth partial sum S_ν of the series (2.1) is given by

$$S_v(x) - f(x) = \frac{1}{2\pi}\int_0^\pi \phi(t)\frac{\sin(v+\frac{1}{2})t}{\sin(t/2)}dt$$

then $$S_{n-k,m} - f(x) = \frac{1}{n-k+1}\sum_{v=m}^{n-k+m}\{S_v(x) - f(x)\}$$

$$= \frac{1}{n-k+1}\sum_{v=m}^{n-k+m}\left\{\frac{1}{2\pi}\int_0^\pi \phi(t)\frac{\sin(v+\frac{1}{2})t}{\sin t/2}dt\right\}$$

$$= \frac{1}{2\pi}\int_0^\pi \phi(t)\left\{\frac{1}{n-k+1}\sum_{v=m}^{n-k+m}\frac{\sin(v+\frac{1}{2})t}{\sin t/2}\right\}dt$$

$$= \frac{1}{2\pi}\int_0^\pi \phi(t)\frac{1}{n-k+1}\frac{\sin(2m{+}n{-}k{+}1)t/2\,\sin(n{-}k{+}1)t/2}{\sin^2(t/2)}dt$$

Now

$$\sum_{k=0}^{n} a_{n,n-k}\{S_{n-k,m}(x) - f(x)\}$$

$$= \frac{1}{2\pi}\int_0^\pi \phi(t)\left(\sum_{k=0}^{n} a_{n,n-k}\frac{\sin(2m+n-k+1)t/2\,\sin(n-k+1)t/2}{(n-k+1)\sin^2(t/2)}\right)dt$$

or $$t_{n,m}(x) - f(x)$$

$$= \frac{1}{2\pi}\left[\int_0^{1/n} + \int_{1/n}^{\pi}\right]\phi(t)\left(\sum_{k=0}^{n} a_{n,n-k}\frac{\sin(2m+n-k+1)t/2\,\sin(n-k+1)t/2}{(n-k+1)\sin^2(t/2)}\right)dt$$

$$= \left[\int_0^{1/n} + \int_{1/n}^{\pi}\right]\phi(t)M_{n,m}(t)dt$$

$$= I_1 + I_2, \quad \text{say.} \tag{5.1}$$

Applying Hölder's inequality and the fact that $\phi(t) \in W(L^p, \xi(t))$, we get

$$I_1 = \int_0^{1/n}\phi(t)M_{n,m}(t)dt$$

$$\leq \left\{\int_0^{1/n}\left(\frac{t|\phi(t)|}{\xi(t)}\sin^\beta t\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{1/n}\left(\frac{\xi(t)}{t}\frac{|M_{n,m}(t)|}{\sin^\beta t}\right)^q dt\right\}^{\frac{1}{q}}$$

$$= O(1)\left\{\int_0^{1/n}\left(\frac{\xi(t)}{t^{1+\beta}}\right)^q dt\right\}^{\frac{1}{q}} \quad \text{by lemma (4.1) and condition (3.2)}$$

$$= O(1)\xi\Big(\frac{1}{n}\Big)\left\{\int_0^{1/n}\left(\frac{dt}{t^{(1+\beta)q}}\right)\right\}^{\frac{1}{q}}, \quad \text{by mean value theorem}$$

$$
\begin{aligned}
&= O(1)\xi\Big(\frac{1}{n}\Big)\left[\left\{\frac{t^{-(1+\beta)q+1}}{-(1+\beta)q+1}\right\}_1^{1/n}\right]^{\frac{1}{q}}\\
&= O(1)\xi\Big(\frac{1}{n}\Big)[n^{(1+\beta)q-1}]^{\frac{1}{q}}\\
&= O(1)\xi\Big(\frac{1}{n}\Big)[n^{(1+\beta)-\frac{1}{q}}]\\
I_1 &= O\Big(\xi\Big(\frac{1}{n}\Big)n^{\beta+\frac{1}{p}}\Big) \quad \Big(\because \frac{1}{p}+\frac{1}{q}=1\Big). \qquad (5.2)
\end{aligned}
$$

Also, similarly, as above,

$$
\begin{aligned}
I_2 &= \left[\left\{\int_{1/n}^{\pi}\left|\frac{t^{-\delta}\sin^\beta t|\phi(t)|}{\xi(t)}\right|^p dt\right\}^{\frac{1}{p}}\left\{\int_{1/n}^{\pi}\Big(\frac{\xi(t)}{t^{-\delta}}\frac{M_{n,m}(t)}{\sin^\beta t}\Big)^q dt\right\}^{\frac{1}{q}}\right]\\
&= \left[\left\{\int_{1/\pi}^{n}\left|\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right|^p dt\right\}^{\frac{1}{p}}\left\{\int_{1/n}^{\pi}\Big(\frac{\xi(t)}{t^{\beta+1-\delta}}A_{n,\tau}\Big)^q dt\right\}^{\frac{1}{q}}\right] \quad \text{by lemma (4.3)}\\
&= O(n^\delta)\left[\left\{\int_{1/\pi}^{n}\Big(\frac{\xi(1/y)A_{n,y}}{y^{\delta-\beta-1}}\Big)^q\frac{dy}{y^2}\right\}^{\frac{1}{q}}\right], \quad \text{taking } t=1/y \text{ by condition (3.3)}\\
&= O(n^\delta)\left[\left\{\int_{1/\pi}^{n}\frac{(\xi(1/y)A_{n,y})^q}{y^{q(\delta-\beta-1)+2}}dy\right\}^{\frac{1}{q}}\right]\\
&= O\Big(n^\delta\xi\Big(\frac{1}{n}\Big)A_{n,\eta}\Big)\left[\left\{\int_{1/\pi}^{n}\frac{dy}{y^{q(\delta-\beta-1)+2}}\right\}^{\frac{1}{q}}\right], \quad \text{by mean value theroem}\\
&= O\Big(n^\delta\xi\Big(\frac{1}{n}\Big)\Big)\left[\left[\frac{y^{-q(\delta-\beta-1)-1}}{-q(\delta-\beta-1)-1}\right]^n\right]^{1/q},\\
&= O\Big(n^\delta\xi\Big(\frac{1}{n}\Big)\Big)(n^{-q(\delta-\beta-1)-1})^{\frac{1}{q}}\\
&= O\Big(n^\delta\xi\Big(\frac{1}{n}\Big)n^{-\delta+\beta+1-\frac{1}{q}}\Big)\\
&= O\Big(\xi\Big(\frac{1}{n}\Big)n^{\beta+\frac{1}{p}}\Big). \qquad (5.3)
\end{aligned}
$$

By (5.1), (5.2) & (5.3), we have

$$
\begin{aligned}
|t_{n,m}(x)-f(x)| &= O\Big(\xi\Big(\frac{1}{n+1}\Big)(n+1)^{\beta+\frac{1}{p}}\Big)\\
\|t_{n,m}(x)-f(x)\|_p &= O\left[\int_0^{2\pi}\Big(\xi\Big(\frac{1}{n+1}\Big)n^{\beta+\frac{1}{p}}\Big)^p dx\right]^{1/p}\\
&= O\Big(\xi\Big(\frac{1}{n+1}\Big)\cdot(n+1)^{\beta+\frac{1}{p}}\Big)\left[\int_0^{2\pi}dx\right]^{1/p}\\
&= O\Big(\xi\Big(\frac{1}{n+1}\Big)\cdot(n+1)^{\beta+\frac{1}{p}}\Big).
\end{aligned}
$$

This completes the proof of the theorem.

6. Corollaries

Following corollaries can be derived from main theorem.

Corollary 1. *If $\beta = 0$, and $\xi(t) = t^{\alpha}$, $0 < \alpha \le 1$, then the degree of approximation of a function f belonging to the class $Lip(\alpha, p)$ class is given by*

$$|t_{n,m}(x) - f(x)| = O\left(\frac{1}{n^{\alpha - \frac{1}{p}}}\right).$$

Proof.

$$\begin{aligned} \|t_{n,m}(x) - f(x)\| &= O\left(\frac{1}{n^{\alpha}} \cdot n^{\frac{1}{p}}\right) \\ &= O\left(\frac{1}{n^{\alpha - \frac{1}{p}}}\right). \end{aligned}$$

Now
$$\|t_{n,m}(x) - f(x)\| = \left\{\int_0^{2\pi} |t_{n,m}(x) - f(x)|^p dx\right\}^{\frac{1}{p}}$$

$$O\left(\frac{1}{n^{\alpha - \frac{1}{p}}}\right) = \left\{\int_0^{2\pi} |t_{n,m}(x) - f(x)|^p dx\right\}^{\frac{1}{p}}$$

or
$$O(1) = \left\{\int_0^{2\pi} |t_{n,m}(x) - f(x)|^p dx\right\}^{\frac{1}{p}} O(n^{\alpha - \frac{1}{p}}).$$

Hence

$$|t_{n,m}(x) - f(x)| = O\left(\frac{1}{n^{\alpha - \frac{1}{p}}}\right)$$

for if the right hand side will be $O(1)$, therefore

$$|t_{n,m}(x) - f(x)| = O\left(\frac{1}{n^{\alpha - \frac{1}{p}}}\right)$$

This completes the proof.

Cororllary 2. *If $p \to \infty$ in Corollary 1, then we have, for $0 < \alpha < 1$,*

$$|t_{n,m}(x) - f(x)| = O\left(\frac{1}{n^{\alpha}}\right).$$

7. Remark

An independent proof of Corollary 1 can be derived along the same lines as the theorem.

8. Particular Cases

1. If $a_{n,k} = \frac{p_{n-k}q_k}{P_n}$, $\beta = 0$, $\xi(t) = t^\alpha$, $0 < \alpha \leq 1$, $p \to \infty$ then the result of Qureshi (1982) becomes the particular case of the main theorem.
2. The result of Qureshi (1981) becomes the particular case of our theorem if $a_{n,k} = \frac{p_{n-k}q_k}{R_n}$. β, p and $\xi(t)$ are defined as in (1).
3. If $a_{n,k} = \frac{p_{n-k}q_k}{R_n}$ where $R_n = \sum_{k=0}^{n} p_k q_{n-k}$ and $\xi(t) = \psi(t)$ the result of Qureshi and Neha (1990) becomes the particular case of our theorem.

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References

[1] A. H. Siddiqui, Ph. D. Thesis, Aligarh Muslim University, Aligarh, 1967.

[2] A. Zygmund, *Trigonometric series*, 2nd rev. ed., Vol.1, Cambridge Univ. Press, Cambridge 1959, 114.

[3] B. N. Sahney and D. S. Goel, *On degree of approximation of continuous functions*, Ranchi Uni. Math. J. **4**(1973), 50-53.

[4] D. Borwein, *On product of sequences*, London Mathematical Society **33**(1958), 332-357.

[5] G. Alexits, *Über die Annanherung einer stetigen function durch die Cesàrochen Mittel in hrer Fourier reihe*, Math. Annal **100**(1928), 264-277.

[6] G. G. Lorentz, *A contribution to the theory of divergent series*, Acta Mathematica **80**(1948), 167-190.

[7] Huzoor H. Khan, *On degree of approximation of function belonging to the class Lip(α, p)*, Indian J. Pure and Appl. Math. **5**(1974), 132-136.

[8] K. Qureshi, *On the degree of approximation of a functions belonging to Lip α*, Indian Jour. Pure. Appl. Maths. **13**(1982), 898-903.

[9] K. Qureshi, *On the degree of approximation of a periodic function f by almost Nörlund means*, Tamkang J. Math. **12**(1981), 35-38.

[10] K. Qureshi and H. K. Neha, *On the degree of approximation of a functions belonging to the weighted class by Ganita*, **41**, 17-22.

[11] L. Mc. Faddden, *Absolute Nörlund summability*, Duke Maths. Journal **9**(1942), 168-207.

[12] N. E. Nörlund, *Sur une application des fonction permutables*, Lunds Universitets Arsskrift **16**(1919), 1-10.
[13] N. Sahney and Gopal Rao, *Errors bound in the approximation function*, Bull. Aust. Math. Soc. **6**(1972), 11-18.
[14] O. Toeplitz, *Überallgemeine linear Mittelbildunger*, P. M. F. **22**(1913), 113-119.
[15] Prem Chandra, *On degree of approximation of functions belonging to Lipchitz class*, Nanta Math. **80**(1975), 88-89.
[16] S. Bernstein, *Sur 1' order de la Meilleure approximation des function continue par des polynomes de degree' donne'e*, Memories Acad. Roy-Belyique **4**(1912), 1-104.
[17] S. M. Mazhar and A. H. Siddiqui, *On almost summability of a trigonometrical sequence*, Act. Mathematica, Acad. Sci., (1-z) **20**(1961), 21-24.
[18] Shyam Lal, *On degree of approximation of functions belonging to the weighted* $W(L^p, \xi(t))$ *class by* $(C,1)(E,1)$ *means*, Tamkang Journal of Mathematics **30**(1999), 47-52.
[19] Shyam Lal, *On degree of approximation of conjugate of a function belonging to weighted* $W(L^p, \xi(t))$ *class by matrix summability means of conjugate series of Fourier series*, Tamkang Journal of Mathematics **31**(2000), 7-16.

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