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A SUBCLASS OF HARMONIC FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY DZIOK-SRIVASTAVA OPERATOR

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Abstract. Making use of the Dziok-Srivastava operator, we introduce the class $\mathscr{R}^{p,q}_{\overline{\mathscr{H}}}([\alpha_1],\lambda,\gamma)$ of complex valued harmonic functions. We investigate the coefficient bounds, distortion inequalities , extreme points and inclusion results for this class.

1. Introduction

A continuous function $f = u + iv$ is a complex- valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply-connected domain $D \subset \Omega$, we can write $f = h+\overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [3]).

Denote by $\mathscr{H}$ the family of functions

$$f = h+\overline{g} \tag{1.1}$$

which are harmonic, univalent and orientation preserving in the open unit disc $U = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0)-1 = 0$. Thus, for $f = h+\overline{g} \in \mathscr{H}$, we may express the analytic functions for h and g in the forms

$$h(z) = z+\sum_{n=2}^{\infty} a_n z^n,\ g(z) = \sum_{n=1}^{\infty} b_n z^n \quad (|b_1| < 1).$$

Hence

$$f(z) = z+\sum_{n=2}^{\infty} a_n z^n+\overline{\sum_{n=1}^{\infty} b_n z^n},\ |b_1| < 1. \tag{1.2}$$

We note that the family $\mathscr{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class S of normalized univalent functions if the co-analytic part of f is identically zero, that is $g \equiv 0$.

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Also denote by $\overline{\mathcal{H}}$ the subfamily of $\mathcal{H}$ consists harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \overline{\sum_{n=1}^{\infty} |b_n| z^n}, \ |b_1| < 1. \tag{1.3}$$

Let the Hadamard product (or convolution) of two power series

$$\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$$

and

$$\psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n$$

be defined by

$$(\phi * \psi)(z) = \phi(z) * \psi(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n.$$

For complex parameters $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_q$ $(\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, q)$ the generalized hypergeometric function ${}_pF_q(z)$ is defined by

$${}_pF_q(z) \equiv {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!}, \tag{1.4}$$

$$(p \leq q+1; \ p, q \in \ N_0 := \ N \cup \{0\}; z \in U)$$

where N denotes the set of all positive integers and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)(a+2)\ldots(a+n-1), & n \in \ N. \end{cases} \tag{1.5}$$

Let

$$H(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q) : A \to A$$

be a linear operator defined by

$$\begin{aligned} [(H(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q))(\phi)](z) := & \ \ z \ {}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z) * \phi(z) \\ = & \ z + \sum_{n=2}^{\infty} \Gamma(\alpha_1, n) \ a_n \ z^n, \end{aligned} \tag{1.6}$$

where

$$\Gamma(\alpha_1, n) = \frac{(\alpha_1)_{n-1} \ldots (\alpha_p)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_q)_{n-1}} \frac{1}{(n-1)!}$$

and

$$\alpha_i > 0 (i = 1, 2, \ldots p), \beta_j > 0 (j = 1, 2, \ldots q), p \leq q+1; \ p, q \in \ N_0 = \ N \cup \{0\}.$$

For notational simplicity, we use a shorter notation $H_q^p[\alpha_1]$ for $H(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q)$ in the sequel. Note that if $q=0$ then

$$H_0^p[\alpha_1] = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}\ldots(\alpha_p)_{n-1}}{(n-1)!}\, a_n\, z^n.$$

It follows from (1.6) that

$$H_0^1[1]f(z) = f(z),\ H_0^1[2]f(z) = zf'(z) \text{ and } H_0^1[3]f(z) = zf'(z) + \frac{1}{2}z^2 f''(z).$$

The linear operator $H_q^p[\alpha_1]$ is the Dziok-Srivastava operator (see [4]), which contains such well known operators as the Hohlov linear operator, Saitoh's generalized linear operator, the Carlson-Shaffer linear operator, the Ruscheweyh derivative operator as well as its generalized versions, the Bernardi-Libera-Livingston operator, and the Srivastave-Owa fractional derivative operator. One may refer to Carlson and Shaffer [2], Dziok and Srivastava[4] and Srivastava and Owa [15] for more details concerning these operators (see also [1, 11, 12, 13]).

Applying the Dziok-Srivastava operator to the harmonic function $f = h + \overline{g}$ given by (1.1), we readily get

$$H_q^p[\alpha_1]f(z) = H_q^p[\alpha_1]h(z) + \overline{H_q^p[\alpha_1]g(z)}. \tag{1.7}$$

Motivated by the earlier works of [5, 6, 7, 8, 14, 16] on the subject of harmonic functions, we introduce here a new subclass $\mathscr{R}_{\mathscr{H}}^{p,q}([\alpha_1],\lambda,\gamma)$ of $\mathscr{H}$.

For $0 \le \gamma < 1$, we let $\mathscr{R}_{\mathscr{H}}^{p,q}([\alpha_1],\lambda,\gamma)$ a subclass of $\mathscr{H}$ of the form $f = h + \overline{g}$ given by (1.2) and satisfying the analytic criteria

$$\operatorname{Re}\left\{\frac{z(H_q^p[\alpha_1]h(z))' - \overline{z(H_q^p[\alpha_1]g(z))'}}{(1-\lambda)\left(H_q^p[\alpha_1]h(z) + \overline{H_q^p[\alpha_1]g(z)}\right) + \lambda\left(z(H_q^p[\alpha_1]h(z))' - \overline{z(H_q^p[\alpha_1]g(z))'}\right)}\right\} \ge \gamma \tag{1.8}$$

where $0 \le \lambda \le 1$, $H_q^p[\alpha_1]f(z)$ is given by (1.2) and $z \in U$.

We also let $\mathscr{R}_{\overline{\mathscr{H}}}^{p,q}([\alpha_1],\lambda,\gamma) = \mathscr{R}_{\mathscr{H}}^{p,q}([\alpha_1],\lambda,\gamma) \cap \overline{\mathscr{H}}$.

The family $\mathscr{R}_{\overline{\mathscr{H}}}^{p,q}([\alpha_1],\lambda,\gamma)$ is of special interest because it reduces to various classes of well-known harmonic univalent functions as well many new ones. For example:

(i) $\mathscr{R}_{\overline{\mathscr{H}}}^{1,0}([\alpha_1],0,\gamma) \equiv \mathscr{V}_{\overline{\mathscr{H}}}(\alpha_1,\gamma)$ (Al-Kharsani and Al-Khal[9]), item $\mathscr{R}_{\overline{\mathscr{H}}}^{1,0}([1],\lambda,\gamma) \equiv \mathscr{T}\mathscr{S}^*_{\mathscr{H}}(\lambda,\gamma)$ (Öztürk *et al.*[10]),

(ii) $\mathscr{R}_{\overline{\mathscr{H}}}^{1,0}([1],0,\gamma) \equiv \mathscr{T}_{\mathscr{H}}(\gamma)$ (Jahangiri [6]),

(iii) $\mathscr{R}_{\overline{\mathscr{H}}}^{1,0}([1],0,0) \equiv \mathscr{T}_{\mathscr{H}}^{*0}$ (Silverman [14]).

In this paper, we obtained coefficient conditions for the classes $\mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\gamma)$ and $\mathcal{R}^{p,q}_{\overline{\mathcal{H}}}([\alpha_1],\lambda,\gamma)$. Coefficient bounds, distortion inequalities, inclusion properties for the class $\mathcal{R}^{p,q}_{\overline{\mathcal{H}}}([\alpha_1],\lambda,\gamma)$ are also established.

2. Coefficient bounds

In our first theorem, we obtain a sufficient coefficient condition for harmonic functions in $\mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\gamma)$.

Theorem 2.1. *Let $f = h+\overline{g}$ be given by (1.2). If*

$$\sum_{n=1}^{\infty}\left[(n-\gamma-\gamma\lambda(n-1))|a_n| + (n+\gamma-\gamma\lambda(n+1))|b_n|\right]\Gamma(\alpha_1,n) \le 2(1-\gamma) \tag{2.1}$$

where $a_1 = 1$ and $0 \le \gamma < 1$, then $f \in \mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\gamma)$.

Proof. We first show that if (2.1) holds for the coefficients of $f = h+\overline{g}$, the required condition (1.8) is satisfied. From (1.8) we can write

$$\begin{aligned}&\operatorname{Re}\left\{\frac{z(H^p_q[\alpha_1]h(z))' - \overline{z(H^p_q[\alpha_1]g(z))'}}{(1-\lambda)(H^p_q[\alpha_1]h(z)+\overline{H^p_q[\alpha_1]g(z)}) + \lambda(z(H^p_q[\alpha_1]h(z))' - \overline{z(H^p_q[\alpha_1]g(z))'})}\right\}\\ &= \operatorname{Re}\left\{\frac{A(z)}{B(z)}\right\} \ge \gamma\end{aligned}$$

where

$$A(z) = z(H^p_q[\alpha_1]h(z))' - \overline{z(H^p_q[\alpha_1]g(z))'} = z + \sum_{n=2}^{\infty} n\Gamma(\alpha_1,n)a_n z^n - \sum_{n=1}^{\infty} n\Gamma(\alpha_1,n)\overline{b}_n\overline{z}^n$$

$$\begin{aligned}\text{and } B(z) &= (1-\lambda)(H^p_q[\alpha_1]h(z)+\overline{H^p_q[\alpha_1]g(z)}) + \lambda(z(H^p_q[\alpha_1]h(z))' - \overline{z(H^p_q[\alpha_1]g(z))'})\\ &= z + \sum_{n=2}^{\infty}(1-\lambda+n\lambda)\Gamma(\alpha_1,n)a_n z^n + \sum_{n=1}^{\infty}(1-\lambda-n\lambda)\Gamma(\alpha_1,n)\overline{b}_n\overline{z}^n.\end{aligned}$$

Using the fact that Re $\{w\} \ge \gamma$ if and only if $|1-\gamma+w| \ge |1+\gamma-w|$, it suffices to show that

$$|A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)| \ge 0. \tag{2.2}$$

Substituting for $A(z)$ and $B(z)$ in (2.2), we get

$$\begin{aligned}&|A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)|\\ &= \Big|\ (2-\gamma)z + \sum_{n=2}^{\infty}[(n+1-\gamma)(1-\lambda+n\lambda)]\Gamma(\alpha_1,n)a_n z^n -\\ &\quad - \sum_{n=1}^{\infty}[n-(1-\gamma)(1-\lambda+n\lambda)]\Gamma(\alpha_1,n)\overline{b}_n\,\overline{z}^n\ \Big|\end{aligned}$$

$$
\begin{aligned}
&-|-\gamma z+\sum_{n=2}^{\infty}[n-(1+\gamma)(1-\lambda+n\lambda)\Gamma(\alpha_1,n)a_n z^n-\\
&-\sum_{n=1}^{\infty}[n+(1+\gamma)(1-\lambda+n\lambda)]\Gamma(\alpha_1,n)\overline{b}_n\overline{z}^n|\\
\geq\ &(2-\gamma)|z|-\sum_{n=2}^{\infty}[n+(1-\gamma)(1-\lambda+n\lambda)\Gamma(\alpha_1,n)|a_n||z|^n-\\
&-\sum_{n=1}^{\infty}[n-(1-\gamma)(1-\lambda-n\lambda)]\Gamma(\alpha_1,n)|b_n|\,|z|^n-\\
&-\gamma|z|-\sum_{n=2}^{\infty}[n-(1+\gamma)(1-\lambda+n\lambda)]\Gamma(\alpha_1,n)|a_n|\,|z|^n-\\
&-\sum_{n=1}^{\infty}[n+(1+\gamma)(1-\lambda-n\lambda)]\Gamma(\alpha_1,n)|b_n|\,|z|^n\\
\geq\ &2(1-\gamma)|z|\left\{2-\sum_{n=1}^{\infty}\left[\frac{n-\gamma-\gamma\lambda(n-1))}{1-\gamma}|a_n|+\frac{n+\gamma-\gamma\lambda(n+1))}{1-\gamma}|b_n|\right]\Gamma(\alpha_1,n)|z|^{n-1}\right\}\\
\geq\ &2(1-\gamma)\left\{2-\sum_{n=1}^{\infty}\left[\frac{n-\gamma-\gamma\lambda(n-1))}{1-\gamma}|a_n|+\frac{n+\gamma-\gamma\lambda(n+1))}{1-\gamma}|b_n|\right]\Gamma(\alpha_1,n)\right\}.
\end{aligned}
$$

The above expression is non negative by (2.1), and so $f(z)\in\mathscr{R}_{\mathscr{H}}^{p,q}([\alpha_1],\lambda,\gamma)$. □

The harmonic function

$$f(z)=z+\sum_{n=2}^{\infty}\frac{1-\gamma}{(n-\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n))}x_n z^n+\sum_{n=1}^{\infty}\frac{1-\gamma}{(n+\gamma-\gamma\lambda(n+1))\Gamma(\alpha_1,n)}\overline{y}_n(\overline{z})^n \tag{2.3}$$

where $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1$ shows that the coefficient bound given by (2.1) is sharp.
The functions of the form (2.3) are in $\mathscr{R}_{\mathscr{H}}^{p,q}([\alpha_1],\lambda,\gamma)$ because

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(\frac{(n-\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n)}{1-\gamma}|a_n|+\frac{(n+\gamma-\gamma\lambda(n+1))\Gamma(\alpha_1,n)}{1-\gamma}|b_n|\right)\\
&=1+\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=2.
\end{aligned}
$$

Next theorem establishes that such coefficient bounds cannot be improved further.

Theorem 2.2. *For $a_1=1$ and $0\leq\gamma<1$, $f=h+\overline{g}\in\overline{\mathscr{R}_{\mathscr{H}}^{p,q}}([\alpha_1],\lambda,\gamma)$ if and only if*

$$\sum_{n=1}^{\infty}\left[(n-\gamma-\gamma\lambda(n-1))|a_n|+(n+\gamma-\gamma\lambda(n+1))|b_n|\right]\Gamma(\alpha_1,n)\leq 2(1-\gamma). \tag{2.4}$$

Proof. Since $\overline{\mathscr{R}_{\mathscr{H}}^{p,q}}([\alpha_1],\lambda,\gamma)\subset\mathscr{R}_{\mathscr{H}}^{p,q}([\alpha_1],\lambda,\gamma)$, we only need to prove the "only if" part of the theorem. To this end, for functions f of the form (1.3), we notice that the condition

$$\operatorname{Re}\left\{\frac{z(H_q^p[\alpha_1]f(z))'}{(1-\lambda)H_q^p[\alpha_1]f(z)+\lambda z(H_q^p[\alpha_1]f(z))'}\right\}\geq\gamma.$$

Equivalently,

$$\operatorname{Re}\left\{\frac{(1-\gamma)z-\sum_{n=2}^{\infty}(n-\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n)a_nz^n-\sum_{n=1}^{\infty}(n+\gamma-\gamma\lambda(n+1))\Gamma(\alpha_1,n)\overline{b}_n\overline{z}^n}{z-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\Gamma(\alpha_1,n)a_nz^n+\sum_{n=1}^{\infty}(1-\lambda-n\lambda)\Gamma(\alpha_1,n)\overline{b}_n\overline{z}^n}\right\}\geq 0.$$

The above required condition must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$\frac{(1-\gamma)-\sum_{n=2}^{\infty}(n-\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n)a_nr^{n-1}-\sum_{n=1}^{\infty}(n+\gamma-\gamma\lambda(n+1))\Gamma(\alpha_1,n)b_nr^{n-1}}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\Gamma(\alpha_1,n)a_nr^{n-1}+\sum_{n=1}^{\infty}(1-\lambda-n\lambda)\Gamma(\alpha_1,n)b_nr^{n-1}}\geq 0. \tag{2.5}$$

If the condition (2.4) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Hence, there exist $z_0=r_0$ in (0,1) for which the quotient of (2.5) is negative. This contradicts the required condition for $f(z)\in\mathscr{R}^{p,q}_{\overline{\mathscr{H}}}([\alpha_1],\lambda,\gamma)$. This completes the proof of the theorem. □

Letting $p=1$ and $q=\lambda=0$ in Theorem 2.2, we have

Corollary 2.3. *([9])For $a_1=1$ and $0\leq\gamma<1$, $f=h+\overline{g}\in\mathscr{V}_{\overline{\mathscr{H}}}(\alpha_1,\gamma)$ if and only if*

$$\sum_{n=1}^{\infty}\left[(n-\gamma)|a_n|+(n+\gamma(n+1))|b_n|\right]\Gamma(\alpha_1,n)\leq 2(1-\gamma). \tag{2.6}$$

Letting $p=1$, $q=0$ and $\alpha_1=1$ in Theorem 2.2, we have

Corollary 2.4. *([10])For $a_1=1$ and $0\leq\gamma<1$, $f=h+\overline{g}\in\mathscr{T}\mathscr{S}^*_{\overline{\mathscr{H}}}(\lambda,\gamma)$ if and only if*

$$\sum_{n=1}^{\infty}\left[(n-\gamma-\gamma\lambda(n-1))|a_n|+(n+\gamma-\gamma\lambda(n+1))|b_n|\right]\leq 2(1-\gamma). \tag{2.7}$$

Letting $p=1$, $q=0$ and $\alpha_1=2$ in Theorem 2.2, we have

Corollary 2.5. *For $a_1=1$ and $0\leq\gamma<1$, $f=h+\overline{g}\in\mathscr{R}^{1,0}_{\overline{\mathscr{H}}}([2],\lambda,\gamma)$ if and only if*

$$\sum_{n=1}^{\infty}n\left[(n-\gamma-\gamma\lambda(n-1))|a_n|+(n+\gamma-\gamma\lambda(n+1))|b_n|\right]\leq 2(1-\gamma). \tag{2.8}$$

Letting $p=1$, $q=0$ and $\alpha_1=3$ in Theorem 2.2, we have

Corollary 2.6. *For $a_1=1$ and $0\leq\gamma<1$, $f=h+\overline{g}\in\mathscr{R}^{1,0}_{\overline{\mathscr{H}}}([3],\lambda,\gamma)$ if and only if*

$$\sum_{n=1}^{\infty}\frac{n(n+1)}{2}\left[(n-\gamma-\gamma\lambda(n-1))|a_n|+(n+\gamma-\gamma\lambda(n+1))|b_n|\right]\leq 2(1-\gamma). \tag{2.9}$$

3. Distortion bounds and extreme points

The following theorem gives the distortion bounds for functions in $\mathcal{R}_{\mathcal{H}}^{p,q}([\alpha_1],\lambda,\gamma)$ which yields a covering result for the class $\mathcal{R}_{\mathcal{H}}^{p,q}([\alpha_1],\lambda,\gamma)$.

Theorem 3.1. *Let* $f \in \mathcal{R}_{\mathcal{H}}^{p,q}([\alpha_1],\lambda,\gamma)$. *Then for* $|z| = r < 1$, *we have*

$$\begin{aligned}(1-b_1)r - \frac{\beta_1}{\alpha_1}\left(\frac{1-\gamma}{2-\gamma-\gamma\lambda} - \frac{1+\gamma}{2-\gamma-\gamma\lambda}b_1\right)r^2 &\le |f(z)| \\ \le (1+b_1)r + \frac{\beta_1}{\alpha_1}\left(\frac{1-\gamma}{2-\gamma-\gamma\lambda} - \frac{1+\gamma}{2-\gamma-\gamma\lambda}b_1\right)r^2&.\end{aligned}$$

Proof. We only prove the right hand inequality. Taking the absolute value of $f(z)$, we obtain

$$\begin{aligned}|f(z)| &= \left|z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n\right| \\ &\le (1+b_1)|z| + \sum_{n=2}^{\infty}(a_n+b_n)|z|^n \\ &\le (1+b_1)r + \sum_{n=2}^{\infty}(a_n+b_n)r^2 \\ &\le (1+b_1)r + \frac{(1-\gamma)\beta_1}{(2-\gamma-\gamma\lambda)\alpha_1}\sum_{n=2}^{\infty}\left(\frac{(2-\gamma-\gamma\lambda)\alpha_1}{(1-\gamma)\beta_1}a_n + \frac{(2-\gamma-\gamma\lambda))\alpha_1}{(1-\gamma)\beta_1}b_n\right)r^2 \\ &\le (1+b_1)r + \frac{(1-\gamma)\beta_1}{(2-\gamma-\gamma\lambda)\alpha_1}\left(1 - \frac{1+\gamma}{1-\gamma}b_1\right)r^2 \\ &\le (1+b_1)r + \frac{\beta_1}{\alpha_1}\left(\frac{1-\gamma}{2-\gamma-\gamma\lambda} - \frac{1+\gamma}{2-\gamma-\gamma\lambda}b_1\right)r^2.\end{aligned}$$

The proof of the left hand inequality follows on lines similar to that of the right hand side inequality. □

The covering result follows from the left hand inequality given in Theorem 3.1.

Corollary 3.2. *If* $f(z) \in \mathcal{R}_{\mathcal{H}}^{p,q}([\alpha_1],\lambda,\gamma)$. *Then*

$$\left\{w : |w| < \frac{2\alpha_1 - \beta_1 - ((1+\lambda)\alpha_1 - \beta_1)\gamma}{(2-\gamma-\gamma\lambda)\alpha_1}(1-b_1)\right\} \subset f(U).$$

Proof. Using the left hand inequality of Theorem 3.1 and letting $r \to 1$, we prove that

$$\begin{aligned}&(1-b_1) - \frac{1}{\Gamma(\alpha_1,2)}\left(\frac{1-\gamma}{2-\gamma-\gamma\lambda} - \frac{1+\gamma}{2-\gamma-\gamma\lambda}b_1\right) \\ &= (1-b_1) - \frac{1}{\Gamma(\alpha_1,2)(2-\gamma-\gamma\lambda)}[1-\gamma-(1+\gamma)b_1] \\ &= \frac{(1-b_1)\Gamma(\alpha_1,2)(2-\gamma-\gamma\lambda) - (1-\gamma) + (1+\gamma)b_1}{\Gamma(\alpha_1,2)(2-\gamma-\gamma\lambda)}\end{aligned}$$

$$= \left\{ \frac{2\alpha_1 - \beta_1 - ((1+\lambda)\alpha_1 - \beta_1)\gamma}{(2-\gamma-\gamma\lambda)\alpha_1}(1-b_1) \right\} \subset f(U). \qquad \square$$

Next we determine the extreme points of closed convex hulls of $\mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\gamma)$ denoted by $clco\mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\gamma)$.

Theorem 3.3. *A function* $f(z) \in \mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\gamma)$ *if and only if* $f(z) = \sum_{n=1}^{\infty} \big(X_n h_n(z) + Y_n g_n(z)\big)$ *where*

$$h_1(z) = z, \; h_n(z) = z - \frac{1-\gamma}{(n-\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n)} z^n; \quad (n \geq 2),$$

$$g_n(z) = z + \frac{1-\gamma}{(n+\gamma-\gamma\lambda(n+1))\Gamma(\alpha_1,n)} \overline{z}^n; (n \geq 2),$$

$$\sum_{n=1}^{\infty} (X_n + Y_n) = 1, \;\; X_n \geq 0 \; and \;\; Y_n \geq 0.$$

In particular, the extreme points of $\mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\gamma)$ *are* $\{h_n\}$ *and* $\{g_n\}$.

Proof. First, we note that for f as in the theorem above, we may write

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \big(X_n h_n(z) + Y_n g_n(z)\big) \\ &= \sum_{n=1}^{\infty} (X_n + Y_n) z - \sum_{n=2}^{\infty} \frac{1-\gamma}{(n-\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n)} X_n z^n \\ &\quad + \sum_{n=1}^{\infty} \frac{1-\gamma}{(n+\gamma-\gamma\lambda(n+1))\Gamma(\alpha_1,n)} Y_n \overline{z}^n \\ &= z - \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \overline{z}^n \end{aligned}$$

where $A_n = \dfrac{1-\gamma}{(n-\gamma-\gamma\lambda(n-1)))\Gamma(\alpha_1,n)} X_n$, and $B_n = \dfrac{1-\gamma}{(n+\gamma-\gamma\lambda(n+1))\Gamma(\alpha_1,n)} Y_n$.

Therefore

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{(n-\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n)}{1-\gamma} A_n + \sum_{n=1}^{\infty} \frac{(n+\gamma-\gamma\lambda(n+1))\Gamma(\alpha_1,n)}{1-\gamma} B_n \\ &= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n \\ &= 1 - X_1 \leq 1, \end{aligned}$$

and hence $f(z) \in clco\mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\gamma)$. Conversely, suppose that $f(z) \in clco\mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\gamma)$. Setting

$$X_n = \frac{(n-\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n)}{1-\gamma} A_n, \;\; (n \geq 2) \text{ and } Y_n = \frac{(n+\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n)}{1-\gamma} B_n, \;\; (n \geq 1)$$

where $\sum_{n=1}^{\infty}(X_n+Y_n)=1$. Then

$$\begin{aligned}
f(z) &= z-\sum_{n=2}^{\infty}a_nz^n+\sum_{n=1}^{\infty}\overline{b}_n\overline{z}^n,\quad a_n,\ b_n\geq 0.\\
&= z-\sum_{n=2}^{\infty}\frac{1-\gamma}{(n-\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n)}X_nz^n+\sum_{n=1}^{\infty}\frac{1-\gamma}{(n+\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n)}Y_n\overline{z}^n\\
&= z-\sum_{n=2}^{\infty}(h_n(z)-z)X_n+\sum_{n=1}^{\infty}(g_n(z)-z)Y_n\\
&= \sum_{n=1}^{\infty}(X_nh_n(z)+Y_ng_n(z))
\end{aligned}$$

as required. □

4. Inclusion results

Now we show that $\mathcal{R}_{\mathcal{H}}^{p,q}([\alpha_1],\lambda,\gamma)$ is closed under convex combinations of its member and also closed under the convolution product.

Theorem 4.1. *The family $\mathcal{R}_{\mathcal{H}}^{p,q}([\alpha_1],\lambda,\gamma)$ is closed under convex combinations.*

Proof. For $i=1,2,\ldots$, suppose that $f_i\in\mathcal{R}_{\mathcal{H}}^{p,q}([\alpha_1],\lambda,\gamma)$ where

$$f_i(z)=z-\sum_{n=2}^{\infty}a_{i,n}z^n+\sum_{n=2}^{\infty}\overline{b}_{i,n}\overline{z}^n.$$

Then, by Theorem 3.1

$$\sum_{n=2}^{\infty}\frac{(n-\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n)}{(1-\gamma)}a_{i,n}+\sum_{n=1}^{\infty}\frac{(n+\gamma-\gamma\lambda(n+1))\Gamma(\alpha_1,n)}{(1-\gamma)}b_{i,n}\leq 1. \tag{4.1}$$

For $\sum_{i=1}^{\infty}t_i$, $0\leq t_i\leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty}t_if_i(z)=z-\sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_ia_{i,n}\right)z^n+\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i\overline{b}_{i,n}\right)\overline{z}^n.$$

Using the inequality (2.4), we obtain

$$\begin{aligned}
&\sum_{n=2}^{\infty}\frac{(n-\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n)}{(1-\gamma)}\left(\sum_{i=1}^{\infty}t_ia_{i,n}\right)+\sum_{n=1}^{\infty}\frac{(n+\gamma-\gamma\lambda(n+1))\Gamma(\alpha_1,n)}{(1-\gamma)}\left(\sum_{i=1}^{\infty}t_ib_{i,n}\right)\\
&=\sum_{i=1}^{\infty}t_i\left(\sum_{n=2}^{\infty}\frac{(n-\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n)}{(1-\gamma)}a_{i,n}+\sum_{n=1}^{\infty}\frac{(n+\gamma-\gamma\lambda(n+1))\Gamma(\alpha_1,n)}{(1-\gamma)}b_{i,n}\right)\\
&\leq\sum_{i=1}^{\infty}t_i=1,
\end{aligned}$$

and therefore $\sum_{i=1}^{\infty}t_if_i\in\mathcal{R}_{\mathcal{H}}^{p,q}([\alpha_1],\lambda,\gamma)$. □

Theorem 4.2. *For* $0 \le \beta \le \gamma < 1$, *let* $f(z) \in \mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\gamma)$ *and* $F(z) \in \mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\delta)$. *Then* $f(z) * F(z) \in \mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\gamma) \subset \mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\delta)$.

Proof. Let $f(z) = z - \sum\limits_{n=2}^{\infty} a_n z^n + \sum\limits_{n=1}^{\infty} \overline{b}_n \overline{z}^n \in \mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\gamma)$ and $F(z) = z - \sum\limits_{n=2}^{\infty} A_n z^n + \sum\limits_{n=1}^{\infty} \overline{B}_n \overline{z}^n \in \mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\delta)$. Then $f(z) * F(z)$ is $f(z) * F(z) = z - \sum\limits_{n=2}^{\infty} a_n A_n z^n + \sum\limits_{n=1}^{\infty} \overline{b}_n \overline{B}_n \overline{z}^n$.

For $f(z) * F(z) \in \mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\delta)$ we note that $|A_n| \le 1$ and $|B_n| \le 1$. Now by Theorem 2.2, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{(n-\delta-\delta\lambda(n-1))\Gamma(\alpha_1,n)}{1-\delta}|a_n|\ |A_n| + \sum_{n=1}^{\infty} \frac{(n+\delta-\delta\lambda(n+1))\Gamma(\alpha_1,n)}{1-\delta}|b_n|\ |B_n| \\
&\le \sum_{n=2}^{\infty} \frac{(n-\gamma-\gamma\lambda(n-1))\Gamma(\alpha_1,n)}{1-\gamma}|a_n| + \sum_{n=1}^{\infty} \frac{[n+\gamma-\gamma\lambda(n+1))\Gamma(\alpha_1,n)}{1-\gamma}|b_n| \le 1,
\end{aligned}$$

by Theorem 2.2, $f(z) \in \mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\gamma)$. Therefore $f(z)*F(z) \in \mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\gamma) \subset \mathcal{R}^{p,q}_{\mathcal{H}}([\alpha_1],\lambda,\delta)$. □

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