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# CHARACTERIZATION OF A CLASS OF GRAPHS WITH UNIQUE MINIMUM GRAPHOIDAL COVER

S. ARUMUGAM, INDRA RAJASINGH AND P. ROUSHINI LEELY PUSHPAM

**Abstract**. A *graphoidal cover* of a graph $G$ is a collection $\psi$ of (not necessarily open) paths in $G$ such that every vertex of $G$ is an internal vertex of at most one path in $\psi$ and every edge of $G$ is in exactly one path in $\psi$. The minimum cardinality of a graphoidal cover of $G$ is called the *graphoidal covering number* of $G$ and is denoted by $\eta$. Two graphoidal covers $\psi_1$ and $\psi_2$ of a graph $G$ are said to be *isomorphic* if there exists an automorphism $f$ of $G$ such that $\psi_2 = \{f(P)/P \in \psi_1\}$. A graph $G$ is said to have a *unique minimum graphoidal cover* if any two minimum graphoidal covers of $G$ are isomorphic. In this paper we characterize the class of all graphs $G$ with a unique minimum graphoidal cover when $\delta = 2$ and no end block of $G$ is a cycle.

## 1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected, connected graph without loops. The order and size of $G$ are denoted by $p$ and $q$ respectively. For graph theoretic terminology we refer to Harary [6].

If $P = (v_0, v_1, \ldots, v_n)$ is a path or a cycle in $G$, $v_1, v_2, \ldots, v_{n-1}$, are called *internal vertices of $P$*. If $P = (v_0, v_1, \ldots, v_n)$ and $Q = (v_n = w_0, w_1, \ldots, w_m)$ are two paths in $G$ then the walk obtained by concatenating $P$ and $Q$ at $v_n$ is denoted by $P \circ Q$ and the path $(v_n, v_{n-1}, \ldots, v_1, v_0)$ is denoted by $P^{-1}$. For any subset $V_1$ of $V$ the subgraph of $G$ induced by $V_1$ is denoted by $\langle V_1 \rangle$.

The concept of graphoidal cover was introduced by Acharya and Sampathkumar [1].

**Definition 1.1.** A *graphoidal cover* of a graph $G$ is a collection $\psi$ of (not necessarily open) paths in $G$ satisfying the following conditions.
(i) Every path in $\psi$ has at least two vertices.
(ii) Every vertex of $G$ is an internal vertex of at most one path in $\psi$.
(iii) Every edge of $G$ is in exactly one path in $\psi$.

The minimum cardinality of a graphoidal cover of $G$ is called the *graphoidal covering number* of $G$ and is denoted by $\eta$.

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Arumugam and Pakkiam introduced the concept of isomorphism between graphoidal covers of a graph. [5]

**Definition 1.2.** Two graphoidal covers $\psi_1$ and $\psi_2$ of a graph $G$ are said to be *isomorphic* if there exists an automorphism $f$ of $G$ such that $\psi_2 = \{f(P)/P \in \psi_1\}$. A graph $G$ is said to have a *unique minimum graphoidal cover* if any two minimum graphoidal covers of $G$ are isomorphic.

An elaborate review of results in graphoidal covers with several interesting applications and a large collection of unsolved problems is given in [2].

**Definition 1.3.** Let $\psi$ be a collection of internally disjoint paths in $G$. A vertex of $G$ is said to be an *interior vertex* of $\psi$ if it is an internal vertex of some path in $\psi$. Any vertex which is not an interior vertex of $\psi$ is said to be an *exterior vertex* of $\psi$.

**Theorem 1.4.**([8]) *For any graphoidal cover $\psi$ of $G$, let $t_\psi$ denote the number of exterior vertices of $\psi$. Let $t = \min t_\psi$, where the minimum is taken over all graphoidal covers of $G$. Then $\eta = q - p + t$.*

**Corollary 1.5.**([8]) *For any graph $G$, $\eta \geq q - p$. Moreover the following are equivalent.*
(i) $\eta = q - p$
(ii) *There exists a graphoidal cover without exterior vertices*
(iii) *There exists a set of internally disjoint and edge disjoint paths without exterior vertices.*

**Corollary 1.6.**([8]) *Let $G$ be a graph with $\eta = q - p$. Then the minimum vertex degree $\delta$ in $G$ is at least two.*

**Corollary 1.7.**([8]) *Let $G$ be a graph with $\eta = q - p$. Then the maximum vertex degree $\triangle$ in $G$ is at least three.*

**Theorem 1.8.**([7]) *For any graph $G$ with $\delta \geq 3$, $\eta = q - p$.*

Let $\mathcal{F}$ denote the class of all connected graphs with $\eta = q - p$.

**Theorem 1.9.**([3]) *Let $G$ be a 2-edge connected graph. Then $G \notin \mathcal{F}$ if and only if every block of $G$ is a cycle or a cycle with exactly one chord or a theta graph and at most one block of $G$ is not a cycle.*

**Theorem 1.10.**([4]) *Let $G$ be a connected graph with $\delta = 2$ and $\kappa' = 1$. Then $G \notin \mathcal{F}$ if and only if there exists a cut edge $e$ of $G$ such that at least one component of $G - e$ is a graph, all of whose blocks are cycles.*

In [5], it has been proved that if a graph $G$ has a unique minimum graphoidal cover, then $\delta \leq 3$ and when $\delta = 3$, $G$ has a unique minimum graphoidal cover if and only if $G = K_4$. Trees and unicyclic graphs having a unique minimum graphoidal cover have also been characterized [5]. In this paper we characterize the class of graphs with a unique minimum graphoidal cover when $\delta = 2$, and no end block of $G$ is a cycle.

## 2. Main Results

**Theorem 2.1.** *Let $G$ be a graph in which $\delta = 2$ and no end block of $G$ is a cycle. Let $G'$ be the graph (allowing multiple edges) obtained from $G$ by contracting all vertices of degree 2. If $G$ has a unique minimum graphoidal cover then $G'$ also has a unique minimum graphoidal cover.*

**Proof.** Suppose $G$ has a unique minimum graphoidal cover. If $\eta > q - p$, since no end block of $G$ is a cycle, it follows from Theorem 1.9 and Theorem 1.10 that $G$ is either a cycle with exactly one chord or a $\theta$-graph. In either of the cases there exist two nonisomorphic minimum graphoidal covers of $G$, which is a contradiction. Hence $\eta = q - p$. Now let $\psi$ be a minimum graphoidal cover of $G$ so that each vertex of $G$ is interior to $\psi$. As we contract the vertices of degree 2 in $G$ we perform the same sequence of opertaions on the paths in $\psi$. Let $\psi'$ be the resulting collection. Then clearly $\psi'$ is a minimum graphoidal cover of $G'$. Now let $\psi_1$ and $\psi_2$ be two minimum graphoidal covers of $G'$. Then there exist two minimum graphoidal covers $\psi_3$ and $\psi_4$ of $G$ such that $\psi_3' = \psi_1$ and $\psi_4' = \psi_2$. Since $G$ has a unique minimum graphoidal cover, there exists an automorphism $\alpha$ of $G$ such that $\psi_4 = \{\alpha(P)/P \in \psi_3\}$. Let $\alpha_{|V(G')} = \alpha'$. Clearly, $\alpha'$ is an automorphism of $G'$ and $\psi_2 = \{\alpha'(P)/P \in \psi_1\}$, so that $G'$ has a unique minimum graphoidal cover.

We observe that $G'$ is a multigraph with $\delta \geq 3$. We first characterize multigraphs with $\delta \geq 3$ having a unique minimum graphoidal cover.

**Theorem 2.2.** *Let $G$ be a multigraph with $\delta \geq 3$. Then $G$ has a unique minimum graphoidal cover if and only if $G$ is isomorphic to one of the following graphs, $G_i$, $1 \leq i \leq 12$ given in Figure 1.*

**Proof.** It can be easily verified that each of the graphs $G_i$, $1 \leq i \leq 12$, has a unique minimum graphoidal cover. Conversely suppose $G$ is a multigraph with $\delta \geq 3$ having a unique minimum graphoidal cover. Let $P_1 = (u_1, u_2, \ldots, u_n)$ be a longest path in $G$ so that vertices adjacent to $u_1$ or $u_n$ are already in $P_1$. In the sequel all the graphoidal covers that are exhibited are minimal by Corollay 1.5.

**Case (i).** $u_1$ and $u_n$ are non-adjacent.

Since $\delta \geq 3$ we can find vertices $u_i$, $u_j$ in $P_1$ such that $2 \leq i \leq j \leq n-1$, and $u_i$, $u_j$ are adjacent to $u_1$. Similarly we can find vertices $u_r$, $u_s$ in $P_1$ such that $2 \leq r \leq s \leq n-1$ and $u_r$, $u_s$ are adjacent to $u_n$. Now let $P_2 = (u_i, u_1.u_j)$ and $P_3 = (u_r, u_n, u_s)$. We claim that $j \leq r$. Suppose $j > r$.

If $i \neq j$, let

$$Q_1 = (u_1, u_2, \ldots, u_j),$$

$$Q_2 = (u_i, u_1, u_j, u_{j+1}, \dots, u_n, u_r),$$

and $\quad Q_3 = (u_s, u_n)$.

If $\quad i = j,$ let

$$Q_1 = (u_i, u_1, \dots, u_j),$$
$$Q_2 = (u_1, u_j, u_{j+1}, \dots, u_n), \text{ and}$$
$$Q_3 = P_3$$

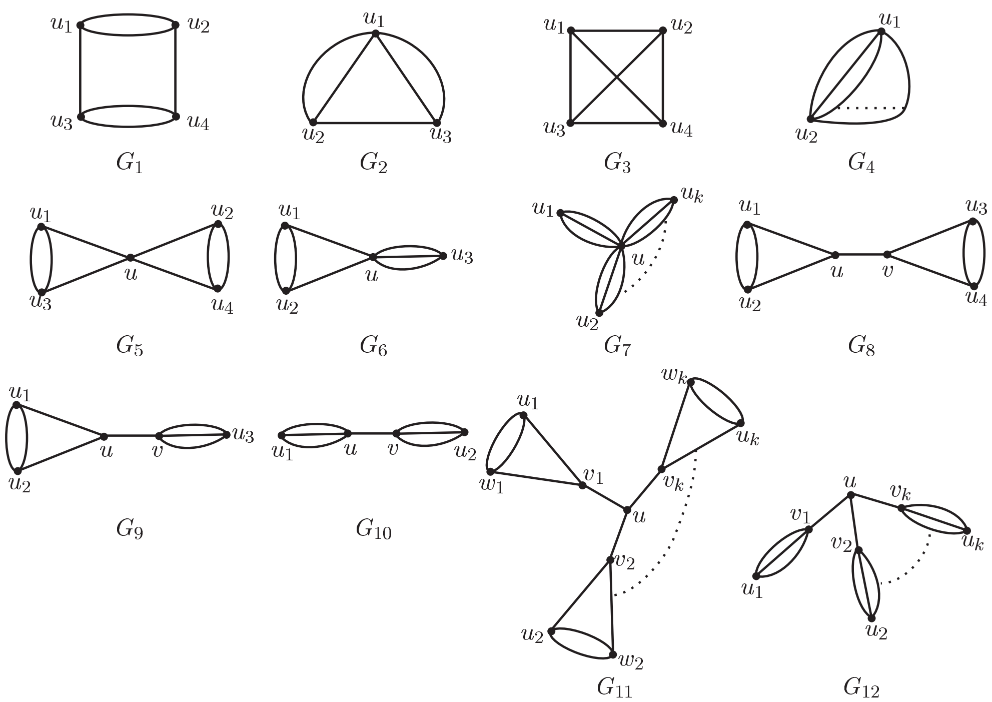


**Figure 1.**

Let $\mathcal{P}$ be a collection of paths in $G$ such that $\psi_1 = \{P_1, P_2, P_3\} \cup \mathcal{P}$ and $\psi_2 = \{Q_1, Q_2, Q_3\} \cup \mathcal{P}$ are minimum graphoidal covers of $G$. Clearly $\psi_1$ and $\psi_2$ give rise to different partitions of $E(G)$ so that $\psi_1$ and $\psi_2$ are non-isomorphic, which is a contradiction. Hence $j \leq r$.

Now let $V(G) = \{u_1, u_2, \dots, u_n\}$, $H = P_1 \cup P_2 \cup P_3$ and $H' = \langle V(P_1) \rangle$. We prove that $H' = H$. If not, there exists an edge $e = u_l u_m \in E(H') \setminus E(H)$ where $1 \leq l < m \leq n$. Since $u_1$ and $u_n$ are non-adjacent $\{l, m\} \neq \{1, n\}$. Suppose $m < n$.

If $m \neq 2$, Let

$$Q_1 = (u_1, u_2, \dots, u_m)$$

and $\quad Q_2 = (u_l, u_m, u_{m+1}, \ldots, u_n)$.

If $\quad m = 2$, let

$$Q_1 = (u_1, u_2, u_1)$$
$$Q_2 = (u_2, u_3, \ldots, u_m)$$

Then $\psi_1 = \{P_1, P_2, P_3, e\}$ and $\psi_2 = \{P_2, P_3, Q_1, Q_2\}$ are two nonisomorphic minimum graphoidal covers of $G$, which is a contradiction. Similarly if $l > 1$ we get a contradiction. Hence $H' = H$. Clearly $3 \le p \le 6$. When $p = 3$, $G$ is isomorphic to $G_7$ with exactly two blocks. When $p = 4$, $G$ is isomorphic to $G_6$ or $G_{10}$; when $p = 5$, $G$ is isomorphic to $G_5$ or $G_9$ and when $p = 6$, $G$ is isomorphic to $G_8$.

If $V(G) \ne \{u_1, u_2, \ldots, u_n\}$, let $w$ be a vertex not in $P_1$. Let $P_4 = (w_1, w_2, \ldots, w, \ldots, w_d)$ be a longest path in $G$ containing $w$ and internally disjoint with the paths $P_1$, $P_2$, $P_3$ and with one of its ends say $w_d$, in $P_1$. Then $w_d = u_k$ for some $k$, $1 < k < n$. Let $R_1$ and $R_2$ be the $(u_1, u_k)$ and $(u_k, u_n)$-sections of $P_1$ [Refer Figure 2]. Let $l_1$, $l_2$ and $l_3$ be the lengths of the paths $R_1$, $R_2$ and $P_4$ respectively.

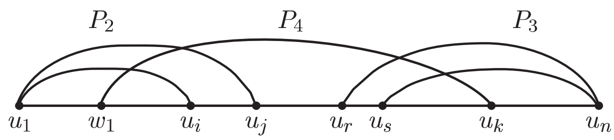


**Figure 2.**

If $w_1 \in V(P_1)$, then there exists a collection of paths $\mathcal{P}$ in $G$ such that $\psi_1 = \{P_1, P_2, P_3, P_4\} \cup \mathcal{P}$ and $\psi_2 = \{P_2, P_3, R_1, P_4 \circ R_2\} \cup \mathcal{P}$ are minimum graphoidal covers of $G$ and $\psi_1$ and $\psi_2$ are nonisomorphic, which is a contradiction. Hence $w_1 \notin V(P_1)$. Since $\delta \ge 3$, we can find vertices $w_a$, $w_b$ in $P_4$ such that $1 \le a \le b \le d$ and $w_a$, $w_b$ are adjacent to $w_1$. Let $P_5 = (w_a, w_1, w_b)$. Now we prove that $l_1 = l_2 = l_3$. There exists a collection $\mathcal{P}$ of paths in $G$ such that

$$\psi = \{P_1, P_2, P_3, P_4, P_5\} \cup \mathcal{P}$$
$$\psi_1 = \{P_2, P_3, P_5, R_1 \circ P_4^{-1}, R_2\} \cup \mathcal{P}$$

and
$$\psi_2 = \{P_2, P_3, P_5, P_4 \circ R_2, R_1\} \cup \mathcal{P}$$

are minimum graphoidal covers of $G$. The paths $P_1$, $P_4$ of $\psi$ are of lengths $l_1 + l_2$, $l_3$ respectively, the paths $R_1 \circ P_4^{-1}$, $R_2$ of $\psi_1$ are of lengths $l_1 + l_3$, $l_2$ respectively and the paths $P_4 \circ R_2$, $R_1$ of $\psi_2$ are of lengths $l_3 + l_2$, $l_1$ respectively. Since $G$ has a unique minimum graphoidal cover we see that $l_1 = l_2 = l_3$. As before we can prove that $\langle V(P_1 \cup P_4)\rangle = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5$. If $V(G) = V(P_1 \cup P_4)$ then $2 \le n \le 7$. Since $l_1 = l_2$, $n$ is an odd number. If $n = 7$ then $G$ is isomorphic to $G_{11}$. Let $n = 5$. If $i = j$, $r = s$, $a = b$ then $G$ is isomorphic to $G_{12}$. Otherwise either $i \ne j$ or $r \ne s$ or $a \ne b$. Let $a \ne b$, then $w_b = w_d$ (Refer Figure 3).

Let $Q_1 = (w_d, w_{d-1}, \ldots, w_1, w_d)$ and $Q_2 = (w_1, w_2)$. Then $\psi_1 = (P_1, P_2, P_3, Q_1, Q_2\} \cup \mathcal{P}$ and $\psi_2 = \{P_1, P_2, P_3, P_4, P_5\} \cup \mathcal{P}$ are nonisomorphic minimum graphoidal covers of $G$, which is a contradiction. In a similar way we get contradictions when $r \neq s$ and $i \neq j$.

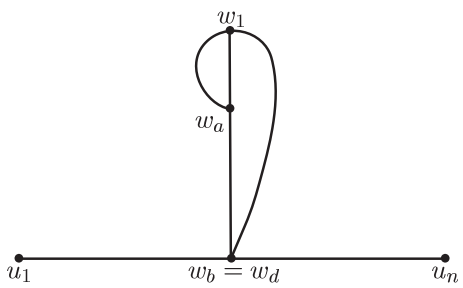


**Figure 3.**

Now if $n = 3$ then $G$ is isomorphic to $G_7$. If $V(G) \neq V(P_1 \cup P_4)$, then repeating the above process we see that $G$ is isomorphic to $G_7$, $G_{11}$ or $G_{12}$.

**Case (ii)** $u_1$ and $u_n$ are adjacent

Let $t$ be the number of edges joining $u_1$ and $u_n$. We prove that $V(G) = \{u_1, u_2, \ldots, u_n\}$. Suppose not, then there exists a vertex $w \notin V(P_1)$ such that $w$ is adjacent to some $u_i$ and hence $G$ contains a path of length greater than $n-1$, which is a contradiction. Hence $V(G) = \{u_1, u_2, \ldots, u_n\}$. Since $\delta \geq 3$ we can find vertices $u_i$, $u_j$ in $P_1$ such that $2 \leq i \leq n-1$ and $2 \leq j \leq n-1$, $u_i$ is adjacent to $u_1$ and $u_j$ is adjacent to $u_n$. Let $P_2 = (u_i, u_1, u_n, u_j)$, $H = P_1 \cup P_2$ and $t = 1$ (Refer Figure 4).

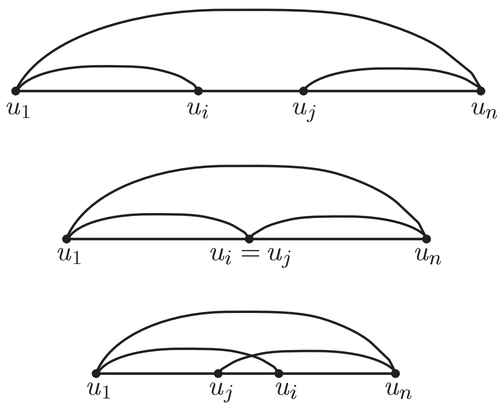


**Figure 4.**

We claim that $G = H$. If not, there exists an edge $e = u_l u_m \in E(G) \setminus E(H)$ where

$1 \le l < m \le n$. Since $t = 1$, $\{l, m\} \neq \{1, n\}$. Supoose $m < n$.

If $m \neq 2$, let

$$Q_1 = (u_1, u_2, \ldots, u_m)$$

and

$$Q_2 = (u_l, u_m, u_{m+1}, \ldots, u_n).$$

If $m = 2$, let

$$Q_1 = (u_1, u_2, u_1)$$

and

$$Q_2 = (u_2, u_3, \ldots, u_n).$$

Let $S$ denote the set of edges not covered by the paths $P_1$ and $P_2$. Then $\psi_1 = \{P_1, P_2\} \cup S$ and $\psi_2 = \{P_2, Q_1, Q_2\} \cup (S \setminus \{e\})$ are nonisomorphic minimum graphoidal covers of $G$, which is a contradiction. Similarly if $l > 1$ we get a contradiction. Hence $G = H$. Therefore when $i < j$, $G$ is isomorphic to $G_1$, when $i = j$, $G$ is isomorphic to $G_2$ and when $i < j$, $G$ is isomorphic to $G_3$.

Suppose $t = 2$, $P_2 = (u_n, u_1, u_n)$ and $H = P_1 \cup P_2$. When $G = H$, $G$ is isomorphic to $G_4$ with $\delta(G_4) = 3$. When $G \neq H$, there exists an edge $e \in E(G) \setminus E(H)$. Let $H_1 = H \cup \{e\}$. Proceeding as before we can prove that $G = H_1$ and $G$ is isomorphic to $G_1$ or $G_2$. Let $t > 2$. We claim that the length of $P_1$ is one. Suppose not, then there exists an edge $e_1 = u_l u_m \in E(G) \setminus E(H)$ where $1 \le l \le m \le n$, $\{l, m\} \neq \{1, n\}$. Since $t > 2$, there exists an edge $e_2 = u_1 u_n$ which does not lie on $P_2$ (Refer Figure 5). Let $S$ be the set of edges not covered by the paths $P_1$, $P_2$, and the edges $e_1$ and $e_2$. Let $\psi_1 = \{(u_1 u_2 \cdots u_n u_1), P_2, e_1\} \cup S$

and
$$\psi_2 = \begin{cases} \{(P_1, (u_1 u_n u_1), (u_m u_1 u_n)\} \cup S & \text{if } 1 = l < m < n \\ \{(P_2, (u_1 u_m u_{m+1} \cdots u_n u_1), (u_1 u_2 \cdots u_m)\} \cup S & \text{if } 1 < l < m < n \end{cases}$$

Then $\psi_1$ and $\psi_2$ are minimum graphoidal covers which are nonisomorphic, which is a contradiction. Hence $G$ is isomorphic to $G_4$ with $\delta(G_4) > 3$.

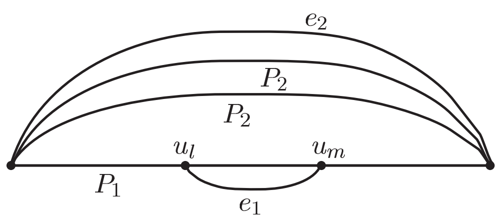


**Figure 5.**

Now we define a series of graphs corresponding to the graphs $G_i$ given in Figure 1 as follows:

For any edge $uv$, let $(uv)^m$ denote the $(u, v)$-path obtained by subdividing $m$ times the edge $uv$.

**Definition 2.3.** Let $H_1$ be the graph obtained from $G_1$ by replacing the multiple edges $u_1u_2$ by $(u_1u_2)^m$, the multiple edges $u_3u_4$ by $(u_3u_4)^n$, the edge $u_3u_1$ by $(u_3u_1)^a$, and the edge $u_2u_4$ by $(u_2u_4)^b$ where $m$, $n \geq 1$ and $a$, $b \geq 0$. Let $H_2$ be the graph obtained from $G_2$ by replacing the multiple edges $u_1u_2$ by $(u_1u_2)^m$, the multiple edges $u_1u_3$ by $(u_1u_3)^n$ and the edge $u_2u_3$ by $(u_2u_3)^a$ where $m$, $n \geq 1$ and $a \geq 0$.

Let $H_3$, $H_4$ be the graphs obtained from $G_3$ and $G_4$, respectively by replacing each edge $uv$ by $(uv)^m$, $m \geq 1$.

Let $H_5$ be the graph obtained from $G_5$ by replacing the multiple edges $u_1u_2$ by $(u_1u_2)^m$, the multiple edges $u_3u_4$ by $(u_3u_4)^n$, the edges $uu_1$, $uu_2$ by $(uu_1)^a$, $(uu_2)^a$ respectively and the edges $uu_3$, $uu_4$ by $(uu_3)^b$, $(uu_4)^b$ respectively, where $m$, $n \geq 1$ and $a$, $b \geq 0$.

Let $H_6$ be the graph obtained from $G_6$ by replacing the multiple edges $u_1u_2$ by $(u_1u_2)^m$, the multiple edges $uu_3$ by $(uu_3)^n$ and the edges $uu_1$, $uu_2$ by $(uu_1)^a$, $(uu_2)^a$ respectively, where $m$, $n \geq 1$ and $a \geq 0$.

When $\Delta(G_7) > 6$, let $H_7$ be the graph obtained from $G_7$ by replacing the multiple edges $uu_1$ by $(uu_1)^m$ and $uu_2$ by $(uu_2)^n$, $m$, $n \geq 1$. When $\Delta(G_7) = 6$ let $H_7$ be the graph obtained from $G_7$ by replacing each edge $uv$ by $(uv)^m$, $m \geq 1$.

Let $H_8$ be the graph obtained from $G_8$ by replacing the multiple edges $u_1u_2$ by $(u_1u_2)^n$, the multiple edges $u_3u_4$ by $(u_3u_4)^m$, the edges $u_1u$, $u_2u$ by $(u_1u)^a$, $(u_2u)^a$ respectively and the edges $vu_3$, $vu_4$ by $(vu_3)^b$, $(vu_4)^b$ respectively and the edges $uv$ by $(uv)^c$ where $m$, $n \geq 1$ and $a$, $b$, $c \geq 0$.

Let $H_9$ be the graph obtained from $G_9$ by replacing the multiple edges $u_1u_2$ by $(u_1u_2)^n$, the multiple edges $vu_3$ by $(vu_3)^m$, the edges $u_1u$, $u_2u$ by $(u_1u)^a$, $(u_2u)^a$ respectively and the edges $uv$ by $(uv)^b$, where $m$, $n \geq 1$ and $a$, $b \geq 0$.

Let $H_{10}$ be the graph obtained from $G_{10}$ by replacing the multiple edges $u_1u$ by $(u_1u)^m$, the multiple edges $u_2u$ by $(u_2u)^n$, and the edge $uv$ by $(uv)^a$ where $m$, $n \geq 1$ and $a \geq 0$.

Let $H_{11}$ be the graph obtained from $G_{11}$ by replacing the multiple edges $u_iw_i$, $1 \leq i \leq k$, by $(u_iw_i)^m$, the edges $uv_i$, $1 \leq i \leq k$, by $(uv_i)^a$, the edges $u_iv_i$, $w_iv_i$, $1 \leq i \leq k$, by $(u_iv_i)^b$, $(w_i, v_i)^b$ respectively, where $m \leq 1$ and $a$, $b \leq 0$.

Let $H_{12}$ be the graph obtained from $G_{12}$ by replacing the multiple edges $u_iv_i$, $1 \leq i \leq k$, by $(u_iv_i)^m$ the edges $uv_i$, $1 \leq i \leq k$, by $(uv_i)^a$ where $m \geq 1$ and $a \geq 0$.

The following theorem gives a characterization of simple graphs $G$ with unique minimum graphoidal cover and with $\delta = 2$ in which no end block of $G$ is a cycle.

**Theorem 2.4.** *Let $G$ be a simple graph with $\delta = 2$ in which no end block is a cycle. Then $G$ has a unique minimum graphoidal cover if and only if $G$ is not a $\theta$-graph and $G$ is isomorphic to one of the graphs $H_i$, $1 \leq i \leq 12$ given in Definition* 2.3.

**Proof.** Suppose $G$ has a unique minimum graphoidal cover. Clearly $G$ is not a $\theta$-graph. Let $G'$ be the graph obtained by contracting all vertices of degree 2 in $G$. By Theorem 2.1 $G'$ has a unique minimum graphoidal cover. By Theorem 2.2 $G'$ is isomorphic to one of the graphs $G_i$, $1 \leq i \leq 12$ given in Figure 1. We claim that if $G'$ is isomorphic to $G_i$, then $G$ is isomorphic to $H_i$.

Let $G'$ be isomorphic to $G_1$. Since any automorphism of $G_1$ maps $u_1$, $u_3$ to $u_2$, $u_4$ or $u_1$, $u_2$ to $u_3$, $u_4$ it follows that the multiple edges $u_1u_2$ must be subdivided the same number of times and the multiple edges $u_3u_4$ must be subdivided the same number of times to obtain $G$. Hence $G$ is isomorphic to $H_1$. The proof is similar if $G'$ is isomorphic to $G_i$, $2 \leq i \leq 12$.

Conversely suppose $G$ is not a $\theta$-graph and $G$ is isomorphic to $H_i$, $1 \leq i \leq 12$. Let $\psi_i$ be a minimum graphoidal cover of $G_i$. $H_i$ is obtained from $G_i$ by subdivding the edges of $G_i$ as given in Definition 2.3. Perform the same sequence of subdivisions on the edges of the paths in $\psi_i$. Let $\psi_i'$ be the resulting collection of paths. Then $\psi_i'$ is a minimum graphoidal cover of $H_i$ and any minimum graphoidal cover of $H_i$ is isomorphic to $\psi_i'$, $1 \leq i \leq 12$. Hence $H_i$, $1 \leq i \leq 12$, has a unique minimum graphoidal cover.

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Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli 627 012, India.

Department of Mathematics, Loyola College, Chennai - 600 034, India.

Department of Mathematics, D. B. Jain College, Chennai - 600 096, India.