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ON NEW GENERALIZATIONS OF HILBERT'S INEQUALITIES

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Abstract. In this paper, some generalizations of Hilbert's inequalities are shown by introducing two real functions $\phi(x)$ and $\psi(x)$.

1. Introduction

The following inequalities are well-known as Hilbert's integral inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq \pi \left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \right)^{1/2} \tag{1}$$

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} dx \right)^2 dy \leq \pi^2 \int_0^\infty f^2(x)dx \tag{2}$$

where π is the best value (Cf. [1, Chap 9]). Their associated double series forms are as follows, respectively. If $\{a_m\}$ and $\{b_n\}$ are sequences of real numbers such that $0 < \sum_{m=1}^\infty a_m^2 < \infty$ and $0 < \sum_{n=1}^\infty b_n^2 < \infty$, then

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} \leq \pi \left(\sum_{m=1}^\infty a_m^2 \sum_{n=1}^\infty b_n^2 \right)^{1/2} \tag{3}$$

$$\sum_{n=1}^\infty \left(\sum_{m=1}^\infty \frac{a_m}{m+n} \right)^2 \leq \pi^2 \sum_{m=1}^\infty a_m^2 \tag{4}$$

In recent years, Hu [2], Gao [3] and Kuang [4] gave some distinct improvements of (1) and (3), and Gao [5] gave (3) a strengthened version. Yang Bicheng [6] gave interesting generalization of (1) by introducing parameters $\lambda \in (0, 1]$.

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_0^\infty t^{1-\lambda} f^2(t)dt \int_0^\infty t^{1-\lambda} g^2(t)dt \right)^{1/2} \tag{5}$$

where $B(p, q)$ is the β function.

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Yang Bicheng [7] made some generalizations of (1) and (3) and (5) by introducing three parameters, A, B and λ.

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax+By)^\lambda}dxdy < \frac{1}{(AB)^{\lambda/2}}B\Big(\frac{\lambda}{2},\frac{\lambda}{2}\Big) \times \left(\int_0^\infty x^{1-\lambda}f^2(x)dx\int_0^\infty y^{1-\lambda}g^2(t)dy\right)^{1/2} \tag{6}$$

$$\int_0^\infty y^{\lambda-1}\left(\int_0^\infty \frac{f(x)}{(Ax+By)^\lambda}dx\right)^2 dy < \frac{1}{(AB)^\lambda}\left(B\Big(\frac{\lambda}{2},\frac{\lambda}{2}\Big)\right)^2\int_0^\infty x^{1-\lambda}f^2(x)dx \tag{7}$$

$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{a_m b_n}{(Am+Bn)^\lambda} < \frac{1}{(AB)^{\lambda/2}}B\Big(\frac{\lambda}{2},\frac{\lambda}{2}\Big)\left(\sum_{m=1}^\infty m^{1-\lambda}a_m^2\sum_{n=1}^\infty n^{1-\lambda}b_n^2\right)^{1/2} \tag{8}$$

$$\sum_{n=1}^\infty n^{\lambda-1}\left(\sum_{m=1}^\infty \frac{a_m}{(Am+Bn)^\lambda}\right)^2 < \frac{1}{(AB)^\lambda}\left(B\Big(\frac{\lambda}{2},\frac{\lambda}{2}\Big)\right)^2\sum_{m=1}^\infty m^{1-\lambda}a_m^2 \tag{9}$$

Where A, B, $\lambda > 0$.

In this paper, we show some new generalizations on above inequalities by introducing two real functions $\phi(x)$ and $\psi(x)$.

First we introduce some Lemmas:

Lemma 1.([9]) *Let $a < 1$, $\lambda > 0$, define $h(y)$ as*

$$h(y) = y^{-1+a}\int_0^y \frac{1}{(1+u)^\lambda}\Big(\frac{1}{u}\Big)^a du, \quad y \in (0,1]$$

Then $h(y) > h(1)$ $(0 < y < 1)$.

Use the same way, we get the following Lemma.

Lemma 2. *Let $a < 1$, $\lambda > 0$, define $h(y)$ as*

$$h(y) = y^{-1+a}\int_0^y \frac{1}{1+u^\lambda}\Big(\frac{1}{u}\Big)^a du, \quad y \in (0,1].$$

Then $h(y) > h(1)$ $(0 < y < 1)$.

Lemma 3. *Let $p > 1$, $1/p + 1/q = 1$, $\phi(x)$ and $\psi(x)$ are differentiable functions, and $\phi(0) \geq 0$, $\phi'(x) > 0$, $\psi(0) \geq 0$, $\psi'(x) > 0$, $\phi'(x)$ and $\psi'(x)$ has infimum, respectively, $\lambda > 2 - \min\{p,q\}$, define $\omega_1(\phi,\psi,q,\lambda,x)$ as*

$$\omega_1(\phi,\psi,q,\lambda,x) = \int_0^\infty \frac{1}{(\phi(x)+\psi(y))^\lambda}\Big(\frac{\phi(x)}{\psi(y)}\Big)^{\frac{2-\lambda}{q}}dy, \quad x > 0$$

then

$$\omega_1(\phi,\psi,q,\lambda,x) \leq \frac{\phi^{1-\lambda}(x)}{\inf\{\psi'(y)\}}B\left(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p}\right). \tag{10}$$

Proof. Putting $u = \frac{\psi(y)}{\phi(x)}$, we have

$$\omega_1(\phi, \psi, q, \lambda, x) \leq \frac{\phi^{1-\lambda}(x)}{\inf\{\psi'(y)\}} \int_0^\infty \frac{1}{(1+u)^\lambda} \left(\frac{1}{u}\right)^{\frac{2-\lambda}{q}} du$$
$$= \frac{\phi^{1-\lambda}(x)}{\inf\{\psi'(y)\}} B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}\right).$$

The lemma is proved.

Lemma 4. *Let $p > 1$, $1/p + 1/q = 1$, $\phi(x)$ and $\psi(x)$ are differentiable functions, and $\phi(0) \geq 0$, $\phi'(x) > 0$, $\psi(0) \geq 0$, $\psi'(x) > 0$, $\phi'(x)$ and $\psi'(x)$ has infimum, respectively, $\lambda > 2 - \min\{p, q\}$, define $\omega_2(\phi, \psi, q, \lambda, x)$ as*

$$\omega_2(\phi, \psi, q, \lambda, x) = \int_0^\infty \frac{1}{(\phi(x))^\lambda + (\psi(y))^\lambda} \left(\frac{\phi(x)}{\psi(y)}\right)^{\frac{2-\lambda}{q}} dy, \quad x > 0$$

then

$$\omega_2(\phi, \psi, q, \lambda, x) \leq \frac{\phi^{1-\lambda}(x)}{\lambda \inf\{\psi'(y)\}} B\left(\frac{q+\lambda-2}{q\lambda}, 1 - \frac{q+\lambda-2}{q\lambda}\right). \tag{11}$$

Proof. Putting $u = \frac{\psi(y)}{\phi(x)}$, we have

$$\omega_2(\phi, \psi, q, \lambda, x) \leq \frac{\phi^{1-\lambda}(x)}{\inf\{\psi'(y)\}} \int_0^\infty \frac{1}{1+u^\lambda} \left(\frac{1}{u}\right)^{\frac{2-\lambda}{q}} du$$
$$= \frac{\phi^{1-\lambda}(x)}{\lambda \inf\{\psi'(y)\}} B\left(\frac{q+\lambda-2}{q\lambda}, 1 - \frac{q+\lambda-2}{q\lambda}\right).$$

The lemma is proved.

2. Main Results

Now we introduce main results.

Theorem 1. *Let $p > 1$, $1/p + 1/q = 1$, $f(x)$, $g(x) \geq 0$, $\phi(x)$ and $\psi(x)$ are differentiable functions, and $\phi(0) \geq 0$, $\phi'(x) > 0$, $\psi(0) \geq 0$, $\psi'(x) > 0$, $\phi'(x)$ and $\psi'(x)$ has infimum, respectively, $\lambda > 2 - \min\{p, q\}$, such that $0 < \int_0^\infty \phi^{1-\lambda}(t) f^p(t) dt < \infty$ and $0 < \int_0^\infty \phi^{1-\lambda}(t) g^q dt < \infty$. Then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(\phi(x)+\psi(y))^\lambda} dxdy \leq \frac{B(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p})}{(\inf\{\phi'(x)\})^{\frac{1}{q}} (\inf\{\psi'(y)\})^{\frac{1}{p}}}$$
$$\times \left(\int_0^\infty \phi^{1-\lambda}(x) f^p(x) dx\right)^{\frac{1}{p}} \left(\int_0^\infty \psi^{1-\lambda}(y) g^q(y) dy\right)^{\frac{1}{q}} \tag{12}$$

$$\int_0^\infty \psi^{(\lambda-1)(p-1)}(y)\left(\int_0^\infty \frac{f(x)}{(\phi(x)+\psi(y))^\lambda}dx\right)^p dy \le \frac{\left(B(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p})\right)^p}{(\inf\{\phi'(x)\})^{p-1}\inf\{\psi'(y)\}} \times \int_0^\infty \phi^{1-\lambda}(x)f^p(x)dx \qquad (13)$$

In particular, when $p = q = 2$, we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(\phi(x)+\psi(y))^\lambda}dxdy \le \frac{B(\frac{\lambda}{2},\frac{\lambda}{2})}{(\inf\{\phi'(x)\}\inf\{\psi'(y)\})^{\frac{1}{2}}} \times \left(\int_0^\infty \phi^{1-\lambda}(x)f^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^\infty \psi^{1-\lambda}(y)g^2(y)dy\right)^{\frac{1}{2}} \qquad (14)$$

$$\int_0^\infty \psi^{\lambda-1}(y)\left(\int_0^\infty \frac{f(x)}{(\phi(x)+\psi(y))^\lambda}dx\right)^2 dy \le \frac{\left(B(\frac{\lambda}{2},\frac{\lambda}{2})\right)^2}{\inf\{\phi'(x)\})\inf\{\psi'(y)\}} \times \int_0^\infty \phi^{1-\lambda}(x)f^2(x)dx. \qquad (15)$$

Proof. By Hölder's inequality and (10), we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(\phi(x)+\psi(y))^\lambda}dxdy\\
&=\int_0^\infty\int_0^\infty \frac{f(x)}{(\phi(x)+\psi(y))^{\frac{\lambda}{p}}}\left(\frac{\phi(x)}{\psi(y)}\right)^{\frac{2-\lambda}{pq}}\frac{g(y)}{(\phi(x)+\psi(y))^{\frac{\lambda}{q}}}\left(\frac{\psi(y)}{\phi(x)}\right)^{\frac{2-\lambda}{pq}}dxdy\\
&\le\left(\int_0^\infty\int_0^\infty \frac{f^p(x)}{(\phi(x)+\psi(y))^\lambda}\left(\frac{\phi(x)}{\psi(y)}\right)^{\frac{2-\lambda}{pq}}dxdy\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^\infty\int_0^\infty \frac{g^q(y)}{(\phi(x)+\psi(y))^\lambda}\left(\frac{\psi(y)}{\phi(x)}\right)^{\frac{2-\lambda}{p}}dxdy\right)^{\frac{1}{q}}\\
&=\left(\int_0^\infty \omega_1(\phi,\psi,q,\lambda,x)f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty \omega_1(\psi,\phi,q,\lambda,x)g^q(y)dy\right)^{\frac{1}{q}}\\
&\le\frac{B(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p})}{(\inf\{\phi'(x)\})^{\frac{1}{q}}(\inf\{\psi'(y)\})^{\frac{1}{p}}}\left(\int_0^\infty \phi^{1-\lambda}(x)f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty \psi^{1-\lambda}(y)g^q(y)dy\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence (12) is valid.

By Hölder's inequality and (10), we have

$$\begin{aligned}
&\int_0^\infty \frac{f(x)}{(\phi(x)+\psi(y))^\lambda}dx \\
&= \int_0^\infty \frac{f(x)}{(\phi(x)+\psi(y))^{\frac{\lambda}{p}}}\left(\frac{\phi(x)}{\psi(y)}\right)^{\frac{2-\lambda}{pq}}\frac{1}{(\phi(x)+\psi(y))^{\frac{\lambda}{q}}}\left(\frac{\psi(y)}{\phi(x)}\right)^{\frac{2-\lambda}{pq}}dx \\
&\leq \left(\int_0^\infty \frac{f^p(x)}{(\phi(x)+\psi(y))^\lambda}\left(\frac{\phi(x)}{\psi(y)}\right)^{\frac{2-\lambda}{q}}dx\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{1}{(\phi(x)+\psi(y))^\lambda}\left(\frac{\psi(y)}{\phi(x)}\right)^{\frac{2-\lambda}{p}}dx\right)^{\frac{1}{q}} \\
&\leq \left(\frac{B(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p})\psi^{1-\lambda}(y)}{\inf\{\phi'(x)\}}\right)^{\frac{1}{q}}\left(\int_0^\infty \frac{f^p(x)}{(\phi(x)+\psi(y))^\lambda}\left(\frac{\phi(x)}{\psi(y)}\right)^{\frac{2-\lambda}{q}}dx\right)^{\frac{1}{p}}
\end{aligned}$$

Then

$$\begin{aligned}
&\int_0^\infty \psi^{(\lambda-1)(p-1)}(y)\left(\int_0^\infty \frac{f(x)}{(\phi(x)+\psi(y))^\lambda}dx\right)^p dy \\
&\leq \left(\frac{B(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p})}{\inf\{\phi'(x)\}}\right)^{\frac{p}{q}}\int_0^\infty\int_0^\infty \frac{f^p(x)}{(\phi(x)+\psi(y))^\lambda}\left(\frac{\phi(x)}{\psi(y)}\right)^{\frac{2-\lambda}{q}}dxdy \\
&\leq \frac{\left(B(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p})\right)^p}{(\inf\{\phi'(x)\})^{p-1}\inf\{\psi'(y)\}}\int_0^\infty \phi^{1-\lambda}(x)(x)f^p(x)dx
\end{aligned}$$

Hence (13) is valid. The theorem is proved.

Theorem 2. *Let $p>1$, $1/p+1/q=1$, $f(x)$, $g(x)\geq 0$, $\phi(x)$ and $\psi(x)$ are differentiable functions, and $\phi(0)\geq 0$, $\phi'(x)>0$, $\psi(0)\geq 0$, $\psi'(x)>0$, $\phi'(x)$ and $\psi'(x)$ has infimum, respectively, $\lambda>2-\min\{p,q\}$, such that $0<\int_0^\infty \phi^{1-\lambda}(t)f^p(t)dt<\infty$ and $0<\int_0^\infty \phi^{1-\lambda}(t)g^q dt<\infty$. Then*

$$\begin{aligned}
\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(\phi(x))^\lambda+(\psi(y))^\lambda}dxdy \leq{}& \frac{\left(B(\frac{q+\lambda-2}{q\lambda},1-\frac{q+\lambda-2}{q\lambda})\right)^{\frac{1}{p}}\left(B(\frac{p+\lambda-2}{p\lambda},1-\frac{p+\lambda-2}{p\lambda})\right)^{\frac{1}{q}}}{\lambda(\inf\{\phi'(x)\})^{\frac{1}{q}}(\inf\{\psi'(y)\})^{\frac{1}{p}}} \\
&\times\left(\int_0^\infty \phi^{1-\lambda}(x)f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty \psi^{1-\lambda}(y)g^q(y)dy\right)^{\frac{1}{q}} \quad (16)
\end{aligned}$$

$$\begin{aligned}
&\int_0^\infty \psi^{(\lambda-1)(p-1)}(y)\left(\int_0^\infty \frac{f(x)}{(\phi(x))^\lambda+(\psi(y))^\lambda}dx\right)^p dy \\
&\leq \frac{\left(B(\frac{p+\lambda-2}{p\lambda},1-\frac{p+\lambda-2}{p\lambda})\right)^{p-1}B(\frac{q+\lambda-2}{q\lambda},1-\frac{q+\lambda-2}{q\lambda})}{\lambda^p(\inf\{\phi'(x)\})^{p-1}\inf\{\psi'(y)\}}\int_0^\infty \phi^{1-\lambda}(x)f^p(x)dx. \quad (17)
\end{aligned}$$

In particular, when $p = q = 2$, we have

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(\phi(x))^\lambda+(\psi(y))^\lambda}dxdy \leq \frac{B(\frac{1}{2},\frac{1}{2})}{\lambda(\inf\{\phi'(x)\}\inf\{\psi'(y)\})^{\frac{1}{2}}} \times \left(\int_0^\infty \phi^{1-\lambda}(x)f^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^\infty \psi^{1-\lambda}(y)g^2(y)dy\right)^{\frac{1}{2}} \quad (18)$$

$$\int_0^\infty \psi^{\lambda-1}(y)\left(\int_0^\infty \frac{f(x)}{(\phi(x))^\lambda+(\psi(y))^\lambda}dx\right)^2 dy \leq \frac{\left(B(\frac{1}{2},\frac{1}{2})\right)^2}{\lambda^2\inf\{\phi'(x)\})\inf\{\psi'(y)\}} \times \int_0^\infty \phi^{1-\lambda}(x)f^2(x)dx \qquad (19)$$

Proof. By Hölder's inequality and (11), we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(\phi(x))^\lambda+(\psi(y))^\lambda}dxdy \\
&= \int_0^\infty\int_0^\infty \frac{f(x)}{((\phi(x))^\lambda+(\psi(y))^\lambda)^{\frac{1}{p}}}\left(\frac{\phi(x)}{\psi(y)}\right)^{\frac{2-\lambda}{pq}} \frac{g(y)}{((\phi(x))^\lambda+(\psi(y))^\lambda)^{\frac{1}{p}}}\left(\frac{\psi(y)}{\phi(x)}\right)^{\frac{2-\lambda}{pq}} dxdy \\
&\leq \left(\int_0^\infty\int_0^\infty \frac{f^p(x)}{(\phi(x))^\lambda+(\psi(y))^\lambda}\left(\frac{\phi(x)}{\psi(y)}\right)^{\frac{2-\lambda}{q}} dxdy\right)^{\frac{1}{p}} \\
&\times \left(\int_0^\infty\int_0^\infty \frac{g^q(y)}{(\phi(x))^\lambda+(\psi(y))^\lambda}\left(\frac{\psi(y)}{\phi(x)}\right)^{\frac{2-\lambda}{p}} dxdy\right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \omega_2(\phi,\psi,q,\lambda,x)f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty \omega_2(\psi,\phi,q,\lambda,x)g^q(y)dydy\right)^{\frac{1}{q}} \\
&\leq \frac{B(\frac{q+\lambda-2}{q\lambda},\frac{p+\lambda-2}{p\lambda})}{\lambda(\inf\{\phi'(x)\})^{\frac{1}{q}}(\inf\{\psi'(y)\})^{\frac{1}{p}}}\left(\int_0^\infty \phi^{1-\lambda}(x)f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty \psi^{1-\lambda}(y)g^q(y)dy\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence (16) is valid.

By Hölder's inequality and (11), we have

$$\begin{aligned}
&\int_0^\infty \frac{f(x)}{(\phi(x))^\lambda+(\psi(y))^\lambda}dx \\
&= \int_0^\infty \frac{f(x)}{((\phi(x))^\lambda+(\psi(y))^\lambda)^{\frac{1}{p}}}\left(\frac{\phi(x)}{\psi(y)}\right)^{\frac{2-\lambda}{pq}} \frac{1}{((\phi(x))^\lambda+(\psi(y))^\lambda)^{\frac{1}{q}}}\left(\frac{\psi(y)}{\phi(x)}\right)^{\frac{2-\lambda}{pq}} dx \\
&\leq \left(\int_0^\infty \frac{f^p(x)}{(\phi(x))^\lambda+(\psi(y))^\lambda}\left(\frac{\phi(x)}{\psi(y)}\right)^{\frac{2-\lambda}{q}} dx\right)^{\frac{1}{p}}\left(\int_0^\infty \frac{1}{(\phi(x))^\lambda+(\psi(y))^\lambda}\left(\frac{\psi(y)}{\phi(x)}\right)^{\frac{2-\lambda}{p}} dx\right)^{\frac{1}{q}}
\end{aligned}$$

$$\leq \left(\frac{B(\frac{p+\lambda-2}{p\lambda},1-\frac{p+\lambda-2}{p\lambda})\psi^{1-\lambda}(y)}{\lambda\inf\{\phi'(x)\}}\right)^{\frac{1}{q}}\left(\int_0^\infty \frac{f^p(x)}{(\phi(x))^\lambda+(\psi(y))^\lambda}\left(\frac{\phi(x)}{\psi(y)}\right)^{\frac{2-\lambda}{q}}dx\right)^{\frac{1}{p}}.$$

Then

$$\int_0^\infty \psi^{(\lambda-1)(p-1)}(y)\left(\int_0^\infty \frac{f(x)}{(\phi(x))^\lambda+(\psi(y))^\lambda}dx\right)^p dy$$
$$\leq \left(\frac{B(\frac{p+\lambda-2}{p\lambda},1-\frac{p+\lambda-2}{p\lambda})}{\lambda\inf\{\phi'(x)\}}\right)^{\frac{p}{q}}\int_0^\infty\int_0^\infty \frac{f^p(x)}{(\phi(x))^\lambda+(\psi(y))^\lambda}\left(\frac{\phi(x)}{\psi(y)}\right)^{\frac{2-\lambda}{q}}dxdy$$
$$\leq \frac{\left(B(\frac{p+\lambda-2}{p\lambda},1-\frac{p+\lambda-2}{p\lambda})\right)^{p-1}B(\frac{q+\lambda-2}{q\lambda},1-\frac{q+\lambda-2}{q\lambda})}{\lambda^p(\inf\{\phi'(x)\})\inf\{\psi'(y)\}}\int_0^\infty \phi^{1-\lambda}(x)f^p(x)dx.$$

Hence (17) is valid. The theorem is proved.

Theorem 3. *Let $p>1$, $1/p+1/q=1$, $\{a_m\}$ and $\{b_n\}$ are two arbitary sequences of nonnegative real numbers. $\phi(x)$ and $\psi(y)$ be differentiable functions, and $\phi(0)\geq 0$, $\phi'(x)>0$, $\psi(0)\geq 0$, $\psi'(x)>0$, $\phi'(x)$ and $\psi'(x)$ has infimum, respectively, $2\geq\lambda>2-\min\{p,q\}$, such that $0<\sum_{m=1}^\infty(\phi(m))^{1-\lambda}a_m<\infty$ and $0<\sum_{n=1}^\infty(\psi(n))^{1-\lambda}b_n<\infty$. Then*

$$\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{a_mb_n}{(\phi(m)+\psi(n))^\lambda}\leq\frac{B(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p})}{(\inf\{\phi'(x)\})^{\frac{1}{q}}(\inf\{\psi'(y)\})^{\frac{1}{p}}}\times\left(\sum_{m=1}^\infty\phi^{1-\lambda}(m)a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty\psi^{1-\lambda}(n)b_n^q\right)^{\frac{1}{q}} \tag{20}$$

$$\sum_{n=1}^\infty\psi^{(\lambda-1)(p-1)}(n)\left(\sum_{m=1}^\infty\frac{a_m}{(\phi(m)+\psi(n))^\lambda}\right)^p<\frac{\left(B(\frac{q+\lambda-2}{q},\frac{p+\lambda-2}{p})\right)^p}{(\inf\{\phi'(x)\})^{p-1}\inf\{\psi'(y)\}}\times\sum_{m=1}^\infty\phi^{1-\lambda}(m)a_m^p. \tag{21}$$

Proof. By Hölder's inequality , we have

$$\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{a_mb_n}{(\phi(m)+\psi(n))^\lambda}$$
$$=\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{a_m}{(\phi(m)+\psi(n))^{\frac{\lambda}{p}}}\left(\frac{\phi(m)}{\psi(n)}\right)^{\frac{2-\lambda}{pq}}\frac{b_n}{(\phi(m)+\psi(n))^{\frac{\lambda}{q}}}\left(\frac{\psi(n)}{\phi(m)}\right)^{\frac{2-\lambda}{pq}}$$
$$\leq\left(\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{a_m^p}{(\phi(m)+\psi(n))^\lambda}\left(\frac{\phi(m)}{\psi(n)}\right)^{\frac{2-\lambda}{q}}\right)^{\frac{1}{p}}$$

$$\times \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{b_n^q}{(\phi(m)+\psi(n))^\lambda} \left(\frac{\psi(n)}{\phi(m)} \right)^{\frac{2-\lambda}{p}} \right)^{\frac{1}{q}}$$
$$= \left(\sum_{m=1}^{\infty} \omega_3(\phi,\psi,q,\lambda,m) a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \omega_3(\psi,\phi,,p,\lambda,n) b_n^q \right)^{\frac{1}{q}}$$

where $\omega_3(\phi,\psi,q,\lambda,m) = \sum_{n=1}^{\infty} \frac{1}{(\phi(m)+\psi(n))^\lambda} (\frac{\phi(m)}{\psi(n)})^{\frac{2-\lambda}{q}}$.

By (10), we have

$$\omega_3(\phi,\psi,q,\lambda,m) < \int_0^{\infty} \frac{1}{(\phi(m)+\psi(y))^\lambda} \left(\frac{\phi(m)}{\psi(y)} \right)^{\frac{2-\lambda}{q}} dy$$
$$\leq \frac{\phi^{1-\lambda}(m)}{\inf\{\psi'(y)\}} B\left(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p} \right). \tag{22}$$

Hence (20) is valid.

By Hölder's inequality and (22), we have

$$\sum_{m=1}^{\infty} \frac{a_m}{(\phi(m)+\psi(n))^\lambda}$$
$$= \sum_{m=1}^{\infty} \frac{a_m}{(\phi(m)+\psi(n))^{\frac{\lambda}{p}}} \left(\frac{\phi(m)}{\psi(n)} \right)^{\frac{2-\lambda}{pq}} \frac{1}{(\phi(m)+\psi(n))^{\frac{\lambda}{q}}} \left(\frac{\psi(n)}{\phi(m)} \right)^{\frac{2-\lambda}{pq}}$$
$$\leq \left(\sum_{m=1}^{\infty} \frac{a_m^p}{(\phi(m)+\psi(n))^\lambda} \left(\frac{\phi(m)}{\psi(n)} \right)^{\frac{2-\lambda}{q}} \right)^{\frac{1}{p}} \left(\sum_{m=1}^{\infty} \frac{1}{(\phi(m)+\psi(n))^\lambda} \left(\frac{\psi(n)}{\phi(m)} \right)^{\frac{2-\lambda}{p}} \right)^{\frac{1}{q}}$$
$$< \left(\frac{B(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p})\psi^{1-\lambda}(n)}{\inf\{\phi'(x)\}} \right)^{\frac{1}{q}} \left(\sum_{m=1}^{\infty} \frac{a_m^p}{(\phi(m)+\psi(n))^\lambda} (\frac{\phi(m)}{\psi(n)})^{\frac{2-\lambda}{q}} \right)^{\frac{1}{p}}.$$

Then

$$\sum_{n=1}^{\infty} \psi^{(\lambda-1)(p-1)}(n) \left(\sum_{m=1}^{\infty} \frac{a_m}{(\phi(m)+\psi(n))^\lambda} \right)^p$$
$$< \left(\frac{B(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p})}{\inf\{\phi'(x)\}} \right)^{\frac{p}{q}} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m^p}{(\phi(m)+\psi(n))^\lambda} (\frac{\phi(m)}{\psi(n)})^{\frac{2-\lambda}{q}}$$
$$< \frac{\left(B(\frac{q+\lambda-2}{q}, \frac{p+\lambda-2}{p}) \right)^p}{(\inf\{\phi'(x)\})^{p-1} \inf\{\psi'(y)\}} \sum_{m=1}^{\infty} \phi^{1-\lambda}(m) a_m^p.$$

Hence (21) is valid. The theorem is proved.

In a similar way to the proof of Theorem 3, the following Theorem 4 can be showed.

Theorem 4. *Let $p > 1$, $1/p + 1/q = 1$, $\{a_m\}$ and $\{b_n\}$ be two arbitary sequences of nonnegative real numbers. $\phi(x)$ and $\psi(y)$ be differentiable functions, and $\phi(0) \geq 0$, $\phi'(x) > 0$, $\psi(0) \geq 0$, $\psi'(x) > 0$, $\phi'(x)$ and $\psi'(x)$ has infimum, respectively, $2 \geq \lambda > 2 - \min\{p, q\}$, such that $0 < \sum_{m=1}^{\infty}(\phi(m))^{1-\lambda} a_m < \infty$ and $0 < \sum_{n=1}^{\infty}(\psi(n))^{1-\lambda} b_n < \infty$. Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(\phi(m))^{\lambda}+(\psi(n))^{\lambda}} < \frac{\left(B(\frac{q+\lambda-2}{q\lambda}, 1-\frac{q+\lambda-2}{q\lambda})\right)^{\frac{1}{p}}\left(B(\frac{p+\lambda-2}{p\lambda}, 1-\frac{p+\lambda-2}{p\lambda})\right)^{\frac{1}{q}}}{\lambda(\inf\{\phi'(x)\})^{\frac{1}{q}}(\inf\{\psi'(y)\})^{\frac{1}{p}}} \times\left(\sum_{m=1}^{\infty}\phi^{1-\lambda}(m)a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\psi^{1-\lambda}(n)b_n^q\right)^{\frac{1}{q}} \tag{23}$$

$$\sum_{n=1}^{\infty}\psi^{(\lambda-1)(p-1)}(n)\left(\sum_{m=1}^{\infty}\frac{a_m}{(\phi(m))^{\lambda}+(\psi(n))^{\lambda}}\right)^{p} < \frac{\left(B(\frac{p+\lambda-2}{p\lambda}, 1-\frac{p+\lambda-2}{p\lambda})\right)^{p-1}B(\frac{q+\lambda-2}{q\lambda}, 1-\frac{q+\lambda-2}{q\lambda})}{\lambda^p(\inf\{\phi'(x)\})^{p-1}\inf\{\psi'(y)\}}\sum_{m=1}^{\infty}\phi^{1-\lambda}(m)a_m^p \tag{24}$$

Remark 1. For $\phi(x) = Ax$, $\psi(y) = By$, $A > 0$, $B > 0$, inequalities (14) and (15) change to (6) and (7), respectively, hence inequalities (14) and (15) are generalizations of (6) and (7), respectively.

Remark 2. For $\phi(x) = Ax$, $\psi(y) = By$, $A > 0$, $B > 0$, $p = q = 2$, inequalities (20) and (21) change to (8) and (9), respectively, hence inequalities (20) and (21) are generalizations of (8) and (9), respectively.

Remark 3. For $\phi(x) = x$, $\psi(y) = y$, $p = q = 2$, inequalities (12), (14), (16) and (18) change to (5), hence inequalities (12), (14), (16) and (18) are generalizations of (5).

Remark 4. For $\phi(x) = x$, $\psi(y) = y$, $p = q = 2$, $\lambda = 1$, inequalities (23) and (24) change to (3) and (4), respectively, hence inequalities (23) and (24) are generalizations of (3) and (4), respectively.

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