FRACTIONAL INTEGRAL OPERATORS INVOLVING THE PRODUCT OF SRIVASTAVA POLYNOMIALS AND SRIVASTAVA-PANDA MULTIVARIABLE $H$-FUNCTIONS OF GENERALIZED ARGUMENTS

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Abstract. The paper deals with two fractional integral formulae involving the product of a general class of polynomials and multivariate $H$-function. The first involves the operator $e I^v_z[f(z)]$ whereas second is associated with the integral operator $I^{v+q}_z[f(z)]$. In our fractional integral formulae we have taken all the functions and polynomials with a generalized argument. The formulae, we have introduced here, are in compact form and basic in nature. A number of known and new results have been obtained by proper choice of parameters. For the sake of illustration, we record here some particular cases of our main results.

1. Introduction

In recent years several authors (see, for example) Chen et al. [13], Lin et al. [16], Soni et al. [15], see also Gaira et al. [12] have made significant contributions to the fractional calculus operators involving various functions and polynomials. Here we are making an attempt to develop extensions of these results.

We start by introducing following definitions:

Oldham and Spanier [8] considered the fractional integral of a function $f(z)$ of complex order $v$

$$e I^v_z[f(z)] = \begin{cases} 
\frac{1}{\Gamma(v)} \int^z_c (z-t)^{v-1} f(t) dt, & Re(v) > 0 \\
\frac{d^q}{dz^q} e I^{v+q}_z[f(z)], & Re(v) \leq 0, 0 < Re(v) + q \leq 1, \quad q = 1, 2, 3, \ldots.
\end{cases} \quad (1.1)
$$

The special case of fractional integral operator $e I^v_z$, when $c = 0$, will be denoted by $I^v_z$. Thus we write

$$I^v_z \equiv a I^v_z$$

and

$$I^v_z[f(z)] = \begin{cases} 
\frac{1}{\Gamma(v)} \int^z_c (z-t)^{v-1} f(t) dt, & Re(v) > 0 \\
\frac{d^q}{dz^q} I^{v+q}_z[f(z)], & Re(v) \leq 0, 0 < Re(v) + q \leq 1, \quad q = 1, 2, 3, \ldots.
\end{cases} \quad (1.2)
$$

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The above integral operator $I^\alpha_v[f](z)$ is called the \textit{Riemann-Liouville fractional integral operator.} Also, Srivastava \cite{3} introduced general classes of polynomials

$$S^m_n[x] = \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-n)^{mk}}{k!} A_{n,k} x^k, \quad n = 0, 1, \ldots$$  \hspace{1cm} (1.3)

where $m$ is an arbitrary positive integer and coefficients $A_{n,k} \ (n, k \geq 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients of known polynomials as its special cases (\cite{14}, p.158-161).

The $H$-function of several complex variables \cite{3} is defined in the following manner:

$$H[z_1, \ldots, z_r] = H_{P,Q,P',Q',\ldots,P'^{(r)},Q'^{(r)}}^{N,M,N',M',\ldots,N'^{(r)},M'^{(r)}} \left[ \frac{z_1^{(1)} \cdots z_r^{(r)}}{\prod_j \Gamma(1 - a_j^{(i)} + \sum_{i=1}^{r} b_j^{(i)} \xi_i)} \prod_j \Gamma(1 - c_j^{(i)} + \sum_{i=1}^{r} d_j^{(i)} \xi_i) \right]$$

$$= \frac{1}{2\pi i} \int_{L_1} \cdots \int_{L_r} \phi_1(\xi_1) \cdots \phi_r(\xi_r) \psi(\xi_1, \ldots, \xi_r) z_1^{\varphi_1} \cdots z_r^{\varphi_r} d\xi_1 \cdots d\xi_r$$ \hspace{1cm} (1.4)

where $\omega = \sqrt{-1}$

$$\phi_i(\xi_i) = \prod_{j=1}^{M^{(i)}} \Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i) \prod_{j=1}^{N^{(i)}} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i)$$

$$\prod_{j=N^{(i)}+1}^{Q^{(i)}} \Gamma(1 - a_j^{(i)} + \sum_{i=1}^{r} b_j^{(i)} \xi_i)$$ \hspace{1cm} (1.5)

and $\psi(\xi_1, \ldots, \xi_r) = \prod_{j=N+1}^{p} \Gamma(a_j - \sum_{i=1}^{r} a_j^{(i)} \xi_i) \prod_{j=1}^{Q} \Gamma(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} \xi_i)$. \hspace{1cm} (1.6)

The convergence conditions and other details of the above function are given by Srivastava, Gupta and Goyal (\cite{4}, p.251, eq. (c.1), also see p252-253, eq. (c.5 and c.6)).
2. Main results

2.A. Fractional integral formula 1

\[
L^p_v \left[ z^p \prod_{i=1}^{l} (z + \alpha_i)^{\sigma_i} \prod_{j=1}^{k} S_{n_j}^m \left[ e_j z^{u_j} \prod_{i=1}^{l} (z + \alpha_i)^{u_i(j)} \right] \right. \\
\times H \left[ x_1 z^{u_1} \prod_{i=1}^{l} (z + \alpha_i)^{u_i(i)} , \ldots , x_r z^{u_r} \prod_{i=1}^{l} (z + \alpha_i)^{u_r(j)} \right] \\
= z^{\rho} (\alpha_1)^{\sigma_1} \cdots (\alpha_r)^{\sigma_r} \sum_{g_1 = 0}^{\infty} \cdots \sum_{g_l = 0}^{\infty} \sum_{k_1 = 0}^{\infty} \cdots \sum_{k_l = 0}^{\infty} \frac{(-n_1/m_1) \cdots (-n_l/m_l)}{k_1! \cdots k_l!} A^{(1)}_{n_1, k_1} \cdots
\]

Also, by the application of following formula in (2.1)

\[
\sum_{R=0}^{\infty} \frac{(-1)^R}{\Gamma(v)(R + v)!} H^{0, N+1; M', N', \ldots, M'(i), N'(i)}_{P+1; Q, \ldots, Q(i), Q(i)} \left[ \begin{array}{c}
\sum_{j=0}^{m} a_{j} z^{(j)}_{P} \\
\vdots \\
\sum_{r=0}^{m} a_{r} z^{(r)}_{Q} \\
\end{array} \right] \left[ \begin{array}{c}
\frac{(i')_{1, P}}{(i')_{1, Q}} \\
\vdots \\
\frac{(i')_{1, P}}{(i')_{1, Q}} \\
\end{array} \right] \\
= H^{0, N+1; M', N', \ldots, M'(i), N'(i)}_{P+1; Q, \ldots, Q(i), Q(i)} \left[ \begin{array}{c}
\sum_{j=0}^{m} a_{j} z^{(j)}_{P} \\
\vdots \\
\sum_{r=0}^{m} a_{r} z^{(r)}_{Q} \\
\end{array} \right] \left[ \begin{array}{c}
\frac{(i')_{1, P}}{(i')_{1, Q}} \\
\vdots \\
\frac{(i')_{1, P}}{(i')_{1, Q}} \\
\end{array} \right].
\]

We can write

\[
L^p_v \left[ z^p \prod_{i=1}^{l} (z + \alpha_i)^{\sigma_i} \prod_{j=1}^{k} S_{n_j}^m \left[ e_j z^{u_j} \prod_{i=1}^{l} (z + \alpha_i)^{u_i(j)} \right] \right. \\
\times H \left[ x_1 z^{u_1} \prod_{i=1}^{l} (z + \alpha_i)^{u_i(i)} , \ldots , x_r z^{u_r} \prod_{i=1}^{l} (z + \alpha_i)^{u_r(j)} \right] \\
= z^{\rho + v} (\alpha_1)^{\sigma_1} \cdots (\alpha_r)^{\sigma_r} \sum_{g_1 = 0}^{\infty} \cdots \sum_{g_l = 0}^{\infty} \sum_{k_1 = 0}^{\infty} \cdots \sum_{k_l = 0}^{\infty} \frac{(-n_1/m_1) \cdots (-n_l/m_l)}{k_1! \cdots k_l!} A^{(1)}_{n_1, k_1} \cdots
\]
Provided that

(i) $\text{Re}(v) > 0$; the quantities $u_1, u_1^{(1)}, \ldots, u_r^{(1)}, u_2, u_2^{(1)}, \ldots, u_s, \mu_1, \mu_1^{(1)}, \ldots, \mu_r, \mu_r^{(1)}, \ldots, \mu_s^{(1)}$ are all positive (some of them may however decrease to zero provided that the resulting integral has a meaning).

(ii) $\text{Re}(\rho) + \sum_{i=1}^{r} \mu_i \min_{1 \leq j \leq M^{(i)}} |\text{Re}(d_j^{(i)} / \delta_j^{(i)})| + 1 > 0.$

2.B. Fractional integral formula 2

$$I_z^{\eta,v} \left\{ z^\rho \prod_{i=1}^{r} (z + \alpha_i)^{\sigma_i} \prod_{j=1}^{s} \left( \epsilon_j z^u_j \prod_{i=1}^{r} (z + \alpha_i)^{u_j^{(i)}} \right) \right\} \times H \left[ x_1 (\alpha_1)^{\mu_1} \ldots, x_r (\alpha_r)^{\mu_r} H \left[ \prod_{i=1}^{r} (z + \alpha_i)^{\mu_i} \ldots, x_r (\alpha_r)^{\mu_r} \prod_{i=1}^{r} (z + \alpha_i)^{\mu_i} \right] \right]$$

$$= z^\rho (\alpha_1)^{\sigma_1} \ldots (\alpha_r)^{\sigma_r} \prod_{i=0}^{\infty} \sum_{g_i=0}^{\rho^{(i)}} \prod_{g_i=0}^{\infty} \sum_{k_i=0}^{\rho^{(i)}} \frac{(-n_i)_{m_i k_i}}{k_i! g_i!} a_i^{(s)} \ldots$$

$$\times z^{\rho} \prod_{i=1}^{r} (z + \alpha_i)^{\sigma_i} \prod_{j=1}^{s} \left( \epsilon_j z^u_j \prod_{i=1}^{r} (z + \alpha_i)^{u_j^{(i)}} \right) \times H \left[ x_1 (\alpha_1)^{\mu_1} \ldots, x_r (\alpha_r)^{\mu_r} H \left[ \prod_{i=1}^{r} (z + \alpha_i)^{\mu_i} \ldots, x_r (\alpha_r)^{\mu_r} \prod_{i=1}^{r} (z + \alpha_i)^{\mu_i} \right] \right]$$

Provided that

(i) $\eta > 0$; the quantities $u_1, u_1^{(1)}, \ldots, u_r^{(1)}, \ldots, u_s, \mu_1, \mu_1^{(1)}, \ldots, \mu_r, \mu_r^{(1)}, \ldots, \mu_s^{(1)}$ are all positive (some of them may however decrease to zero provided that the resulting integral has a meaning),
\[ \text{(ii) } \text{Re}(\rho) + \sum_{i=1}^{r} \mu_i \min_{1 \leq j \leq M(i)} |\text{Re}(d_{j}^{(i)} / \delta_{j}^{(i)})| + \eta > 0. \]

**Proof.** In order to prove (2.1), we first express a general class of polynomials in series form given by (1.3) and multivariate \( H \)-function in terms of Mellin-Barnes type of contour integrals and interchanging the order of summations, integration and taking the fractional integral operator inside, which is permissible under the stated conditions. Now, using binomial expansion along with the use of the known formula (1.1) and interpreting the multiple Mellin-Barnes contour integral so obtained in terms of \( H \)-function, we easily arrive at the desired formula (2.1).

Also, using the same method adopted in the proof of the result (2.1) and making use of the formula [12, eq.(2.10)]

\[ I_{\eta,\nu}^{\lambda} [z^\lambda] = \frac{\Gamma(\lambda + \eta)}{\Gamma(\lambda + \eta + \nu)} z^\lambda, \quad \text{Re}(\lambda) > -\eta \]

we can prove the result (2.4).

### 3. Special cases

**3.A.** If we put \( t = 2 \) and \( s = 2 \) in our integral formula (2.1), it reduces to the known result recently obtained by Gaira and Dhami [12, p.2, eq.(2.1)].

**3.B.** On taking \( t = 2 \) and \( s = 2 \) our integral formula (2.4) reduces to another known result obtained by Gaira and Dhami [12, p.5, eq.(2.9)].

**3.C.** Letting \( t = 1 \) and \( s = 1 \) in our result (2.4), it leads to a known result given by Gupta and Agarwal [9].

**3.D.** Particularly, when we substitute \( t = 1 \) and \( n_j = 0 \) (\( j = 0, 1, \ldots, s \)) in our integral formula (2.3), we arrive on the result obtained by Srivastava et al. [7].

**3.E.** If we put \( m_j = n_j = k_j = 0 \) (for \( j = 2, 3, \ldots, s \)) and \( t = 1 \) in our integral formula (2.1), we can derive another result obtained by Gupta et al. [9].

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**References**


