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ASYMPTOTIC BEHAVIOR FOR A CLASS OF DELAY DIFFERENTIAL EQUATIONS WITH A FORCING TERM*

YUJI LIU

Abstract. We study the asymptotic behavior of solutions of the following forced delay differential equation

$$x'(t) = -p(t)f(x(t-\tau)) + r(t), \quad t \geq 0. \tag{$*$}$$

It is show that if f is increasing and $|f(x)| \leq |x|$ for all $x \in R$, $\lim_{t\to+\infty} \frac{r(t)}{p(t)} = 0$, $\int_0^{+\infty} p(s)ds = +\infty$ and $\limsup_{t\to+\infty} \int_{t-\tau}^{t} p(s)ds < \frac{3}{2}$ for sufficiently large t, then every solution of the Eq.($*$) tends to zero as t tends to infinity. Our result improves the recent results obtained by Graef and Qian.

1. Introduction

In this paper, we study the asymptotic behavior of solutions of the forced delay differential equation

$$x'(t) = -p(t)f(x(t-\tau)) + r(t), \quad t \geq 0, \tag{1}$$

where $p \in C([0,+\infty),(0,+\infty))$, $r \in C([0,+\infty),R)$, $\tau > 0$, $f : R \to R$ is increasing. We suppose

$$\lim_{x\to 0} \frac{f(x)}{x} = b \in (0,\infty), \tag{2}$$

and

$$|f(x)| \leq |x|, \quad x \in R. \tag{3}$$

Obviously, the equation

$$x'(t) = -p(t)x(t-\tau) + r(t), \quad t \geq 0, \tag{4}$$

studied in [1, 5] is a special case of Eq.(1). Although the more general case

$$x'(t) + \sum_{i=1}^{n} q_j(t)f(x(t-\tau_j)) = r(t), \tag{5}$$

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was studied in [1], the results and their proofs are different. The following theorem is our main result.

Theorem 1. *Suppose that* (2), (3) *hold, and*

$$\int_0^{+\infty} p(s)ds = +\infty, \tag{6}$$

$$\mu = \limsup_{t\to+\infty} \int_{t-\tau}^{t} p(s)ds < \frac{3}{2}, \tag{7}$$

and

$$\lim_{t\to+\infty} \frac{r(t)}{p(t)} = 0. \tag{8}$$

Then every solution of Eq(1) *tends to zero as* $t \to +\infty$*.*

Example 1. Consider the equation

$$x'(t) = -\frac{1}{4}\Big(\frac{6}{5} + \cos t\Big)x(t-\pi) + \frac{1}{1+t}.$$

Since $p(t) = \frac{1}{4}(\frac{6}{5} + \cos t)$, we get

$$\limsup_{t\to+\infty} \int_{t-\pi}^{t} p(s)ds = \limsup_{t\to\infty} \frac{1}{4}\Big(\frac{6}{5}\pi + 2\sin t\Big) = \frac{2 + \frac{6}{5}\pi}{4} < \frac{3}{2}.$$

It is easy to see that (6) and (8) are valid. Then by Theorem 1, every solution of the equation tends to zero as t tends to infinity.

Remark. In Theorem 1 [5], it was proved that if (6), (8) hold, and

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} p(s)ds < 1, \qquad \int_0^{\infty} r(s)ds \ \text{ converges},$$

then every solution of Eq.(4) tends to zero. Theorem 1 improves the result in [5].

In Theorem 2 [5], it is proved that if (6), (8) hold, and

$$\lim_{t\to+\infty} \int_{t-\tau}^{t} p(s)ds = \beta < \frac{\pi}{2},$$

then every solution of Eq. (4) tends to zero. The equation in Example 1 does not satisfy the above conditions, owing to

$$\int_{t-\tau}^{t} p(s)ds = \frac{1}{4}\Big(\frac{6}{5}\pi + 2\sin t\Big)$$

being not asymptotically constant.

2. Some Lemmas

Clearly, conditions (2) and (8) imply that there exist $\alpha > 0$ such that

$$\frac{f(x)}{x} > \frac{b}{2} \quad \text{for} \quad |x| < \alpha,$$

and for any $\epsilon \in (0, \alpha)$, there is $T > 0$ such that

$$\Big|\frac{r(t)}{p(t)}\Big| < \frac{b\epsilon}{2}, \quad t > T. \tag{9}$$

In order to prove Theorem 1, we need the following lemmas.

Lemma 1. *Suppose that* (2), (8) *hold, $x(t)$ is an oscillatory solution of Eq.*(1) *and $A > 0$, $\delta > 1$ such that $x(t)$ satisfies that*

$$x'(t) \le Ap(t) + r(t), \quad t \ge T. \tag{10}$$

$$x'(t) \le -p(t)x(t-\tau) + r(t), \quad \textit{if} \;\; x(t-\tau) \le 0, \quad \textit{and} \quad t \ge T + \tau. \tag{11}$$

$$\int_{t-\tau}^{t} p(s)ds \le \delta, \quad \textit{for all} \quad t \ge T + \tau. \tag{12}$$

If $c > T + 2\tau$, $x'(c) \ge 0$, then we have

$$x(c) \le \Big(\delta - \frac{1}{2}\Big)A + \epsilon\Big(b\delta + \frac{b\delta^2}{2} + 1\Big). \tag{13}$$

Proof. By (2), (8), we know (9) holds. Since $x'(c) \ge 0$, we claim that $x(c-\tau) \le \epsilon$. In fact if $x(c-\tau) > \epsilon$, then by (1), noting that $f(x)$ is increasing, we get

$$0 \le x'(c) = p(c)\Big(-f(x(c-\tau)) + \frac{r(c)}{p(c)}\Big) < \epsilon p(c)\Big(-\frac{f(\epsilon)}{\epsilon} + \frac{b}{2}\Big) < \epsilon p(c)\Big(-\frac{b}{2} + \frac{b}{2}\Big) = 0.$$

This is impossible. Now we consider two cases.

Case 1. If $0 < x(c-\tau) \le \epsilon$. For $t \in [c-\tau, c]$, we have $t - \tau \le c - \tau$. Integrating (10) from $t - \tau$ to $c - \tau$, we get

$$\begin{aligned}
-x(t-\tau) &\le -x(c-\tau) + A\int_{t-\tau}^{c-\tau} p(s)ds + \int_{t-\tau}^{c-\tau} r(s)ds \\
&\le A\int_{t-\tau}^{c-\tau} p(s)ds + \frac{b\epsilon}{2}\int_{t-\tau}^{c-\tau} p(s)ds \\
&\le A\int_{t-\tau}^{c-\tau} p(s)ds + \frac{b\epsilon}{2}\delta.
\end{aligned}$$

If $x(t-\tau)\le 0$, then by (11) we get

$$x'(t)\le Ap(t)\int_{t-\tau}^{c-\tau}p(s)ds+p(t)\delta\frac{b\epsilon}{2}+r(t),\quad t\in[c-\tau,c]. \tag{14}$$

If $x(t-\tau)>0$, then (1) implies $x'(t)\le r(t)$, and hence (14) is also valid.

Subcase 1.1. $\int_{c-\tau}^{c}p(s)ds\le 1$. Integrating (14) from $c-\tau$ to c, and applying (9), (12) we get

$$\begin{aligned}
x(c)&\le x(c-\tau)+A\int_{c-\tau}^{c}p(t)\int_{t-\tau}^{c-\tau}p(s)dsdt+\frac{b\epsilon}{2}\delta\int_{c-\tau}^{c}p(s)ds+\int_{c-\tau}^{c}r(t)dt\\
&\le \epsilon\Big(\frac{b\delta}{2}+\frac{b\delta^2}{2}+1\Big)+A\int_{c-\tau}^{c}p(t)\Big(\delta-\int_{c-\tau}^{t}p(s)ds\Big)dt\\
&=\epsilon\Big(\frac{b\delta}{2}+\frac{b\delta^2}{2}+1\Big)+A\delta\int_{c-\tau}^{c}p(t)dt-A\int_{c-\tau}^{t}p(t)\int_{c-\tau}^{t}p(s)dsdt\\
&=\epsilon\Big(\frac{b\delta}{2}+\frac{b\delta^2}{2}+1\Big)+A\delta\int_{c-\tau}^{c}p(t)dt-\frac{1}{2}A\Big(\int_{c-\tau}^{c}p(s)ds\Big)^2.
\end{aligned}$$

Since $\delta x-\frac{1}{2}x^2$ is increasing for $0\le x\le 1<\delta$, then

$$x(c)\le\epsilon\Big(\frac{b\delta}{2}+\frac{b\delta^2}{2}+1\Big)+\Big(\delta-\frac{1}{2}\Big)A.$$

Subcase 1.2. $\int_{c-\tau}^{c}p(s)ds>1$. Choosing $\eta\in(c-\tau,c)$ such that $\int_{\eta}^{c}p(s)ds=1$, we get in applying (10), (14), (9), (12), that

$$\begin{aligned}
x(c)&=x(c-\tau)+\int_{c-\tau}^{\eta}x'(s)ds+\int_{\eta}^{c}x'(s)ds\\
&\le x(c-\tau)+\int_{c-\tau}^{\eta}(Ap(s)+r(s))ds+\int_{\eta}^{c}\Big[Ap(t)\int_{t-\tau}^{c-\tau}p(s)ds+\delta\frac{b\epsilon}{2}p(t)+r(t)\Big]dt\\
&\le\epsilon\Big(1+b\delta+\frac{b\delta^2}{2}\Big)+A\int_{c-\tau}^{\eta}p(t)dt+A\int_{\eta}^{c}p(t)\int_{t-\tau}^{c-\tau}p(s)dsdt\\
&=\epsilon\Big(1+b\delta+\frac{b\delta^2}{2}\Big)+A\int_{c-\tau}^{\eta}p(t)dt+A\int_{\eta}^{c}p(t)\Big[\int_{t-\tau}^{t}p(s)ds-\int_{c-\tau}^{t}p(s)ds\Big]dt\\
&\le\epsilon\Big(1+b\delta+\frac{b\delta^2}{2}\Big)+A\int_{c-\tau}^{\eta}p(t)dt+A\int_{\eta}^{c}p(t)\Big[\delta-\int_{c-\tau}^{t}p(s)ds\Big]dt\\
&=\epsilon\Big(1+b\delta+\frac{b\delta^2}{2}\Big)+A\int_{c-\tau}^{\eta}p(t)dt+A\delta\int_{\eta}^{c}p(t)dt-A\int_{\eta}^{c}p(t)\int_{c-\tau}^{t}p(s)dsdt\\
&=\epsilon\Big(1+b\delta+\frac{b\delta^2}{2}\Big)+A\int_{c-\tau}^{\eta}p(t)dt+A\delta\int_{\eta}^{c}p(s)ds-\frac{1}{2}A\Big\{\Big(\int_{c-\tau}^{c}p(t)dt\Big)^2-\Big(\int_{c-\tau}^{\eta}p(s)ds\Big)^2\Big\}\\
&=\epsilon\Big(1+b\delta+\frac{b\delta^2}{2}\Big)+\Big(\delta-\frac{1}{2}\Big)A.
\end{aligned}$$

Case 2. $x(c-\tau) \le 0$. There exists $\xi \in (c-\tau, c]$ such that $x(\xi) = 0$. Then for $t \in [\xi, c]$, we have $t - \tau \le \xi$. By use of (9), we get from (10)

$$-x(t-\tau) \le A\int_{t-\tau}^{\xi} p(s)ds + \frac{b\epsilon}{2}\epsilon, \tag{15}$$

If $x(t-\tau) \le 0$, we get based on (11)

$$x'(t) \le Ap(t)\int_{t-\tau}^{\xi} p(s)ds + \frac{b\delta}{2}\epsilon p(t) + r(t), \quad t \in [\xi, c]. \tag{16}$$

If $x(t-\tau) > 0$, then by (1), we have $x'(t) \le r(t)$, and (16) is also valid. By the method of that in the proof of subcase 1.1 and 1.2, we get

$$x(c) \le \epsilon\Big(\frac{b\delta}{2} + \frac{b\delta^2}{2} + 1\Big) + \Big(\delta - \frac{1}{2}\Big)A. \tag{17}$$

or

$$x(c) \le \epsilon\Big(1 + b\delta + \frac{b\delta^2}{2} + 1\Big) + \Big(\delta - \frac{1}{2}\Big)A. \tag{18}$$

This completes the proof.

Lemma 2. *Suppose that* (2), (8) *hold.* $x(t)$ *is a solution of Eq.*(1), $B < 0$, *such that*

$$x'(t) \ge Bp(t) + r(t), \quad r \ge T,$$

$$x'(t) \ge -p(t)x(t-\tau) + r(t), \quad \text{if } x(t-\tau) \ge 0, \quad \text{and} \quad t \ge T + r,$$

(11) *holds,* $x(c) > 0$ *and* $x'(c) \le 0$, *then we have that*

$$x(c) \ge \Big(\delta - \frac{1}{2}\Big)B - \epsilon\Big(1 + b\delta + \frac{b\delta^2}{2}\Big).$$

Proof. We omit the proof since it is similar to that of Lemma 1.

Lemma 3. *Suppose that* $x(t)$ *is an eventually non-negative solution of Eq.*(1), *and* (2), (6), (8) *hold. Then* $x(t)$ *tends to zero as* t *tends to infinity.*

Proof. Let $\limsup_{t\to+\infty} x(t) = v$. If $v = 0$, then the proof is complete. If $v > 0$, we have two cases to consider.

Case 1. If $x'(t)$ is eventually negative, then there is $T_1 > T + \tau$ such that $x(t)$ is decreasing for $t \ge T_1$. The assumption $\limsup_{t\to+\infty} x(t) = v$ implies $x(t-\tau) \ge v$ for all $t \ge T_1$. By (1), we have

$$x'(t) \le -p(t)f(v) + r(t), \quad t \ge T_1. \tag{19}$$

Integrating (19) from T_1 to t, we get

$$x(t) - x(T_1) \le -f(v)\int_{T_1}^{t} p(s)ds + \int_{T_1}^{t} r(s)ds.$$

Since $v > 0$, we get $f(v) > 0$. Choosing $\epsilon \in (0, f(v))$, (8) implies there is $T_2 > T_1$ such that $|r(t)| \le \epsilon p(t)$ for $t \ge T_2$. Hence

$$x(t) - x(T_1) \le (-f(v) + \epsilon)\int_{T_2}^{t} p(s)ds - f(v)\int_{T_1}^{T_2} p(s)ds + \int_{T_1}^{T_2} r(s)ds. \tag{20}$$

Let $t \to +\infty$, by (20), we get $v - x(T_1) \le -\infty$, a contradiction. Therefore $v = 0$.

Case 2. Suppose $x'(t)$ is not eventually negative. Choosing $T_1 > T$ such that $x(t-\tau) \ge 0$ for $t \ge T_1$, we get

$$x'(t) \le r(t), \quad t \ge T_1. \tag{21}$$

Supoose that $t^* > T_1 + \tau$ is any left maximum point of $x(t)$, then we have $x'(t^*) \ge 0$. From now on, we prove that $x(t^* - \tau) \le \epsilon$. Otherwise, we have $x(t^* - \tau) > \epsilon$, using $|r(t)| \le \frac{b\epsilon}{2} p(t)$ and (9), we have

$$\begin{aligned} 0 \le x'(t^*) &= -p(t^*)f(x(t^* - \tau)) + r(t^*) \\ &< p(t^*)\Big(- f(x(t^* - \tau)) + \frac{b\epsilon}{2}\Big) \\ &= \epsilon p(t^*)\Big(- \frac{f(\epsilon)}{\epsilon} + \frac{b}{2}\Big) \\ &< \epsilon p(t^*)\Big(- \frac{b}{2} + \frac{b}{2}\Big) = 0, \end{aligned}$$

a contradiction. Integrating (21) from $t^* - \tau$ to t^*, by (9), (12), we get

$$x(t^*) \le x(t^* - \tau) + \int_{t^*-\tau}^{t^*} r(t)dt \le \frac{b\delta}{2}\epsilon + \epsilon.$$

This shows that $x(t)$ is bounded above and then $v < +\infty$. Choosing $\{t_n\}$ such that $T_2 + \tau < t_1 < t_2 < \cdots$, $\lim_{n\to+\infty} t_n = +\infty$, $x'(t_n) \ge 0$, $\lim_{n\to+\infty} x(t_n) = v$, we get $x(t_n - \tau) \le \epsilon$. By a similar method in case 2, $f(x(t-\tau)) > 0$ implies $x'(t) \le r(t)$. Integrating this inequality from $t_n - \tau$ to t_n, we get

$$x(t_n) \le x(t_n - \tau) + \int_{t_n-\tau}^{t_n} r(t)dt \le \epsilon\Big(1 + \frac{b\delta}{2}\Big).$$

Let $n \to +\infty$, $\epsilon \to 0$, we have $v = 0$. This completes the proof.

Lemma 4. *Suppose that $x(t)$ is any eventually non-positive solution of Eq.(1), and (2), (6), (8) hold. Then $x(t)$ tends to zero.*

The proof is similar to that of Lemma 3 and then omitted.

3. Proof of the Theorem

Proof of Theorem 1. By (2), (7), (8), we choose $\alpha > 0$, such that $\frac{f(x)}{x} > \frac{b}{2}$ for $|x| < \alpha$. For any $\epsilon \in (0, \alpha)$, $\epsilon < 1$, we choose $T > 0$, such that (9) holds and

$$\int_{t-\tau}^{t} p(s)ds \leq \mu + \epsilon = a.$$

By Lemma 3, 4, we need to prove that every oscillatory solution $x(t)$ of Eq.(1) tends to zero. First we prove that $x(t)$ is bounded, to the contrary, there is $t^* > T + \tau$ such that $|x(t)| < |x(t^*)|$ for $t < t^*$. Without loss of generality, we suppose that $x(t^*) > \frac{1}{\epsilon}$. Then we get

$$x'(t) \leq p(t)f(x(t^*)) + r(t) \quad \text{for} \quad t \leq t^*. \tag{22}$$

Then by Lemma 1 and (22), we get

$$x(t^*) \leq \Big(a - \frac{1}{2}\Big)f(x(t^*)) + \epsilon\Big(ab + \frac{ba^2}{2} + 1\Big) \leq \Big(a - \frac{1}{2}\Big)f(x(t^*)) + \epsilon M, \tag{23}$$

where $M = b(1+\mu) + \frac{b(1+\mu)^2}{2} + 1$, (since $\epsilon < 1$, then $a = \delta = \mu + \epsilon < 1 + \mu$). By $\mu < \frac{3}{2}$, without loss of generality, we suppose that $\epsilon < \frac{\frac{3}{2} - \mu}{1+M}$, thus (23) implies that

$$1 < \Big(\mu + \epsilon - \frac{1}{2}\Big) + \epsilon M = \mu - \frac{1}{2} + \epsilon M + \epsilon.$$

This is impossible. Then $x(t)$ is bounded. Now we suppose that $\limsup_{t\to+\infty} x(t) = v$, $\liminf_{t\to+\infty} x(t) = u$, then $-\infty < \mu \leq 0 \leq v < +\infty$. Then there is $T_1 > T$ such that $u_1 = u - \epsilon < x(t-\tau) < v + \epsilon = v_1$ for $t > T_1$. Thus by (1) we get

$$x'(t) \leq -p(t)f(u_1) + r(t), \quad t \geq T_1. \tag{24}$$

$$x'(t) \geq -p(t)f(v_1) + r(t), \quad t \geq T_1. \tag{25}$$

We choose $\{s_n\}$, $\{t_n\}$ such that

$$T_1 + \tau < s_1 < s_2 < \cdots, \quad s_n \to +\infty, \quad x'(s_n) \geq 0, \quad x(s_n) \to v \;\text{ as }\; n \to +\infty.$$
$$T_1 + \tau < t_1 < t_2 < \cdots, \quad t_n \to +\infty, \quad x'(t_n) \leq 0, \quad x(t_n) \to u \;\text{ as }\; n \to +\infty.$$

If $x(t-\tau) \leq 0$, by $|f(x)| \leq |x|$ and (1) we get

$$x'(t) \leq -p(t)x(t-\tau) + r(t), \tag{26}$$

By Lemma 1, we get

$$x(s_n) \leq \epsilon\Big(b(\mu+\epsilon) + \frac{b(\mu+\epsilon)^2}{2} + 1\Big) - \Big(\mu - \frac{1}{2} + \epsilon\Big)f(u_1), \quad n = 1, 2, \ldots,$$

Let $n \to +\infty$, $\epsilon \to 0$, we get $v \leq -(u - \frac{1}{2})f(u)$. Similarly, we get

$$x(t_n) \geq -\Big(u - \frac{1}{2} + \epsilon\Big) f(v_1) - \epsilon\Big(b(\mu + \epsilon) + \frac{b(\mu+\epsilon)^2}{2} + 1\Big),$$

then $u \geq -(\mu - \frac{1}{2})f(v)$. Since $\mu < \frac{3}{2}$, if $v \neq 0$, then $v > 0$. Hence $v < -f(u) \leq u \leq (\mu - \frac{1}{2})f(v) < f(v) \leq v$, which is impossible. We have $u = v = 0$. The proof is complete.

References

[1] J. Y. Yan, *Asymptotic behavior of solutions of forced nonlinear delay differential equations*, Publications Mathematical Debrecen **45**(1994), 283-291.

[2] G. Ladas and Y. G. Sficas, *Asymptotic behavior of oscillatory solutions*, Hiroshima Math. J. **18**(1988), 351-359.

[3] K. Gapalsamy, Stability and Oscillations in Delay Differential Equations of Populction Dynamics, Kluwer Academic Publisher, Boston, 1992.

[4] J. S. Yu, *Global attractivity of zero solution of a closs of functional differential equation and its applications*, Science in China (Series A) **39**(1996), 225-23.

[5] J. R. Graef and C. Qian, *Global attractivity in differential equations with variable delays*, J. of Austral. Math. Soc. Ser. B **41**(2000), 568-579.

[6] T. Y. Oneyama, *On the $\frac{3}{2}$-stability theorem for one dimensional delay differential equations*, J. Math. Anal. Appl. **125**(1987), 161-173.

[7] H. Matsunaga, R. Miyaziki and T. Hara, *Global attractivity for delay differential equations*, J. Math. Anal. Appl. **234**(1999), 77-90.

Department of Mathematics, Hunan Institute of Technology, Hunan 414000, P.R.China.