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ON OSTROWSKI AND GRÜSS TYPE DISCRETE INEQUALITIES FOR SECOND FORWARD DIFFERENCES

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Abstract. The aim of the present article is to establish two new discrete Ostrowski and Grüss type inequalities involving functions and their first second forward differences.

1. Introduction

In [10, p.468], Ostrowski proved the following interesting inequality.

Let $f : [a, b] \to R$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative is bounded on (a, b) i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a, b]$.

In [1], P. Ceroue, S. S. Dragomir and J. Roumeliotíes proved the following Ostrowski type inequality with twice differentiable mapping.

Let $f : [a, b] \to R$ be continuous and twice differentiable mapping on (a, b) whose second derivate are bounded on (a, b), i.e. $\|f''\|_\infty := \sup_{t\in(a,b)} |f''(t)| < \infty$. Then we have the inequality:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(x - \frac{a+b}{2}\right) f'(x) \right|$$
$$\le \left[\frac{1}{24}(b-a)^2 + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 \right] \|f''\|_\infty \le \frac{(b-a)^2}{6}\|f''\|_\infty \tag{1.2}$$

for all $x \in [a, b]$.

Another celebrated inequality that gives estimation for the integral of a product in terms of the product of integrals, is Grüss inequality [9, p.296].

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If f and g are two integrable functions on $[a,b]$ and ϕ, Φ, γ and Γ are constants such that

$$\phi \le f(x) \le \Phi, \quad \gamma \le g(x) \le \Gamma$$

for all $x \in [a,b]$, then

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx \right| \le \frac{(\Phi-\phi)(\Gamma-\gamma)}{4}. \quad (1.3)$$

In [14], B. G. Pachpatte proved the following Grüss type inequality twice differentiable mapping.

Let $f,\ g : [a,b] \to R$ be continuous on $[a,b]$ and twice differentiable on (a,b), whose second derivatives are bounded on (a,b), i.e.

$$\|f''\|_\infty := \sup_{t\in(a,b)} |f''(t)| < \infty \quad \text{and} \quad \|g''\|_\infty := \sup_{t\in(a,b)} |g''(t)| < \infty.$$

Then

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx - \frac{1}{2(b-a)}\int_a^b \left(x - \frac{a+b}{2}\right)(fg)'(x)dx \right|$$
$$\le \frac{1}{2(b-a)^2}\int_a^b (\|f''\|_\infty |g(x)| + \|g''\|_\infty |f(x)|)B(x)dx \quad (1.4)$$

where $B(x) = \int_a^b |K(x,t)|dt$.
for $x \in [a,b]$, in which

$$K(x,t) = \begin{cases} \dfrac{(t-a)^2}{2} & \text{if } t \in [a,x] \\ \dfrac{(b-a)^2}{2} & \text{if } t \in (x,b] \end{cases}.$$

In the past years, a large number of papers have appeared in the literature that deal with various extensions and applications of inequalities (1.1), (1.2), (1.3) and (1.4), see [2-14] and the reference given therein. Recently, in [15], B. G. Pachpatte established two new discrete types of the inequality (1.1) and (1.3) involving functions and their first order forward differences. The main purpose of the present note is to establish two new discrete types of the inequality (1.2) and (1.4) involving functions and their first and second forward differences.

2. Statement of Results

In what followings, R and N denote the set of real numbers and natural numbers, respectively, and $N_{a,b} = \{a, a+1, \ldots, a+n = b\}$ for $a \in R$, $n \in N$. For any function

$u(t)$, $t \in N_{a,b}$, we define the operator Δ by $\Delta u(t) = u(t+1) - u(t)$ and Δ^2 by $\Delta^2 u(t) = \Delta u(t+1) - \Delta u(t)$. We use the usual convention that empty sum is taken to be zero.

The first result reads as follows.

Theorem 1. *Let f, g be real-valued functions defined on $N_{a,b+1}$ for which $\Delta f(t)$, $\Delta g(t)$, $\Delta^2 f(t)$, $\Delta^2 g(t)$ exist and $|\Delta^2 f(t)| \le A$, $|\Delta^2 g(t)| \le B$ on $N_{a,b+1}$. Then*

$$\begin{aligned}&\Bigg| f(t+1)g(t+1) - \frac{1}{2(b-a)}\left[g(t+1)\sum_{s=a}^{b-1} f(s+2) + f(t+1)\sum_{s=a}^{b-1} g(s+2)\right] \\ &-\frac{\left(t-\frac{a+b}{2}\right)}{2}[g(t+1)\Delta f(t) + f(t+1)\Delta g(t)] \\ &+\frac{1}{4(b-a)}[(f(b+1)-f(a+1))\cdot g(t+1) + (g(b+1)-g(a+1)\cdot f(t+1))]\Bigg| \\ &\le \frac{1}{2(b-a)}[Ag(t+1) + Bf(t+1)]\cdot H(t) \end{aligned} \tag{2.1}$$

and

$$\left| f(t+1) - \frac{1}{b-a}\sum_{s=a}^{b-1} f(s+2) - \left(t - \frac{a+b}{2}\right)\Delta f(t) + \frac{f(b+1)-f(a+1)}{2(b-a)}\right| \le \frac{A}{b-a}\cdot H(t) \tag{2.2}$$

for $n \in N_{a,b}$, where

$$H(t) = \sum_{s=a}^{b-1} |r(t,s)|, \tag{2.3}$$

in which

$$r(t,s) = \begin{cases} \dfrac{(s-a)^2}{2} & \text{if } s \in [a, t-1] \\ \dfrac{(b-s)^2}{2} & \text{if } s \in [t, b] \end{cases} \tag{2.4}$$

for all t, $s \in N_{a,b}$ and A, B are nonnegative constants.

Theorem 2. *Let f, g, Δf, Δg, $\Delta^2 f$, $\Delta^2 g$, A and B be as in Theorem 1. Then*

$$\begin{aligned}&\Bigg| \frac{1}{b-a}\sum_{t=a}^{b-1} f(t+1)g(t+1) \\ &-\frac{1}{2(b-a)^2}\left[\left(\sum_{t=a}^{b-1} g(t+1)\right)\left(\sum_{t=a}^{b-1} f(t+2)\right) + \left(\sum_{t=a}^{b-1} f(t+1)\right)\left(\sum_{t=a}^{b-1} g(t+2)\right)\right] \\ &-\frac{1}{2(b-a)}\sum_{t=a}^{b-1}\left[\left(t-\frac{a+b}{2}\right)(g(t+1)\Delta f(t) + f(t+1)\Delta g(t))\right]\end{aligned}$$

$$+\frac{1}{4(b-a)^2}\left[(f(b+1)-f(a+1))\sum_{t=a}^{b-1} g(t+1)+(g(b+1)-g(a+1))\sum_{t=a}^{b-1} f(t+1)\right]\Bigg|$$

$$\leq \frac{1}{2(b-a)^2}\sum_{t=a}^{b-1}[(A|g(t+1)|+B|f(t+1)|)\cdot H(t)] \tag{2.5}$$

3. Proof of Theorem 1

Using the summation by parts formula,

$$\sum_{s=\alpha}^{\beta-1} u(s)\Delta v(s)=[u(\beta)v(\beta)-u(\alpha)v(\alpha)]-\sum_{s=\alpha}^{\beta-1} v(s+1)\Delta u(s),$$

we have

$$\begin{aligned}&\sum_{s=\alpha}^{\beta-1} u(s)\Delta^2 v(s)\\ &=[u(\beta)\Delta v(\beta)-u(\alpha)\Delta v(\alpha)]-\sum_{s=\alpha}^{\beta-1}\Delta u(s)\Delta v(s+1)\\ &=u(\beta)\Delta v(\beta)-u(\alpha)\Delta(\alpha)-\Delta u(\beta)v(\beta+1)+\Delta u(\alpha)v(\alpha+1)+\sum_{s=\alpha}^{\beta-1} v(s+2)\Delta^2 u(s)\end{aligned}$$

where α, $\beta \in N_{a,b}$ and u, v are real-valued functions defined on $N_{a,b+1}$.

Let $u(s)=\frac{(s-a)^2}{2}$, $v(s)=f(s)$, $\beta=t$, $\alpha=a$. Then $u(a)=0$, $\Delta u(s)=(s-a+\frac{1}{2})$ and $\Delta^2 u(s)=1$, so that for $t\in N_{a,b}$, we have

$$\sum_{s=a}^{t-1}\frac{(s-a)^2}{2}\cdot\Delta^2 f(s)=\frac{(t-a)^2}{2}\cdot\Delta f(t)-\left(n-a+\frac{1}{2}\right)\cdot f(t+1)+\frac{1}{2}f(a+1)+\sum_{s=a}^{t-1} f(s+2). \tag{3.1}$$

Similarly, we have

$$\sum_{s=t}^{b-1}\frac{(b-s)^2}{2}\cdot\Delta^2 f(s)=-\frac{(b-t)^2}{2}\cdot\Delta f(t)-\frac{1}{2}f(b+1)+\left(n-b+\frac{1}{2}\right)\cdot f(t+1)+\sum_{s=t}^{b-1} f(s+2). \tag{3.2}$$

Adding (3.1), (3.2) and using (2.4), we get

$$\begin{aligned}&\sum_{s=a}^{b-1} r(t,s)\Delta^2 f(s)\\ &=(b-a)\left(t-\frac{a+b}{2}\right)\Delta f(t)-(b-a)f(t+1)+\frac{f(a+1)-f(b+1)}{2(b-a)}+\sum_{s=a}^{b-1} f(s+2)\end{aligned}$$

i.e.

$$f(t+1)-\frac{1}{b-a}\sum_{s=a}^{b-1}f(s+2)-\left(t-\frac{a+b}{2}\right)\Delta f(t)+\frac{f(b+1)-f(a+1)}{2(b-a)}=\frac{-1}{b-a}\sum_{s=a}^{b-1}r(t,s)\Delta^2 f(s), \tag{3.3}$$

for all $t \in N_{a,b}$.

Similarly, we hvae

$$g(t+1)-\frac{1}{b-a}\sum_{s=a}^{b-1}g(s+2)-\left(t-\frac{a+b}{2}\right)\Delta g(t)+\frac{g(b+1)-g(a+1)}{2(b-a)}=\frac{-1}{b-a}\sum_{s=a}^{b-1}r(t,s)\Delta^2 g(s), \tag{3.4}$$

for all $t \in N_{a,b}$.

Multiplying (3.3) by $g(t+1)$ and (3.4) by $f(t+1)$, $t \in N_{a,b}$, adding the resulting identities and rewriting we get

$$\begin{aligned}
&f(t+1)g(t+1)-\frac{1}{2(b-a)}g(t+1)\sum_{s=a}^{b-1}f(s+2)-\frac{1}{2(b-a)}f(t+1)\sum_{s=a}^{b-1}g(s+2)\\
&-\frac{1}{2}\left(t-\frac{a+b}{2}\right)g(t+1)\Delta f(t)-\frac{1}{2}\left(t-\frac{a+b}{2}\right)f(t+1)\Delta g(t)\\
&\frac{f(b+1)-f(a+1)}{4(b-a)}\cdot g(t+1)+\frac{g(b+1)-g(a+1)}{4(b-a)}\cdot f(t+1)\\
=&\frac{-1}{2(b-a)}g(t+1)\sum_{s=a}^{b-1}r(t,s)\Delta^2 f(s)+\frac{-1}{2(b-a)}f(t+1)\sum_{s=a}^{b-1}r(t,s)\Delta^2 g(s)
\end{aligned} \tag{3.5}$$

From (3.5), we have

$$\begin{aligned}
&\Bigg|f(t+1)g(t+1)-\frac{1}{2(b-a)}\left[g(t+1)\sum_{s=a}^{b-1}f(s+2)+f(t+1)\sum_{s=a}^{b-1}g(s+2)\right]\\
&-\frac{\left(t-\frac{a+b}{2}\right)}{2}[g(t+1)\Delta f(t)+f(t+1)\Delta g(t)]\\
&+\frac{1}{4(b-a)}[(f(b+1)-f(a+1))g(t+1)+(g(b+1)-g(a+1))f(t+1)]\Bigg|\\
\le&\frac{1}{2(b-a)}[A|g(t+1)|+B|f(t+1)|]\cdot\sum_{s=a}^{b-1}|r(t,s)|\\
=&\frac{1}{2(b-a)}[A|g(t+1)|+B|f(t+1)|]\cdot H(t)
\end{aligned}$$

which is the required inequality in (2.1).

The inequality (2.2) follows immediately from (3.3). The proof is complete.

4. Proof of Theorem 2

From the hypotheses, as in the proof of Theorem 1, the identities (3.3), (3.4) and (3.5) hold. Summing both sides of (3.5) over n from a to $b-1$, and rewriting we get

$$\begin{aligned}
&\sum_{t=a}^{b-1} f(t+1)g(t+1)\\
&-\frac{1}{2(b-a)}\left[\left(\sum_{t=a}^{b-1} g(t+1)\right)\left(\sum_{t=a}^{b-1} f(t+2)\right)+\left(\sum_{t=a}^{b-1} f(t+1)\right)\left(\sum_{t=a}^{b-1} g(t+2)\right)\right]\\
&-\frac{1}{2}\sum_{t=a}^{b-1}\left[\left(t-\frac{a+b}{2}\right)(g(t+1)\Delta f(t)+f(t+1)\Delta g(t))\right]\\
&+\frac{1}{4(b-a)}\left[(f(b+1)-f(a+1))\sum_{t=a}^{b-1} g(t+1)+(g(b+1)-g(a+1))\sum_{t=a}^{b-1} f(t+1)\right]\\
=&\frac{-1}{2(b-a)}\left[\left(\sum_{t=a}^{b-1} g(t+1)\sum_{s=a}^{b-1} r(t,s)\Delta^2 f(s)\right)+\left(\sum_{t=a}^{b-1} f(t+1)\sum_{s=a}^{b-1} r(t,s)\Delta^2 g(s)\right)\right]. \quad (4.1)
\end{aligned}$$

From (4.1) and using the properties of modulus, we observe that

$$\begin{aligned}
&\left|\frac{1}{b-a}\sum_{t=a}^{b-1} f(t+1)g(t+1)\right.\\
&-\frac{1}{2(b-a)^2}\left[\left(\sum_{t=a}^{b-1} g(t+1)\right)\left(\sum_{t=a}^{b-1} f(t+2)\right)+\left(\sum_{t=a}^{b-1} f(t+1)\right)\left(\sum_{t=a}^{b-1} g(t+2)\right)\right]\\
&-\frac{1}{2(b-a)}\sum_{t=a}^{b-1}\left[\left(t-\frac{a+b}{2}\right)(g(t+1)\Delta f(t)+f(t+1)\Delta g(t))\right]\\
&\left.+\frac{1}{4(b-a)^2}\left[(f(b+1)-f(a+1))\sum_{t=a}^{b-1} g(t+1)+(g(b+1)-g(a+1))\sum_{t=a}^{b-1} f(t+1)\right]\right|\\
\le&\frac{1}{2(b-a)^2}\left[\sum_{t=a}^{b-1}\left(|g(t+1)|\sum_{s=a}^{b-1}|r(t,s)|\cdot|\Delta^2 f(s)|\right)+\sum_{t=a}^{b-1}\left(|f(t+1)|\sum_{s=a}^{b-1}|r(t,s)|\cdot|\Delta^2 g(s)|\right)\right]\\
\le&\frac{1}{2(b-a)^2}\left[\sum_{t=a}^{b-1}(|g(t+1)|\cdot H(t)\cdot A)+\sum_{s=a}^{b-1}(|f(t+1)|H(t)\cdot B)\right]\\
\le&\frac{1}{2(b-a)^2}\sum_{t=a}^{b-1}[(Ag(t+1)|+Bf(t+1))\cdot H(t)].
\end{aligned}$$

This proves the inequality (2.5). The proof is complete.

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