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STRONG CONVERGENCE THEOREMS FOR EQUILIBRIUM PROBLEMS INVOLVING BREGMAN FUNCTIONS IN BANACH SPACES

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Abstract. In this paper, using Bregman functions, we introduce new Halpern-type iterative algorithms for finding a solution of an equilibrium problem in Banach spaces. We prove the strong convergence of a modified Halpern-type scheme to an element of the set of solution of an equilibrium problem in a reflexive Banach space. This scheme has an advantage that we do not use any Bregman projection of a point on the intersection of closed and convex sets in a practical calculation of the iterative sequence. Finally, some application of our results to the problem of finding a minimizer of a continuously Fréchet differentiable and convex function in a Banach space is presented. Our results improve and generalize many known results in the current literature.

1. Introduction

The equilibrium problem, introduced by Blum and Oettli [1] in 1994, has been attracting a growing attention of researchers; see, e.g., [2, 3, 4, 5] and the references therein. Numerous problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem. There have appeared many papers in this subject with different approach [6, 7]. Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let E be a Banach space with the norm $\|.\|$ and the dual space E^*. For any $x \in E$, we denote the value of $x^* \in E^*$ at x by $\langle x, x^*\rangle$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in E, we denote the strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in E$ as $n \to \infty$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is denoted by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x - y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let $S_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in S_E$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

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exists. In this case, E is called *smooth*. If the limit (1.1) is attained uniformly for all $x, y \in S_E$, then E is called *uniformly smooth*. The Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S_E$ and $x \neq y$. It is well known that E is uniformly convex if and only if E^* is uniformly smooth. It is also known that if E is reflexive, then E is strictly convex if and only if E^* is smooth; for more details, see [8, 9].

Let C be a nonempty subset of E. Let $T : C \to E$ be a mapping. We denote the set of fixed points of T by $F(T)$, *i.e.*, $F(T) = \{x \in C : Tx = x\}$. Recall that the Halpern iteration is given by the following formula

$$\begin{cases} u \in C,\ x_1 \in C \quad \text{chosen arbitrarily,} \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \end{cases} \tag{1.2}$$

where the sequences $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfy some appropriate conditions. The construction of fixed points of nonexpansive mappings via Halpern's algorithm [10] has been extensively investigated recently in the current literature (see, for example, [11] and the references therein).

Let E be a smooth, strictly convex and reflexive Banach space and let J be the normalized duality mapping of E. Let C be a nonempty, closed and convex subset of E. The generalized projection Π_C from E onto C is defined and denoted by

$$\Pi_C(x) = \operatorname{argmin}_{y\in C}\phi(y, x)$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$. Let C be a nonempty, closed and convex subset of a smooth Banach space E, let T be a mapping from C into itself. A point $p \in C$ is said to be an *asymptotic fixed point* [7, 13] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges weakly to p and $\lim_{n\to\infty} \|x_n - T x_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$.

Let C be a nonempty, closed and convex subset of a Banach space E. Let $f : C \times C \to \mathbb{R}$ be a bifunction. Consider the following equilibrium problem: Find $p \in C$ such that

$$f(p, y) \geq 0, \qquad \forall y \in C. \tag{1.3}$$

For solving the equilibrium problem, let us assume that $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $y \in C$, the function $x \longmapsto f(x, y)$ is upper semicontinuous;

(A4) for each $x \in C$, the function $y \longmapsto f(x, y)$ is convex and lower semicontinuous.

The set of solutions of problem (1.3) is denoted by $EP(f)$.

Following Matsushita and Takahashi [7], a mapping $T : C \to C$ is said to be *relatively nonexpansive* if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $\phi(u, Tx) \le \phi(u, x)$, $\forall u \in F(T)$, $x \in C$;

(3) $\hat{F}(T) = F(T)$.

Recently, Takahashi and Zembayashi [4] proved the following strong convergence theorem for relatively nonexpansive mappings in a Banach space.

Theorem 1.1. *Let E be a uniformly smooth and strictly convex Banach space. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and T be a relatively nonexpansive mapping from C into itself such that $F(T) \cap EP(f) \neq \emptyset$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ u_n \in C \quad \text{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \ge 0, \quad \forall y \in C, \\ C_n = \{z \in C_n : \phi(z, u_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x \end{cases} \tag{1.4}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the normalized duality mapping on E, $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ satisfies $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and $\{r_n\}_{n\in\mathbb{N}} \subset [a, \infty)$ for some $a > 0$. Then, $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\Pi_{F(T)\cap EP(f)} x$ as $n \to \infty$.*

In 2010, Plubtieng and Ungchiterakool [3] proved the following strong convergence theorem for relatively nonexpansive mappings in a Banach space.

Theorem 1.2. *Let E be a uniformly smooth and uniformly convex Banach space and let $\hat{C}$ and C be a nonempty, closed and convex subset of E. Let f be a bifunction from $C \times C \to \mathbb{R}$ satisfying* (A1)–(A4) *and $EP(f) \neq \emptyset$. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ be sequences generated by*

$$\begin{cases} x_0 \in E, \\ u_n \in C = C_1 \quad \textit{such that} \;\; f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J x_n\rangle \ge 0, \quad \forall y \in C, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J u_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \qquad n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ satisfies either

(a) $0 \leq \alpha_n < 1$ *for all* $n \in \mathbb{N}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$ *or,*

(b) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$.

Let $\{r_n\}_{n\in\mathbb{N}}$ *be a sequence in* $(0,\infty)$ *such that* $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. *Then* $\{x_n\}_{n\in\mathbb{N}}$, $\{u_n\}_{n\in\mathbb{N}}$, *and* $\{y_n\}_{n\in\mathbb{N}}$ *converge strongly to* $\Pi_{EP(f)} x_0$ *as* $n \to \infty$.

1.1. Some facts about gradients

Let E be a real Banach space and $g : E \to (-\infty, +\infty]$ be a convex function. The domain of g is denoted by $\operatorname{dom} g = \{x \in E : g(x) < \infty\}$. Let $x \in \operatorname{int} \operatorname{dom} g$ and $y \in E$. The *right-hand derivative* of g at x in the direction y is defined and denoted by

$$g^o(x, y) = \lim_{t\downarrow 0} \frac{g(x+ty) - g(x)}{t}. \tag{1.6}$$

The function g is called *Gâteaux differentiable* at x if $\lim_{t\to 0} \frac{g(x+ty)-g(x)}{t}$ exists for any y. In this case $g^o(x, y)$ coincides with $\nabla g(x)$, the value of the *gradient* ∇g of g at x. The function g is said to be *Gâteaux differentiable* if it is Gâteaux differentiable everywhere. The function g is said to be *Fréchet differentiable* at x if this limit is attained uniformly in $\|y\| = 1$. The function g is said to be *Fréchet differentiable* if it is Fréchet differentiable everywhere. It is well known that if a continuous convex function $g : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous (see, for example, [14](Proposition 1.1.10)). Also, it is known that if g is Fréchet differentiable, then ∇g is norm-to-norm continuous (see, [15]). The function g is said to be *strongly coercive* if

$$\lim_{\|x_n\|\to\infty} \frac{g(x_n)}{\|x_n\|} = \infty.$$

It is also said to be *bounded on bounded subsets of* E if $g(U)$ is bounded for each bounded subset U of E. Finally, g is said to be *uniformly Fréchet differentiable* on a subset X of E if the limit (1.6) is attained uniformly for all $x \in X$ and $\|y\| = 1$.

Let $A : E \to 2^{E^*}$ be a set-valued mapping. We define the domain and range of A by $\operatorname{dom} A = \{x \in E : Ax \neq \emptyset\}$ and $\operatorname{ran} A = \cup_{x\in E} Ax$, respectively. The graph of A is denoted by $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. The mapping $A \subset E \times E^*$ is said to be *monotone* [16] if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in A$. It is also said to be *maximal monotone* [17] if its graph is not contained in the graph of any other monotone operator on E. If $A \subset E \times E^*$ is maximal monotone, then we can show that the set $A^{-1}0 = \{z \in E : 0 \in Az\}$ is closed and convex. A mapping $A : \operatorname{dom} A \subset E \to E^*$ is called γ-inverse strongly monotone if there exists a positive real number γ such that for all $x, y \in \operatorname{dom} A$, $\langle x - y, Ax - Ay \rangle \geq \gamma \|Ax - Ay\|^2$.

1.2. Some facts about conjugate functions

Let E be a reflexive Banach space and $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. The *conjugate function* g^* of g is defined by

$$g^*(x^*) = \sup_{x \in E}\{\langle x, x^* \rangle - g(x)\}$$

for all $x^* \in E^*$. It is well known that $g(x) + g^*(x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in E \times E^*$. It is also known that $(x, x^*) \in \partial g$ is equivalent to

$$g(x) + g^*(x^*) = \langle x, x^* \rangle. \tag{1.7}$$

Here, ∂g is the subdifferential of g [18, 19]. We also know that if $g : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous and convex function, then $g^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semicontinuous and convex function; see [9] for more details on convex analysis.

1.3. Some facts about Bregman distances

Let E be a Banach space and let E^* be the dual space of E. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the *Bregman distance* [22, 23] corresponding to g is the function $D_g : E \times E \to \mathbb{R}$ defined by

$$D_g(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y) \rangle, \quad \forall x, y \in E. \tag{1.8}$$

It is clear that $D_g(x, y) \geq 0$ for all $x, y \in E$. In that case when E is a smooth Banach space, setting $g(x) = \|x\|^2$ for all $x \in E$, we obtain that $\nabla g(x) = 2Jx$ for all $x \in E$ and hence $D_g(x, y) = \phi(x, y)$ for all $x, y \in E$.

Let E be a Banach space and let C be a nonempty and convex subset of E. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then, we know from [24] that for $x \in E$ and $x_0 \in C$, $D_g(x_0, x) = \min_{y \in C} D_g(y, x)$ if and only if

$$\langle y - x_0, \nabla g(x) - \nabla g(x_0) \rangle \leq 0, \qquad \forall y \in C. \tag{1.9}$$

Furthermore, if C is a nonempty, closed and convex subset of a reflexive Banach space E and $g : E \to \mathbb{R}$ is a strongly coercive Bregman function, then for each $x \in E$, there exists a unique $x_0 \in C$ such that

$$D_g(x_0, x) = \min_{y \in C} D_g(y, x).$$

The *Bregman projection* proj_C^g from E onto C is defined by $\operatorname{proj}_C^g(x) = x_0$ for all $x \in E$. It is also well known that proj_C^g has the following property:

$$D_g\left(y, \operatorname{proj}_C^g x\right) + D_g\left(\operatorname{proj}_C^g x, x\right) \leq D_g(y, x) \tag{1.10}$$

for all $y \in C$ and $x \in E$ (see [14] for more details).

1.4. Some facts about uniformly convex functions

Let E be a Banach space and let $B_s := \{z \in E : \|z\| \le s\}$ for all $s > 0$. Then a function $g : E \to \mathbb{R}$ is said to be *uniformly convex on bounded subsets of E* ([25] (pp. 203, 221)) if $\rho_s(t) > 0$ for all $s, t > 0$, where $\rho_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_s(t) = \inf_{x,y \in B_s, \|x-y\|=t, \alpha \in (0,1)} \frac{\alpha g(x) + (1-\alpha) g(y) - g(\alpha x + (1-\alpha) y)}{\alpha(1-\alpha)} \tag{1.11}$$

for all $t \ge 0$. The function ρ_s is called the gauge of uniform convexity of g. The function g is also said to be *uniformly smooth on bounded subsets of E* ([25](pp. 207, 221)) if $\lim_{t \downarrow 0} \frac{\sigma_s(t)}{t} = 0$ for all $s > 0$, where $\sigma_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_s(t) = \sup_{x \in B_s, y \in S_E, \alpha \in (0,1)} \frac{\alpha g(x + (1-\alpha) t y) + (1-\alpha) g(x - \alpha t y) - g(x)}{\alpha(1-\alpha)}$$

for all $t \ge 0$.

1.5. Some facts about Bregman quasi-nonexpansive mappings

Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Recall that a mapping $T : C \to C$ is said to be *Bregman quasi-nonexpansive* [28], if $F(T) \ne \emptyset$ and

$$D_g(p, Tx) \le D_g(p, x), \qquad \forall x \in C,\ p \in F(T).$$

A mapping $T : C \to C$ is said to be *Bregman relatively nonexpansive* [28] if the following conditions are satisfied:

(1) $F(T)$ is nonempty;

(2) $D_g(p, Tv) \le D_g(p, v), \forall p \in F(T),\ v \in C$;

(3) $\hat{F}(T) = F(T)$.

The theory of fixed points with respect to Bregman distances has been well developed in the last 15 years and much intensively in the last ten years. For some recent articles on the existence of and the construction of fixed points for Bregman nonexpansive type mappings, we refer the readers to [27, 28, 30].

Remark 1.1. Though the iteration processes (1.4)–(1.5) and the algorithms in [27], as introduced by the authors mentioned above worked, it is easy to see that these processes seem cumbersome and complicated in the sense that at each stage of iteration, two different sets C_n and Q_n are computed and the next iterate taken as the Bregman projection of x_0 on the intersection of C_n and Q_n. This seems difficult to do in application.

But it is worth mentioning that, in all the above results for Bregman nonexpansive type mappings, the computation of closed and convex sets C_n and Q_n for each $n \in \mathbb{N}$ are required. So, the following question arises naturally in a Banach space setting.

Question 1.1. *Is it possible to obtain strong convergence of modified Halpern-type schemes to a solution of equilibrium problem* (1.3) *without using the Bregman projection of a point on the intersection of closed and convex sets C_n and Q_n in a Banach space E?*

In this paper, we deal with an equilibrium problem in a reflexive Banach space. First, we consider disadvantages of the iterative sequences in known results. Namely, Bregman projections are not always available in a practical calculation. We attempt to improve these schemes and, by combining them with iterative method of the Halpern type, we obtain a new type of strong convergence theorem, which overcomes the drawbacks of the previous results. Next, we study Halpern type iterative schemes for finding a solution of an equilibrium problem in a reflexive Banach space. Then, we apply our results to the problem of finding a minimizer of a continuously Fréchet differentiable and convex function in a Banach space under suitable assumptions. The computation of closed and convex sets C_n and Q_n for each $n \in \mathbb{N}$ are not required. Consequently, the above Question 1.1 is answered in the affirmative in a reflexive Banach space setting. Our results improve and generalize many known results in the current literature; see, for example, [3, 7, 20, 27].

2. Preliminaries

In this section, we begin by recalling some preliminaries and lemmas which will be used in the sequel.

The following definition is slightly different from that in Butnariu and Iusem [14].

Definition 2.1 ([15])**.** Let E be a Banach space. The function $g : E \to \mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:

(1) g is continuous, strictly convex and Gâteaux differentiable;

(2) the set $\{y \in E : D_g(x, y) \le r\}$ is bounded for all $x \in E$ and $r > 0$.

The following lemma follows from Butnariu and Iusem [14] and Zălinscu [25].

Lemma 2.1. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function. Then*

(1) *$\nabla g : E \to E^*$ is one-to-one, onto and norm-to-weak* continuous;*

(2) *$\langle x - y, \nabla g(x) - \nabla g(y)\rangle = 0$ if and only if $x = y$;*

(3) *$\{x \in E : D_g(x, y) \le r\}$ is bounded for all $y \in E$ and $r > 0$;*

(4) *dom* $g^* = E^*$, g^* *is Gâteaux differentiable and* $\nabla g^* = (\nabla g)^{-1}$.

We know the following two results; see [25](Proposition 3.6.4).

Theorem 2.1. *Let* E *be a reflexive Banach space and* $g : E \to \mathbb{R}$ *a convex function which is bounded on bounded subsets of* E. *Then the following assertions are equivalent:*

(1) g *is strongly coercive and uniformly convex on bounded subsets of* E;

(2) *dom* $g^* = E^*$, g^* *is bounded on bounded subsets and uniformly smooth on bounded subsets of* E^*;

(3) *dom* $g^* = E^*$, g^* *is Fréchet differentiable and* ∇g^* *is uniformly norm-to-norm continuous on bounded subsets of* E^*.

Theorem 2.2. *Let* E *be a reflexive Banach space and* $g : E \to \mathbb{R}$ *a continuous convex function which is strongly coercive. Then the following assertions are equivalent:*

(1) g *is bounded on bounded subsets and uniformly smooth on bounded subsets of* E;

(2) g^* *is Fréchet differentiable and* ∇g^* *is uniformly norm-to-norm continuous on bounded subsets of* E^*;

(3) *dom* $g^* = E^*$, g^* *is strongly coercive and uniformly convex on bounded subsets of* E^*.

Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the Bregman distance [21] (see also [22, 23]) satisfies the *three point identity* that is

$$D_g(x,z) = D_g(x,y) + D_g(y,z) + \langle x-y, \nabla g(y) - \nabla g(z)\rangle, \qquad \forall x,y,z \in E. \tag{2.1}$$

In particular, it can be easily seen that

$$D_g(x,y) = -D_g(y,x) + \langle y-x, \nabla g(y) - \nabla g(x)\rangle, \qquad \forall x,y \in E. \tag{2.2}$$

Indeed, by letting $z = x$ in (2.1) and taking into account that $D_g(x,x) = 0$, we get the desired result.

The following lemma has been proved in [14] (see also [15, 29, 30]).

Lemma 2.2. *Let* E *be a Banach space and* $g : E \to \mathbb{R}$ *a Gâteaux differentiable function which is uniformly convex on bounded subsets of* E. *Let* $\{x_n\}_{n\in\mathbb{N}}$ *and* $\{y_n\}_{n\in\mathbb{N}}$ *be bounded sequences in* E. *Then the following assertions are equivalent:*

(1) $\lim_{n\to\infty} D_g(x_n, y_n) = 0$.

(2) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

The following result was first proved in [26] (see also [15]).

Lemma 2.3. *Let E be a reflexive Banach space, $g : E \to \mathbb{R}$ a strongly coercive Bregman function and V_g the function defined by*

$$V_g(x, x^*) = g(x) - \langle x, x^* \rangle + g^*(x^*), \quad x \in E,\ x^* \in E^*.$$

Then the following assertions hold:

(1) $D_g(x, \nabla g^*(x^*)) = V_g(x, x^*)$ *for all* $x \in E$ *and* $x^* \in E^*$.

(2) $V_g(x, x^*) + \langle \nabla g^*(x^*) - x, y^* \rangle \le V_g(x, x^* + y^*)$ *for all* $x \in E$ *and* $x^*, y^* \in E^*$.

The following lemma has been proved in [34].

Lemma 2.4. *Let E be a Banach space, $s > 0$ be a constant, ρ_s be the gauge of uniform convexity of g and $g : E \to \mathbb{R}$ be a convex function which is uniformly convex on bounded subsets of E. Then*

(i) *For any $x, y \in B_s$ and $\alpha \in (0,1)$*

$$g(\alpha x + (1-\alpha)y) \le \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha)\rho_s(\|x - y\|).$$

(ii) *For any $x, y \in B_s$*

$$\rho_s(\|x - y\|) \le D_g(x, y).$$

(iii) *If, in addition, g is bounded on bounded subsets and uniformly convex on bounded subsets of E then, for any $x \in E$, $y^*, z^* \in B_s$ and $\alpha \in (0,1)$*

$$V_g(x, \alpha y^* + (1-\alpha)z^*) \le \alpha V_g(x, y^*) + (1-\alpha)V_g(x, z^*) - \alpha(1-\alpha)\rho_s^*(\|y^* - z^*\|).$$

Lemma 2.5 ([3])**.** *Let C be a subset of a real Banach space E and $\{T_n\}_{n\in\mathbb{N}}$ be a family of mappings from C into E. Suppose that for any bounded subset B of C there exists a continuous increasing function $h_B : [0,\infty) \to [0,\infty)$ such that $h_B(0) = 0$ and*

$$\lim_{k,l\to\infty} \sigma_l^k = 0,$$

where $\sigma_l^k := \sup\{h_B(\|T_k z - T_l z\|) : z \in B\} < \infty$, for all $k, l \in \mathbb{N}$. Then, for each $x \in C$, $\{T_n x\}_{n\in\mathbb{N}}$ converges strongly to some point of E. Moreover, let the mapping T be defined by

$$Tx = \lim_{n\to\infty} T_n x, \qquad \forall x \in C.$$

Then, $\limsup_{n\to\infty}\{h_B(\|T_n z - Tz\|) : z \in B\} = 0$.

Lemma 2.6 ([31]). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a subsequence $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1} \quad \text{and} \quad a_k \le a_{m_k+1}.$$

In fact, $m_k = \max\{j \le k : a_j < a_{j+1}\}$.

Lemma 2.7 ([32]). *Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers satisfying the inequality:*

$$s_{n+1} \le (1-\gamma_n)s_n + \gamma_n\delta_n, \qquad \forall n \ge 1,$$

where $\{\gamma_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ satisfy the conditions:

(i) $\{\gamma_n\}_{n\in\mathbb{N}} \subset [0,1]$ *and* $\sum_{n=1}^{\infty}\gamma_n = \infty$, *or equivalently,* $\Pi_{n=1}^{\infty}(1-\gamma_n) = 0$;

(ii) $\limsup_{n\to\infty}\delta_n \le 1$, *or*

(ii)′ $\sum_{n=1}^{\infty}\gamma_n\delta_n < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.8 ([27]). *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a Bregman quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 2.9. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E and $\{T_n\}_{n\in\mathbb{N}}$ an infinite family of Bregman quasi-nonexpansive mappings from C into itself such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let the mapping $T : C \to C$ be defined by*

$$Tx = \lim_{n\to\infty} T_n x,$$

Then, T is a Bregman quasi-nonexpansive mapping.

Proof. Let $x \in C$ and $p \in F(T)$ be fixed. Then we have $\{T_n x\}_{n\in\mathbb{N}}$ is a bounded sequence in E. The function g is bounded on bounded subsets of E and, thus, ∇g is also bounded on bounded subsets of E^* (see, for example, [14] for more details). This implies that the sequence $\{\nabla g(T_n x)\}_{n\in\mathbb{N}}$ is bounded in E^*. Since ∇g is uniformly norm-to-norm continuous on any bounded subset of E, we obtain

$$D_g(p, Tx) = g(p) - g(Tx) - \langle x - Tx, \nabla g(Tx)\rangle$$

$$\begin{aligned}
&= g(p) - g(\lim_{n\to\infty} T_n x) - \langle x - \lim_{n\to\infty} T_n x, \nabla g(\lim_{n\to\infty} T_n x)\rangle \\
&= \lim_{n\to\infty} [g(p) - g(T_n x) - \langle x - T_n x, \nabla g(T_n x)\rangle] \\
&= \lim_{n\to\infty} D_g(p, T_n x) \\
&\le D_g(p, x).
\end{aligned}$$

Thus, T is a Bregman Quasi-nonexpansive mapping, which completes the proof. □

Corollary 2.1 ([25]). *Let E be a Banach space, $g : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function and $p, q \in \mathbb{R}$, with $1 \le p \le 2 \le q$ and $p^{-1} + q^{-1} = 1$. Then the following statements are equivalent:*

(1) *There exists $c_1 > 0$ such that g is ρ-convex with $\rho(t) := \frac{c_1}{q} t^q$ for all $t \ge 0$.*

(2) *There exists $c_2 > 0$ such that for all (x, x^*), $(y, y^*) \in G(\partial g)$; $\|x^* - y^*\| \ge \frac{2c_2}{q} \|x - y\|^{q-1}$.*

3. Equilibrium problem

In this section, we prove strong convergence theorems in a reflexive Banach space. Let E be a Banach space and C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)−(A4) and $EP(f) \neq \emptyset$. Let $g : E \to \mathbb{R}$ be a Legendre function. For $r > 0$, we define a mapping $T_r : E \to C$ as follows:

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, \nabla g(z) - \nabla g(x)\rangle \ge 0 \text{ for all } y \in C \right\} \tag{3.1}$$

for all $x \in E$.

Lemma 3.1 ([28]). *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a Legendre function. Let C be a nonempty, closed and convex subset of E and $f : C \times C \to \mathbb{R}$ a bifunction satisfying (A1)−(A4). For $r > 0$, let $T_r : E \to C$ be the mapping defined by (3.1). Then, dom $(T_r) = E$.*

Lemma 3.2 ([28]). *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a convex, continuous and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E and $f : C \times C \to \mathbb{R}$ a bifunction satisfying (A1)−(A4) and $EP(f) \neq \emptyset$. For $r > 0$, let $T_r : E \to C$ be the mapping defined by (3.1). Then, the following statements hold:*

(1) *T_r is single-valued;*

(2) *T_r is a Bregman firmly nonexpansive mapping [28], i.e., for all x, $y \in E$,*

$$\langle T_r x - T_r y, \nabla g(T_r x) - \nabla g(T_r y)\rangle \le \langle T_r x - T_r y, \nabla g(x) - \nabla g(y)\rangle;$$

(3) *$F(T_r) = EP(f)$;*

(4) *$EP(f)$ is closed and convex;*

(5) *T_r is a Bregman quasi-nonexpansive mapping;*

(6) *$D_g(q, T_r x) + D_g(T_r x, x) \leq D_g(q, x)$, $\forall q \in F(T_r)$.*

The following theorem establishes the strong convergence of the Halpern algorithm.

Theorem 3.1. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function which is bounded on bounded subsets, and uniformly convex and uniformly smooth on bounded subsets of E. Let C be a nonempty, closed and convex subset of E and $f : C \times C \to \mathbb{R}$ a bifunction satisfying* (A1)−(A4)*. For $r > 0$, let $T_r : E \to C$ be the mapping defined by* (3.1)*. Suppose that $EP(f)$ is a nonempty subset of C, where $EP(f)$ is the set of solutions to the equilibrium problem* (1.3)*. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $[0,1]$ satisfying the following control conditions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ such that $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{cases} u \in C,\ x_1 \in C \quad \text{chosen arbitrarily,} \\ u_n \in C \quad \text{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, \nabla g(u_n) - \nabla g(x_n)\rangle \geq 0, \quad \forall y \in C, \\ y_n = \nabla g^*[\beta_n \nabla g(x_n) + (1-\beta_n)\nabla g(u_n)], \\ x_{n+1} = \mathrm{proj}_C^g(\nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)]) \text{ and } n \in \mathbb{N}, \end{cases} \tag{3.2}$$

where ∇g is the gradient of g. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (3.2) *converges strongly to $\mathrm{proj}_{EP(f)}^g u$ as $n \to \infty$.*

Proof. We divide the proof into several steps. In view of Lemma 3.2, we conclude that $EP(f)$ is closed and convex. Set

$$z = \mathrm{proj}_{EP(f)}^g u.$$

Step 1. We show that there exists a mapping $T : C \to C$ such that

$$Tx = \lim_{n\to\infty} T_{r_n} x, \quad (x \in C) \text{ and } F(T) = \bigcap_{n=1}^{\infty} F(T_{r_n}) = \bigcap_{n=1}^{\infty} \hat{F}(T_{r_n}) = \hat{F}(T).$$

Since T_{r_n} is a Bregman quasi-nonexpansive mapping, so we have

$$D_g(z, T_{r_n} v) \leq D_g(z, v), \quad \forall v \in E,\ n \in \mathbb{N}.$$

This, together with Definition 2.1, implies that, for any bounded subset B of E, $\{T_{r_n} v : v \in B\}$ is bounded. Taking any $v \in E$ and setting $v_n = T_{r_n} v$, we get

$$f(v_l, y) + \frac{1}{r_l}\langle y - v_l, \nabla g(v_l) - \nabla g(v)\rangle \geq 0 \quad \forall y \in C \tag{3.3}$$

and

$$f(v_k, y) + \frac{1}{r_k}\langle y - v_k, \nabla g(v_k) - \nabla g(v)\rangle \geq 0 \quad \forall y \in C. \tag{3.4}$$

Letting $y = v_k$ in (3.3) and $y = v_l$ in (3.4), we arrive at

$$f(v_l, v_k) + \frac{1}{r_l}\langle v_k - v_l, \nabla g(v_l) - \nabla g(v)\rangle \geq 0 \text{ and } f(v_k, v_l) + \frac{1}{r_k}\langle v_l - v_k, \nabla g(v_k) - \nabla g(v)\rangle \geq 0.$$

Now, adding up the previous inequalities and taking into account (A2) we get

$$\left\langle v_k - v_l, \frac{\nabla g(v_l) - \nabla g(v)}{r_l} - \frac{\nabla g(v_k) - \nabla g(v)}{r_k}\right\rangle \geq 0$$

and hence

$$\left\langle v_k - v_l, \nabla g(v_l) - \nabla g(v) - \frac{r_l}{r_k}(\nabla g(v_k) - \nabla g(v))\right\rangle \geq 0.$$

Therefore,

$$\langle v_k - v_l, \nabla g(v_k) - \nabla g(v_l)\rangle + \left\langle v_k - v_l, \left(1 - \frac{r_l}{r_k}\right)(\nabla g(v_k) - \nabla g(v))\right\rangle \geq 0.$$

Without loss of generality, we may assume that there exists a real number a such that $r_n > a$ for all $n \in \mathbb{N} \cup \{0\}$. So we obtain

$$\begin{aligned}\langle v_k - v_l, \nabla g(v_k) - \nabla g(v_l)\rangle &\leq \left\langle v_k - v_l, \left(1 - \frac{r_l}{r_k}\right)(\nabla g(v_l) - \nabla g(v))\right\rangle \\ &\leq \frac{1}{a}\|v_k - v_l\||r_k - r_l|\|\nabla g(v_k) - \nabla g(v)\| \\ &= \frac{1}{a}\|T_{r_k} v - T_{r_l} v\|\|\nabla g(T_{r_k} v) - \nabla g(T_{r_l} v)\||r_k - r_l|.\end{aligned}$$

By Lemma 3.2, we receive $EP(f) = \bigcap_{n=1}^{\infty} F(T_{r_n})$. Let

$$M_0 = \sup\left\{\frac{1}{a}\|T_{r_k} v - T_{r_l} v\|\|\nabla g(T_{r_k} v) - \nabla g(T_{r_l} v)\| : v \in B,\ k, l \in \mathbb{N} \cup \{0\}\right\}.$$

Putting $s_1 = \sup\{\|v_k\|, \|v_l\|, \|\nabla g(v_k)\|, \|\nabla g(v_l)\| : k, l \in \mathbb{N} \cup \{0\}\}$, in view of Lemma 2.4(i), there exists a strictly increasing, continuous and convex function $\rho_{s_1} : [0, \infty) \to [0, \infty)$ such that for all $v \in B$,

$$\begin{aligned}\rho_{s_1}(\|T_{r_k} v - T_{r_l} v\|) &= \rho_{s_1}(\|v_k - v_l\|) \leq D_g(v_k, v_l) \\ &= -D_g(v_l, v_k) + \langle v_k - v_l, \nabla g(v_k) - \nabla g(v_l)\rangle\end{aligned}$$

$$\leq \frac{1}{a}\|T_{r_k}v - T_{r_l}v\|\|\nabla g(T_{r_k}v) - \nabla g(T_{r_l}v)\||r_k - r_l|$$
$$\leq M_0|r_k - r_l| \leq M_0\sum_{n=l}^{k-1}|r_{n+1} - r_n| \leq M_0\sum_{n=l}^{\infty}|r_{n+1} - r_n| < \infty.$$

Let

$$\sigma_l^k := \sup\{\rho_{s_1}(\|T_{r_k}v - T_{r_l}v\|) : v \in B\} \leq M_0\sum_{n=l}^{\infty}|r_{n+1} - r_n| < \infty.$$

Letting $l \to \infty$ in the above inequality, we get $\lim_{k,l\to\infty}\sigma_l^k = 0$. Let us define the function $T : C \to C$ by

$$Tx = \lim_{n\to\infty} T_{r_n}x, \qquad (x \in C).$$

We prove that

$$F(T) = \bigcap_{n=1}^{\infty} F(T_{r_n}) = \bigcap_{n=1}^{\infty} \hat{F}(T_{r_n}) = \hat{F}(T). \tag{3.5}$$

Let us mention first the following observations are obvious:

(1) $\bigcap_{n=1}^{\infty} F(T_{r_n}) = \bigcap_{n=0}^{\infty} \hat{F}(T_{r_n}) = EP(f)$.

(2) $\bigcap_{n=1}^{\infty} F(T_{r_n}) \subset F(T)$ and $\bigcap_{n=1}^{\infty} \hat{F}(T_{r_n}) \subset \hat{F}(T)$.
It remains to prove that (3) $F(T) \subset \bigcap_{n=1}^{\infty} F(T_{r_n})$ and (4) $\hat{F}(T) \subset \bigcap_{n=1}^{\infty} \hat{F}(T_{r_n})$.

(3) Let $p \in F(T)$ be fixed. By the definition of T_r (see (3.1)), we see that

$$f(T_{r_n}p, y) + \frac{1}{r_n}\langle y - T_{r_n}p, \nabla g(T_{r_n}p) - \nabla g(p)\rangle \geq 0 \qquad \forall y \in C.$$

In view of (A2), we obtain

$$\frac{1}{r_n}\langle y - T_{r_n}p, \nabla g(T_{r_n}p) - \nabla g(p)\rangle \geq f(y, T_{r_n}p) \qquad \forall y \in C.$$

Since $T_{r_n}p \to Tp$ as $n \to \infty$, ∇g is uniformly continuous on bounded subsets of E and $f(y,\cdot)$ is lower semicontinuous, we conclude that $f(y,p) \leq 0$ for all $y \in C$. Takin any $y \in C$ and setting $x_t = ty + (1-t)p$, for $t \in (0,1]$ we see that

$$0 \leq f(x_t, x_t) \leq tf(x_t, y) + (1-t)f(x_t, p) \leq tf(x_t, y).$$

This amounts to $f(x_t, y) \geq 0$. Letting $t \downarrow 0$ and taking into account (A3), we get $f(p, y) \geq 0$ for all $y \in C$ and hence $p \in EP(f) = \bigcap_{n=1}^{\infty} F(T_{r_n})$.

(4) Let $q \in \hat{F}(T)$. Then, there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subset E$ such that $v_n \rightharpoonup q$ as $n \to \infty$ and $\lim_{n\to\infty}\|v_n - Tv_n\| = 0$. Observe now that $Tv_n \rightharpoonup q$ as $n \to \infty$ and hence $q \in C$. Since ∇g is uniformly continuous on bounded subsets of E, we conclude that $\lim_{n\to\infty}\|\nabla g(v_n) - \nabla g(Tv_n)\| = 0$. For any $m \in \mathbb{N}$, it follows from the definition of T_{r_m} that

$$f(T_{r_m}v_n, y) + \frac{1}{r_m}\langle y - T_{r_m}v_n, \nabla g(T_{r_m}v_n) - \nabla g(v_n)\rangle \geq 0 \qquad \forall y \in C.$$

Applying again (A2) and taking into account $\frac{1}{r_m} \le \frac{1}{a}$, we obtain

$$\begin{aligned} f(y, T_{r_m} v_n) &\le \frac{1}{r_m} \langle y - T_{r_m} v_n, \nabla g(T_{r_m} v_n) - \nabla g(v_n) \rangle \\ &\le \frac{1}{a} \| y - T_{r_m} v_n \| \| \nabla g(T_{r_m} v_n) - \nabla g(v_n) \| \qquad \forall y \in C. \end{aligned}$$

Since $\lim_{m\to\infty} T_{r_m} v_n = T v_n$ and $f(y, \cdot)$ is lower semicontinuous, we arrive at

$$f(y, T v_n) \le \frac{1}{a} \| y - T v_n \| \| \nabla g(T v_n) - \nabla g(v_n) \| \qquad \forall y \in C.$$

Since $T v_n \rightharpoonup q$ as $n \to \infty$, $\lim_{n\to\infty} \| v_n - T v_n \| = 0$ and $f(y, \cdot)$ is lower semicontinuous, we deduce that $f(y, q) \le 0$ for all $y \in C$. Proceeding with the same process as above we conclude that $f(q, y) \ge 0$ for all $y \in C$. Therefore, $q \in EP(f) = \bigcap_{n=1}^{\infty} \hat{F}(T_{r_n})$.

Step 2. We prove that $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ are bounded sequences in E.

We first show that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Let $p \in EP(f)$ be fixed. In view of Lemmas 2.4, 3.2 and (3.2), we have

$$\begin{aligned} D_g(p, y_n) &= D_g(p, \nabla g^* [(1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(T_{r_n} x_n)) \\ &= V_g(p, (1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(T_{r_n} x_n)]) \\ &\le (1-\beta_n) V_g(p, \nabla g(x_n)) + \beta_n V_g(p, \nabla g(T_{r_n} x_n)) \\ &= (1-\beta_n) D_g(p, x_n) + \beta_n D_g(p, T_{r_n} x_n) \\ &\le (1-\beta_n) D_g(p, x_n) + \beta_n D_g(p, x_n) \\ &= D_g(p, x_n). \end{aligned} \tag{3.6}$$

This implies that

$$\begin{aligned} D_g(p, x_{n+1}) &= D_g(p, \mathrm{proj}_C^g (\nabla g^* [\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)])) \\ &\le D_g(p, \nabla g^* [\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)]) \\ &= V_g(p, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)) \\ &\le \alpha_n V_g(p, \nabla g(u)) + (1-\alpha_n) V_g(p, \nabla g(y_n)) \\ &= \alpha_n D_g(p, u) + (1-\alpha_n) D_g(p, y_n) \\ &\le \alpha_n D_g(p, u) + (1-\alpha_n) D_g(p, y_n) \\ &\le \alpha_n D_g(p, u) + (1-\alpha_n) D_g(p, x_n) \\ &\le \max\{D_g(p, u), D_g(p, x_n)\}. \end{aligned} \tag{3.7}$$

By induction, we obtain

$$D_g(p, x_{n+1}) \le \max\{D_g(p, u), D_g(p, x_1)\} \tag{3.8}$$

for all $n \in \mathbb{N}$. It follows from (3.8) that the sequence $\{D_g(p, x_n)\}_{n\in\mathbb{N}}$ is bounded and hence there exists $M_0 > 0$ such that

$$D_g(p, x_n) \le M_1, \quad \forall n \in \mathbb{N}. \tag{3.9}$$

In view of Lemma 2.2 (3), we deduce that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Since $\{T_{r_n}\}_{n\in\mathbb{N}}$ is an infinite family of Bregman relatively nonexpansive mappings from C into itself, we conclude that

$$D_g(p, u_m) = D_g(p, T_{r_n} x_m) \le D_g(p, x_m), \quad \forall n, m \in \mathbb{N}. \tag{3.10}$$

This, together with Definition 2.1 and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, implies that $\{T_{r_n} x_n\}_{n\in\mathbb{N}}$ is bounded. The function g is bounded on bounded subsets of E and therefore ∇g is also bounded on bounded subsets of E^* (see, for example, [14] for more details). This, together with Step 1, implies that the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$ and $\{\nabla g(T_{r_n} x_n)\}_{n\in\mathbb{N}}$ are bounded in E^*. In view of Theorem 2.2 (3), we obtain that dom $g^* = E^*$ and g^* is strongly coercive and uniformly convex on bounded subsets of E. Let $s_2 = \sup\{\|\nabla g(x_n)\|, \|\nabla g(T_{r_n} x_n)\| : n \in \mathbb{N}\}$ and let $\rho^*_{s_2} : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*.

Step 3. We prove that for any $n \in \mathbb{N}$

$$D_g(z, y_n) \le D_g(z, x_n) - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n) - \nabla g(T_{r_n} x_n)\|). \tag{3.11}$$

Let us show (3.11). For each $n \in \mathbb{N}$, in view of the definition of Bregman distance (see (1.8)), Lemma (2.4) and (3.6), we obtain

$$\begin{aligned}
D_g(z, y_n) &= g(z) - g(y_n) - \langle z - y_n, \nabla g(y_n)\rangle \\
&= g(z) + g^*(\nabla g(y_n)) - \langle y_n, \nabla g(y_n)\rangle - \langle z, \nabla g(y_n)\rangle + \langle y_n, \nabla g(y_n)\rangle \\
&= g(z) + g^*((1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(T_{r_n} x_n)) \\
&\quad - \langle z, (1-\beta_n)\nabla g(x_n) + \beta_n \nabla g(T_{r_n} x_n))\rangle \\
&\le (1-\beta_n) g(z) + \beta_n g(z) + (1-\beta_n) g^*(\nabla g(x_n)) + \beta_n g^*(\nabla g(T_{r_n} x_n)) \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n) - \nabla g(T_{r_n} x_n)\|) \\
&\quad - (1-\beta_n)\langle z, \nabla g(x_n)\rangle - \beta_n \langle z, \nabla g(T_{r_n} x_n)\rangle \\
&= (1-\beta_n)[g(z) + g^*(\nabla g(x_n)) - \langle z, \nabla g(x_n)\rangle] \\
&\quad + \beta_n [g(z) + g^*(\nabla g(T_{r_n} x_n)) - \langle z, \nabla g(T_{r_n} x_n)\rangle] \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n) - \nabla g(T_{r_n} x_n)\|) \\
&= (1-\beta_n)[g(z) - g(x_n) + \langle x_n, \nabla g(x_n)\rangle - \langle z, \nabla g(x_n)\rangle] \\
&\quad + \beta_n [g(z) - g(T_{r_n} x_n) + \langle T_{r_n} x_n, \nabla g(T_{r_n} x_n)\rangle - \langle z, \nabla g(T_{r_n} x_n)\rangle] \\
&\quad - \beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n) - \nabla g(T_{r_n} x_n)\|) \\
&= (1-\beta_n) D(z, x_n) + \beta_n D(z, T_{r_n} x_n)
\end{aligned}$$

$$
\begin{aligned}
&-\beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n)-\nabla g(T_{r_n}x_n)\|)\\
&\leq (1-\beta_n)D_g(z,x_n)+\beta_n D_g(z,x_n)\\
&\quad -\beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n)-\nabla g(T_{r_n}x_n)\|)\\
&= D(z,x_n)-\beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n)-\nabla g(T_{r_n}x_n)\|).
\end{aligned}
$$

In view of Lemma 2.5 and (3.11), we obtain

$$
\begin{aligned}
D_g(z,x_{n+1}) &= D_g(p,\mathrm{proj}^g_C(\nabla g^*[\alpha_n\nabla g(u)+(1-\alpha_n)\nabla g(y_n)]))\\
&\leq D_g(z,\nabla g^*[\alpha_n\nabla g(z)+(1-\alpha_n)\nabla g(y_n)])\\
&= V_g(z,\alpha_n\nabla g(u)+(1-\alpha_n)\nabla g(y_n))\\
&\leq \alpha_n V(z,\nabla g(u))+(1-\alpha_n)V_g(z,\nabla g(y_n))\\
&= \alpha_n D_g(z,u)+(1-\alpha_n)D_g(z,y_n)\\
&\leq \alpha_n D_g(z,u)+(1-\alpha_n)D_g(z,y_n)\\
&\leq \alpha_n D_g(z,u)+(1-\alpha_n)[D_g(z,x_n)-\beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n)-\nabla g(T_{r_n}x_n)\|)]. \quad (3.12)
\end{aligned}
$$

Let $M_2 := \sup\{|D_g(z,u)-D_g(z,x_n)|+\beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n)-\nabla g(T_{r_n}x_n)\|) : n\in\mathbb{N}\}$. It follows from (3.12) that

$$
\beta_n(1-\beta_n)\rho^*_{s_2}(\|\nabla g(x_n)-\nabla g(T_{r_n}x_n)\|)\leq D_g(z,x_n)-D_g(z,x_{n+1})+\alpha_n M_2. \quad (3.13)
$$

Let $z_n=\nabla g^*[\alpha_n\nabla g(u)+(1-\alpha_n)\nabla g(y_n)]$. Then $x_{n+1}=\mathrm{proj}^g_C(z_n)$ for all $n\in\mathbb{N}$. In view of Lemma 2.3 and (3.11) we obtain

$$
\begin{aligned}
D_g(z,x_{n+1}) &= D_g(p,\mathrm{proj}^g_C(\nabla g^*[\alpha_n\nabla g(u)+(1-\alpha_n)\nabla g(y_n)]))\\
&\leq D_g(z,\nabla g^*[\alpha_n\nabla g(u)+(1-\alpha_n)\nabla g(y_n)])\\
&= V_g(z,\alpha_n\nabla g(u)+(1-\alpha_n)\nabla g(y_n))\\
&\leq V_g(z,\alpha_n\nabla g(u)+(1-\alpha_n)\nabla g(y_n)-\alpha_n(\nabla g(u)-\nabla g(z)))\\
&\quad -\langle\nabla g^*[\alpha_n\nabla g(u)+(1-\alpha_n)\nabla g(y_n)]-z,-\alpha_n(\nabla g(u)-\nabla g(z))\rangle\\
&= V_g(z,\alpha_n\nabla g(z)+(1-\alpha_n)\nabla g(y_n))+\alpha_n\langle z_n-z,\nabla g(u)-\nabla g(z)\rangle\\
&= D_g(z,\nabla g^*[\alpha_n\nabla g(z)+(1-\alpha_n)\nabla g(y_n)])+\alpha_n\langle z_n-z,\nabla g(u)-\nabla g(z)\rangle\\
&\leq \alpha_n D_g(z,z)+(1-\alpha_n)D_g(z,y_n)+\alpha_n\langle z_n-z,\nabla g(u)-\nabla g(z)\rangle\\
&= (1-\alpha_n)D_g(z,x_n)+\alpha_n\langle z_n-z,\nabla g(u)-\nabla g(z)\rangle. \quad (3.14)
\end{aligned}
$$

Step 4. We show that $x_n\to z$ as $n\to\infty$.

The rest of the proof will be divided into two parts:
Case 1. If there exists $n_0\in\mathbb{N}$ such that $\{D_g(z,x_n)\}_{n=n_0}^{\infty}$ is nonincreasing, then $\{D_g(z,x_n)\}_{n\in\mathbb{N}}$ is

convergent. Thus, we have $D_g(z,x_n)-D_g(z,x_{n+1})\to 0$ as $n\to\infty$. This, together with condition (c), implies that

$$\lim_{n\to\infty}\rho^*_{s_2}(\|\nabla g(x_n)-\nabla g(T_{r_n}x_n)\|)=0.$$

Therefore, from the property of $\rho^*_{s_2}$ we deduce that

$$\lim_{n\to\infty}\|\nabla g(x_n)-\nabla g(T_{r_n}x_n)\|=0.$$

Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at

$$\lim_{n\to\infty}\|x_n-T_{r_n}x_n\|=0.$$

Let $s_3:=\sup\{\|x_n\|,\|T_{r_n}x_n\|\}<\infty$ and let B_1 be a bounded subset of E such that $\{x_n,T_{r_n}x_n\}_{n\in\mathbb{N}}\subset B_1$. By the same argument as in Step 1 we conclude that $\lim_{k,l\to\infty}\sigma_l^k=0$, where $\sigma_l^k:=\sup\{\rho_{s_1}(\|T_{r_k}v-T_{r_l}v\|):v\in B_1\}$. On the other hand, we have

$$\frac{1}{2}\|x_n-Tx_n\|\le\frac{1}{2}\|x_n-T_{r_n}x_n\|+\frac{1}{2}\|T_{r_n}x_n-Tx_n\|.$$

Since $\rho_{s_3}:[0,+\infty)\to[0,+\infty)$ is an increasing, continuous and convex function, so we have

$$\begin{aligned}\rho_{s_3}(\tfrac{1}{2}\|x_n-Tx_n\|)&\le\rho_{s_3}(\tfrac{1}{2}\|x_n-T_{r_n}x_n\|)+\rho_{s_3}(\tfrac{1}{2}\|T_{r_n}x_n-Tx_n\|)\\&\le\tfrac{1}{2}\rho_{s_3}(\|x_n-T_{r_n}x_n\|)+\tfrac{1}{2}\sup\{\rho_{s_3}(\|T_{r_n}v-Tv\|):v\in B_1\}.\end{aligned}$$

Exploiting Lemma 2.4, we obtain

$$\lim_{n\to\infty}\rho_{s_3}(\|x_n-Tx_n\|)=0.$$

By the properties of ρ_{s_3}, we conclude that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.15}$$

This, together with Lemma 3.2 and (3.12), implies that $z\in F(T)=\bigcap_{n=1}^{\infty}F(T_{r_n})=EP(f)$. Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n_i}\rightharpoonup y\in C$ and

$$\limsup_{n\to\infty}\langle x_n-z,\nabla g(x)-\nabla g(z)\rangle=\lim_{i\to\infty}\langle x_{n_i}-z,\nabla g(x)-\nabla g(z)\rangle.$$

This, together with (1.9), implies that

$$\limsup_{n\to\infty}\langle x_n-z,\nabla g(x)-\nabla g(z)\rangle=\langle y-z,\nabla g(x)-\nabla g(z)\rangle\le 0.$$

From Lemma 3.2, we have that

$$\limsup_{n\to\infty}\langle z_n-z,\nabla g(u)-\nabla g(z)\rangle=\limsup_{n\to\infty}\langle x_n-z,\nabla g(u)-\nabla g(z)\rangle\le 0.$$

Thus we have the desired result by Lemma 2.7.

Case 2. If there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ such that

$$D_g(z, x_{n_i}) < D_g(z, x_{n_i+1})$$

for all $i \in \mathbb{N}$, then by Lemma 2.6, there exists a nondecreasing sequence $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$D_g(z, x_{m_k}) < D_g(z, x_{m_k+1}) \quad \text{and} \quad D_g(z, x_k) \le D_g(z, x_{m_k+1})$$

for all $k \in \mathbb{N}$. This, together with (3.13), implies that

$$\beta_{m_k}(1-\beta_{m_k})\rho^*_{s_2}(\|\nabla g(x_{m_k}) - \nabla g(T_{r_{m_k}} x_{m_k})\|) \le D_g(z, x_{m_k}) - D_g(z, x_{m_k+1}) + \alpha_{m_k} M_2 \le \alpha_{m_k} M_2$$

for all $k \in \mathbb{N}$. Then, by conditions (a) and (c), we get

$$\lim_{k\to\infty} \rho^*_{s_2}(\|\nabla g(x_{m_k}) - \nabla g(T_{m_k} x_{m_k})\|) = 0.$$

By the same argument as Case 1, we arrive at

$$\limsup_{k\to\infty} \langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\rangle = \limsup_{k\to\infty} \langle x_{m_k} - z, \nabla g(u) - \nabla g(z)\rangle \le 0.$$

It follows from (3.14) that

$$D_g(z, x_{m_k+1}) \le (1-\alpha_{m_k}) D_g(z, x_{m_k}) + \alpha_{m_k} \langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\rangle \tag{3.16}$$

Since $D_g(z, x_{m_k}) \le D_g(z, x_{m_k+1})$, we have that

$$\begin{aligned} \alpha_{m_k} D_g(z, x_{m_k}) &\le D_g(z, x_{m_k}) - D_g(z, x_{m_k+1}) + \alpha_{m_k}\langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\rangle \\ &\le \alpha_{m_k}\langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\rangle. \end{aligned} \tag{3.17}$$

In particular, since $\alpha_{m_k} > 0$, we obtain

$$D_g(z, x_{m_k}) \le \langle z_{m_k} - z, \nabla g(u) - \nabla g(z)\rangle.$$

In view of (3.16), we deduce that

$$\lim_{k\to\infty} D_g(z, x_{m_k}) = 0.$$

This, together with (3.17), implies that

$$\lim_{k\to\infty} D_g(z, x_{m_k+1}) = 0.$$

On the other hand, we have $D_g(z, x_k) \le D_g(z, x_{m_k+1})$ for all $k \in \mathbb{N}$ which implies that $x_k \to z$ as $k \to \infty$. Thus, we have $x_n \to z$ as $n \to \infty$. □

Remark 3.1. We propose a new type of iterative scheme for the solution of an equilibrium problem in a reflexive Banach space. This scheme has an advantage that we do not use any projections which creates some difficulties in a practical calculation of the iterative sequence. A strong convergence theorem by a new Halpern-type method for the approximation of solution of an equilibrium problem in a reflexive Banach space is also derived.

Remark 3.2. Theorem 3.1 improves Theorems 1.1 and 1.2 in the following aspects.

(1) For the structure of Banach spaces, we extend the duality mapping to more general case, that is, a convex, continuous and strongly coercive Bregman function which is bounded on bounded subsets, and uniformly convex and uniformly smooth on bounded subsets.

(2) For the algorithm, we remove the sets C_n and Q_n in Theorems 1.1 and 1.2.

4 Applications

In this section, we propose Halpern-type iterative schemes for finding common solutions of an equilibrium problem and null spaces of a γ-inverse strongly monotone operator in a 2-uniformly convex Banach space and prove two strong convergence theorems.

Theorem 4.1. *Let E be a 2-uniformly convex Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function which is bounded on bounded subsets, and uniformly convex and uniformly smooth on bounded subsets of E. Assume that there exists $c_1 > 0$ such that g is ρ-convex with $\rho(t) := \frac{c_1}{2}t^2$ for all $t \geq 0$. Let C be a nonempty, closed and convex subset of E and f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)−(A4). *Assume that $A : C \to E^*$ is a γ-inverse strongly monotone for some $\gamma > 0$. Suppose that $F := A^{-1}(0) \cap EP(f)$ is a nonempty subset of C, where $EP(f)$ is the set of solutions to the equilibrium problem* (1.3). *Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $[0,1]$ satisfying the following control conditions:*

(a) $\lim_{n\to\infty} \alpha_n = 0$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ such that $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} u \in E,\ x_1 \in C \quad \textit{chosen arbitrarily}, \\ w_n = \operatorname{proj}_C^g(\nabla g^*[\nabla g(x_n) - \beta A x_n]) \\ u_n \in C \quad \textit{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, \nabla g(u_n) - \nabla g(w_n)\rangle \geq 0, \quad \forall y \in C, \\ y_n = \nabla g^*[\beta_n \nabla g(w_n) + (1-\beta_n)\nabla g(u_n)], \\ x_{n+1} = \operatorname{proj}_C^g(\nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)]) \ \textit{and}\ n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where ∇g is the gradient of g. Let λ be a constant such that $0 < \lambda < \frac{c_2^2\gamma}{2}$, where c_2 is the 2-uniformly convex constant of E satisfying Corollary 2.1 (2). Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (4.1) converges strongly to $\operatorname{proj}_F^g u$ as $n \to \infty$.

Proof. We divide the proof into several steps.
Set $z = \operatorname{proj}_F^g u$.
Step 1. We prove that $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$, $\{w_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ are bounded sequences in C. We first show that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Let $p \in F$ be fixed. In view of (1.9), Lemma 2.3 and (4.1), we obtain

$$\begin{aligned}
D_g(p, w_n) &= D_g(p, \nabla g^*[\nabla g(x_n) - \beta A x_n]) \\
&= V_g(p, \nabla g(x_n) - \lambda A x_n) \\
&\le V_g(p, \nabla g(x_n) - \lambda A x_n + \lambda A x_n) - \langle \nabla g^*(\nabla g(x_n) - \lambda A x_n) - p, \lambda A x_n\rangle \\
&= V(p, \nabla g(x_n)) - \lambda\langle \nabla g^*(\nabla g(x_n) - \lambda A x_n) - p, A x_n\rangle \\
&= D_g(p, x_n) - \lambda\langle x_n - p, A x_n\rangle - \lambda\langle \nabla g^*(\nabla g(x_n) - \lambda A x_n) - x_n, A x_n\rangle \\
&\le D_g(p, x_n) - \lambda\gamma\|A x_n\|^2 + \lambda\|\nabla g^*(\nabla g(x_n) - \lambda A x_n) - \nabla g^*\nabla g(x_n)\|\|A x_n\| \\
&\le D_g(p, x_n) - \lambda\gamma\|A x_n\|^2 + \frac{4\lambda^2}{c_2^2}\|A x_n\|^2 \\
&\le D_g(p, x_n) + \lambda\left(\frac{4\lambda}{c_2^2} - \gamma\right)\|A x_n\|^2.
\end{aligned} \tag{4.2}$$

This, together with $\frac{4\lambda}{c_2^2} - \gamma < 0$, implies that

$$D_g(p, w_n) \le D_g(p, x_n).$$

By the same argument, as in the proof of Theorem 3.1, for each $n \in \mathbb{N}$, we obtain

$$D_g(p, y_n) \le D_g(p, x_n). \tag{4.3}$$

This implies that

$$D_g(p, x_{n+1}) \le \max\{D_g(p, u), D_g(p, x_1)\} \tag{4.4}$$

for all $n \in \mathbb{N}$. It follows from (4.4) that the sequence $\{D_g(x_n, x)\}_{n\in\mathbb{N}}$ is bounded and hence there exists $M_2 > 0$ such that

$$D_g(x_n, x) \le M_2, \quad \forall n \in \mathbb{N}. \tag{4.5}$$

In view of Lemma 2.2 (3), we have that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Since $\{T_{r_n}\}_{n\in\mathbb{N}}$ is an infinite family of Bregman relatively nonexpansive mappings from E into C, we conclude that

$$D_g(p, T_{r_n} w_m) \le D_g(p, w_n) \le D_g(p, x_n), \quad \forall n \in \mathbb{N}. \tag{4.6}$$

This, together with Definition 2.2 and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, implies that $\{T_{r_n} w_n\}_{n\in\mathbb{N}}$ is bounded for each $i = 1,2,\ldots,N$. The function g is bounded on bounded subsets of E and therefore ∇g is also bounded on bounded subsets of E^* (see, for example, [14] for more details). This, together with Step 1, implies that the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$ and $\{\nabla g(T_{r_n} w_n)\}_{n\in\mathbb{N}}$ are bounded in E^*. In view of Theorem 2.2 (3), we obtain that dom $g^* = E^*$ and g^* is strongly coercive and uniformly convex on bounded subsets of E. Let $s_3 = \sup\{\|\nabla g(w_n)\|, \|\nabla g(T_{r_n} w_n)\| : n \in \mathbb{N}\}$ and $\rho^*_{s_3} : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*.

Step 2. Continuing in the same manner of proof in Theorem 3.1 we can prove that for any $n \in \mathbb{N}$

$$D_g(z, y_n) \le D_g(z, x_n) - \beta_n(1-\beta_n)\rho^*_{s_3}(\|\nabla g(w_n) - \nabla g(T_{r_n} w_n)\|). \tag{4.7}$$

Let us show (4.7). Also, in view of Lemma 2.4 and (4.7), we obtain

$$D_g(z, x_{n+1}) \le \alpha_n D_g(z,u) + (1-\alpha_n)[D_g(z,x_n) - \beta_n(1-\beta_n)\rho^*_{s_3}(\|\nabla g(x_n) - \nabla g(T_{r_n} x_n)\|)]. \tag{4.8}$$

Let $M_3 := \sup\{|D_g(z,u) - D_g(z,x_n)| + \beta_n(1-\beta_n)\rho^*_{s_3}(\|\nabla g(w_n) - \nabla g(T_{r_n} w_n)\|) : n \in \mathbb{N}\}$. It follows from (4.8) that

$$\beta_n(1-\beta_n)\rho^*_{s_3}(\|\nabla g(w_n) - \nabla g(T_{r_n} w_n)\|) \le D_g(z,x_n) - D_g(z,x_{n+1}) + \alpha_n M_3. \tag{4.9}$$

Let $z_n = \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(T_n y_n)]$. Then $x_{n+1} = \mathrm{proj}^g_C(z_n)$ for all $n \in \mathbb{N}$. In view of Lemma 2.3 and (4.7) we obtain

$$\begin{aligned}
D_g(z, x_{n+1}) &= D_g(p, \mathrm{proj}^g_C(\nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)]) \\
&\quad D_g(z, \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)]) \\
&= V_g(z, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)) \\
&\le V_g(z, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n) - \alpha_n(\nabla g(u) - \nabla g(z))) \\
&\quad - \langle g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)] - z, -\alpha_n(\nabla g(u) - \nabla g(z))\rangle \\
&= V_g(z, \alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)) + \alpha_n\langle z_n - z, \nabla g(u) - \nabla g(z)\rangle \\
&= D_g(z, \nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)]) + \alpha_n\langle z_n - z, \nabla g(u) - \nabla g(z)\rangle \\
&\le \alpha_n D_g(z,z) + (1-\alpha_n)D_g(z,y_n) + \alpha_n\langle z_n - z, \nabla g(u) - \nabla g(z)\rangle \\
&= (1-\alpha_n)D_g(z,x_n) + \alpha_n\langle z_n - z, \nabla g(u) - \nabla g(z)\rangle.
\end{aligned} \tag{4.10}$$

Step 3. By the same argument as in the proof of Theorem 3.1 and using (4.9)−(4.10), we conclude that

$$\lim_{n\to\infty} \rho^*_{s_3}(\|\nabla g(w_n) - \nabla g(T_{r_n} w_n)\|) = 0.$$

From the properties of $\rho^*_{s_3}$, we conclude that

$$\lim_{n\to\infty} \|\nabla g(w_n) - \nabla g(T_{r_n} w_n)\| = 0.$$

Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at

$$\lim_{n\to\infty} \|w_n - T_{r_n} w_n\| = 0.$$

From the boundedness of $\{w_n\}_{n\in\mathbb{N}}$ and $\{T_{r_n} w_n\}_{n\in\mathbb{N}}$, it follows that there exists a bounded subset B of C such that $\{w_n, T_{r_n} w_n\}_{n\in\mathbb{N}} \subset B$. Let Let $s_4 = \sup\{\|w_n\|, \|T_{r_n} w_n\| : n \in \mathbb{N}\}$ and $\rho^*_{s_3} : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g. Let $Tx = \lim_{n\to\infty} T_{r_n} x$ for all $x \in C$. In view of Lemma 2.9, T is a Bregman quasi-nonexpansive mapping. On the other hand, we have

$$\frac{1}{2}\|w_n - Tw_n\| \le \frac{1}{2}\|w_n - T_{r_n} w_n\| + \frac{1}{2}\|T_{r_n} w_n - Tw_n\|.$$

Since $\rho_4 : [0,+\infty) \to [0,+\infty)$ is an increasing, continuous and convex function, so we have

$$\begin{aligned}\rho_{s_4}(\frac{1}{2}\|w_n - Tw_n\|) &\le \rho_{s_4}(\frac{1}{2}\|w_n - T_{r_n} w_n\|) + \rho_{s_4}(\frac{1}{2}\|T_{r_n} w_n - Tw_n\|)\\ &\le \frac{1}{2}\rho_{s_4}(\|w_n - T_{r_n} w_n\|) + \frac{1}{2}\sup\{\rho_{s_4}(\|T_{r_n} z - Tz\|) : z \in B\}.\end{aligned}$$

Exploiting Lemma 2.5 and (4.10), we obtain

$$\lim_{n\to\infty} \rho_{s_4}(\|w_n - Tw_n\|) = 0.$$

By the properties of ρ_{s_4}, we conclude that

$$\lim_{n\to\infty} \|w_n - Tw_n\| = 0. \tag{4.11}$$

In view of Lemma 2.2 and (4.11) we obtain that

$$\lim_{n\to\infty} D_g(T_{r_n} w_n, w_n) = 0.$$

This implies that

$$D_g(T_{r_n} w_n, y_n) \le (1-\beta_n) D_g(T_{r_n} w_n, w_n) + \beta_n D_g(T_{r_n} w_n, T_{r_n} w_n) = (1-\beta_n) D_g(T_{r_n} w_n, w_n) \to 0 \tag{4.12}$$

as $n \to \infty$. Also, we have

$$D_g(y_n, z_n) \le \alpha_n D_g(y_n, u) + (1-\alpha_n) D_g(y_n, y_n) = \alpha_n D_g(y_n, u) \to 0 \tag{4.13}$$

as $n \to \infty$ and hence

$$D_g(y_n, x_{n+1}) \le D_g(y_n, z_n) \to 0 \tag{4.14}$$

as $n \to \infty$. In view of Lemma 2.2 and (4.12)–(4.14), we conclude that

$$\lim_{n\to\infty} \|y_n - w_n\| = \lim_{n\to\infty} \|y_n - T_{r_n} w_n\| = 0 \text{ and } \lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \quad (4.15)$$

From (4.12)–(4.15), we deduce that

$$\lim_{n\to\infty} \|x_n - z_n\| = \lim_{n\to\infty} \|y_n - z_n\| = 0. \quad (4.16)$$

By a similar argument, as in the proof of Theorem 3.1, we have the desired result which completes the proof. □

At the end of the paper, we study the problem of finding a minimizer of a continuously Fréchet differentiable and convex function in a Banach space. We begin with the following lemma which has been proved in [33].

Lemma 4.1. *Let E be a Banach space. Suppose that h is a continuously Fréchet differentiable and convex function on E. If the gradient ∇h of g is $\frac{1}{\alpha}$-Lipschitz continuous, then ∇h is α-inverse strongly monotone.*

Theorem 4.2. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function which is bounded on bounded subsets, and uniformly convex and uniformly smooth on bounded subsets of E. Assume that there exists $c_1 > 0$ such that g is ρ-convex with $\rho(t) := \frac{c_1}{2} t^2$ for all $t \geq 0$. Let C be a nonempty, closed and convex subset of E and f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Assume that a function $h : E \to \mathbb{R}$ satisfies the following conditions:*

(1) *h is continuously Fréchet differentiable and convex on E and ∇f is $\frac{1}{\alpha}$-Lipschitz continuous;*

(2) *$\Omega := \arg\min_{y\in E} h(y) = \{z \in E : h(z) = \min_{y\in C} h(y)\} \neq \emptyset$. Suppose that $F := \Omega \cap EP(f)$ is a nonempty subset of C, where $EP(f)$ is the set of solutions to the equilibrium problem (1.3). Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences in $[0,1]$ satisfying the following control conditions:*

   (a) $\lim_{n\to\infty} \alpha_n = 0$;

   (b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

   (c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

   Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,\infty)$ such that $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$\begin{cases} u \in E,\ x_1 \in C \quad \text{chosen arbitrarily,} \\ w_n = \mathrm{proj}_C^g(\nabla g^*[\nabla g(x_n) - \beta \nabla h(x_n)]) \\ u_n \in C \quad \text{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, \nabla g(u_n) - \nabla g(w_n)\rangle \geq 0, \qquad \forall y \in C, \\ y_n = \nabla g^*[\beta_n \nabla g(w_n) + (1-\beta_n)\nabla g(u_n)], \\ x_{n+1} = \mathrm{proj}_C^g(\nabla g^*[\alpha_n \nabla g(u) + (1-\alpha_n)\nabla g(y_n)]) \text{ and } n \in \mathbb{N}, \end{cases} \quad (4.17)$$

where ∇g *is the gradient of* g*. Let* λ *be a constant such that* $0 < \lambda < \frac{c_2^2\gamma}{2}$*, where* c_2 *is the* 2*-uniformly convex constant of* E *satisfying Corollary 2.1 (2). Then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *defined in* (4.17) *converges strongly to* $\mathrm{proj}_F^g u$ *as* $n \to \infty$.

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