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FACTORIZED ENHANCEMENT OF COPSON'S INEQUALITY

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Abstract. This paper dealt with the factorized enhancement of Copson's inequality and improves one of the results given by Leindler.

1. Introduction

Let $\mathbf{x} = \{x_n\}$ be a sequence of real numbers and $p > 1$. Then the following well-known Hardy's inequality

$$\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|x_k|\right)^p \le \left(\frac{p}{p-1}\right)^p \sum_{n=0}^{\infty}|x_n|^p \tag{1}$$

has several generalizations and extensions (see e.g. [1], [4]). Bennett (see [1]) in his notable monograph has given a systematic approach to obtain factorization of several inequalities, which include the Hardy's inequality. The classical Hardy's inequality simply asserts that

$$l_p \subseteq \text{ces}(p), \quad p > 1, \tag{2}$$

where the sequence spaces l_p and $\text{ces}(p)$, which was studied initially by Shiue (see [13]) and are defined as follows:

$$l_p = \left\{x : \sum_{n=1}^{\infty}|x_n|^p < \infty\right\},$$
$$\text{ces}(p) = \left\{x : \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p < \infty\right\}.$$

G. Bennett raised a question that how much room is there between l_p and $\text{ces}(p)$. To answer this question, naturally he consider the multipliers from l_p into $\text{ces}(p)$, that is, those sequences $\mathbf{z}$, with the property that $\mathbf{y}.\mathbf{z} \in \text{ces}(p)$ whenever $\mathbf{y} \in l_p$. The set Z consisting of all multipliers must satisfy the following inclusion:

$$l_p.Z \subseteq \text{ces}(p).$$

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The surprising thing is that the above inclusion becomes an identity if Z is described in a very simple terms and this investigation has been carried out by Bennett [1] himself and obtained the result stated in the following Theorem 1.

Theorem 1. *Let $p > 1$. A sequence $\boldsymbol{x}$ belongs to ces(p) if and only if it admits a factorization*

$$\boldsymbol{x} = \boldsymbol{y}.\boldsymbol{z} \ \ (x_n = y_n.z_n), \tag{3}$$

with

$$\boldsymbol{y} \in l_p \text{ and } \sum_{k=1}^{n} |z_k|^{p^*} = \mathbf{O}(n), \ \ p^* = \frac{p}{p-1}. \tag{4}$$

The theorem may be stated as $\text{ces}(p) = l_p.g(p^*) \ \ (p > 1)$, where

$$g(p) = \Big\{\mathbf{z} : \sum_{k=1}^{n} |z_k|^p = \mathbf{O}(n)\Big\}.$$

From then 'factorization of inequalities' is a new area of research. Leindler (see [8], [9], [10], [11] and references cited therein) studied and obtained results on the factorization of generalized Hardy's inequality.

The inequality (1) is generalized by Copson (see [2], Theorem 1.1; [3], Theorem **A**) and established the following inequality, known as Copson's inequality:

Theorem 2. *Let $1 < p < \infty$, $\boldsymbol{q} = \{q_n\}$ be a positive sequence of real numbers and $Q_n = q_0 + q_1 + \cdots + q_n$. Then*

$$\sum_{n=0}^{\infty} q_n \Big(\frac{1}{Q_n} \sum_{k=0}^{n} q_k |x_k|\Big)^p \leq \Big(\frac{p}{p-1}\Big)^p \sum_{n=0}^{\infty} q_n |x_n|^p \text{ holds.} \tag{5}$$

The following sequence spaces are introduced and studied by Johnson and Mohapatra (see [5]):

$$\mathbf{q}^{-1} l_p = \Big\{\mathbf{x} : \sum_{n=0}^{\infty} (q_n |x_n|)^p < \infty\Big\},$$

$$\mathbf{q}^{-\frac{1}{p}} l_p = \Big\{\mathbf{x} : \sum_{n=0}^{\infty} q_n |x_n|^p < \infty\Big\},$$

$$\text{nor-}C_q^{-1}(\mathbf{q}^{-\frac{1}{p}} l_p) = \Big\{\mathbf{x} : \sum_{n=0}^{\infty} q_n \Big(\frac{1}{Q_n} \sum_{k=0}^{n} q_k |x_k|\Big)^p < \infty\Big\},$$

and

$$\text{ces}[p, \mathbf{q}] = \Big\{\mathbf{x} : \sum_{n=0}^{\infty} \Big(\frac{1}{Q_n} \sum_{k=0}^{n} q_k |x_k|\Big)^p < \infty\Big\}.$$

Corollary 1. *With these notations, inequality (5) immediately gives $\boldsymbol{q}^{-\frac{1}{p}} l_p \subseteq \text{nor}-C_q^{-1}(\boldsymbol{q}^{-\frac{1}{p}} l_p)$.*

Additional assumption on the positive sequence $\mathbf{q}$, Theorem 2 gives the following result (see [5], Theorem 1, p. 196):

Theorem 3. *If $\boldsymbol{q} = \{q_n\}$ is bounded away from zero and $1 < p < \infty$, then*

$$\boldsymbol{q}^{-1} l_p \subset \boldsymbol{q}^{-\frac{1}{p}} l_p \subset \text{ nor -} C_q^{-1}(\boldsymbol{q}^{-\frac{1}{p}} l_p) \subset \text{ces}[p, \boldsymbol{q}].$$

Several mathematicians such as Maddox (see [12]), Johnson and Mohapatra (see [5], [6] and [7]) studied the following generalizations of the sequence spaces l_p and $\text{ces}(p)$, for example

$$l(p_n) = \left\{\mathbf{x} : \sum_{n=0}^{\infty} |x_n|^{p_n} < \infty\right\},$$

$$\text{ces}(p_n) = \left\{\mathbf{x} : \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |x_k|\right)^{p_n} < \infty\right\},$$

and

$$\text{ces}[\{p_n\}, \{q_n\}] = \left\{\mathbf{x} : \sum_{n=0}^{\infty} \left(\frac{1}{Q_n} \sum_{k=0}^{n} q_k |x_k|\right)^{p_n} < \infty\right\}.$$

Bennett (see [1], Theorem 6.9) obtained the factorization of the spaces $cop(p)$, $0 < p < 1$ but it is observed that no factorization was derived for the Copson's inequality (5). This paper is aimed to fulfill this gap in the literature. Indeed, we shall factorized the inclusion $\mathbf{q}^{-1} l(p_n) \subset \text{ces}[\mathbf{p}, \mathbf{q}]$, which in particular case, that is, when $p_n = p$ for all n gives $\mathbf{q}^{-1} l_p \subset \text{ces}[p, \mathbf{q}]$.

To obtain our result, we consider the sequence space $\text{ces}[\mathbf{p}, \mathbf{q}]$ is non-trivial. Johnson and Mohapatra (see [7]) obtained equivalent condition for sequence space $\text{ces}[\mathbf{p}, \mathbf{q}]$ to be non-trivial as stated below:

Theorem 4. *The following are equivalent:*

(i) $\text{ces}[\{p_n\}, \{q_n\}] \neq \boldsymbol{0}$.

(ii) $\frac{\boldsymbol{1}}{\boldsymbol{Q}} \in l(p_n)$.

We need one more definition to establish our result:

$$g(p_n) = \left\{\mathbf{z} : \sum_{k=0}^{n} |z_k|^{p_k} \leq K(p)^{p_n - 1} Q_n\right\},$$

where $K(p) \geq 1$ is a constant depending only on the sequence $\mathbf{p}$. The constant $K(p)$ will vary in different occurrences. We denote $\mathbf{p}^* = \{p_n^*\}$ as conjugate of $\mathbf{p}$, that is $1/\mathbf{p} + 1/\mathbf{p}^* = 1$ must holds. We shall use the idea given by Bennett (see [1]) to establish our generalized result but it needs deeper investigation and sincere calculation to reach.

The following lemma is well-known and will be required to prove our result:

Lemma 1.1 ([1], Lemma 3.6)**.** *Let* $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ *be sequences with non-negative terms and suppose that* w_k *decreases with* k. *If*

$$\sum_{k=0}^{n} u_k \le \sum_{k=0}^{n} v_k \qquad (n=0,1,2,\ldots),$$

$$\text{then} \qquad \sum_{k=0}^{n} u_k w_k \le \sum_{k=0}^{n} v_k w_k \qquad (n=0,1,2,\ldots).$$

2. Main results

Now, we state our result:

Theorem 5. *Let* $\boldsymbol{q}=\{q_n\}$ *be a positive sequence of real numbers is bounded away from zero and sequence* $\boldsymbol{Q}=\{Q_n\}$, *where* $Q_n=\sum_{k=0}^{n} q_k$ *diverges to* ∞.

(i) *If* $\boldsymbol{p}=\{p_n\}$ *is a non-increasing sequence of positive numbers, all* $p_n>1$; *and* $\boldsymbol{x}\in \operatorname{ces}[\{p_n\},\{q_n\}]$, *then* $\boldsymbol{x}$ *admits a factorization* (3) *with*

$$\boldsymbol{y}\in q_n^{-1} l(p_n) \text{ and } \boldsymbol{z}\in g(p_n^*). \tag{6}$$

(ii) *Conversely, if* $\boldsymbol{p}$ *is a non-decreasing and bounded sequence of numbers such that* $p_0>1$, *furthermore* (6) *holds, then the product sequence* $\boldsymbol{x}=\boldsymbol{y}.\boldsymbol{z}\in \operatorname{ces}[\{p_n\},\{q_n\}]$.

Proof.

(**i**) Let it be assumed that $\mathbf{x}\neq\mathbf{0}=\{0,0,\ldots\}$. For $\mathbf{x}=\mathbf{0}$, the statement is trivial.
For $\mathbf{x}\in \operatorname{ces}[\{p_n\},\{q_n\}]$, put

$$b_n=\sum_{k=n}^{\infty}\frac{1}{Q_k^{p_k}}\Big(\sum_{i=0}^{k} q_i|x_i|\Big)^{p_k-1}. \tag{7}$$

We assert that the sequence $\mathbf{b}=\{b_n\}$ is monotonically tends to zero. In fact, by the following well-known inequality:

$$ab\le \frac{a^p}{p}+\frac{b^{p^*}}{p^*},\quad p>1,$$

one obtains

$$\begin{aligned} b_n &= \sum_{k=n}^{\infty}\frac{1}{Q_k}\Big(\frac{1}{Q_k}\sum_{i=0}^{k} q_i|x_i|\Big)^{p_k-1} \\ &\le \sum_{k=n}^{\infty}\frac{1}{p_k}\frac{1}{Q_k^{p_k}}+\sum_{k=n}^{\infty}\frac{1}{p_k^*}\Big(\frac{1}{Q_k}\sum_{i=0}^{k} q_i|x_i|\Big)^{p_k} \\ &= S_1+S_2. \end{aligned} \tag{8}$$

Since $p_k > 1$ and sequence space $\text{ces}[\{p_n\},\{q_n\}]$ is non-trivial, so

$$S_1 < \sum_{k=n}^{\infty} \frac{1}{Q_k^{p_k}} \to 0 \text{ as } n \to \infty.$$

Again, since $p_k > 1$ and $\mathbf{x} \in \text{ces}[\{p_n\},\{q_n\}]$, one gets

$$S_2 < \sum_{k=n}^{\infty} \Big(\frac{1}{Q_k} \sum_{i=0}^{k} q_i |x_i|\Big)^{p_k} \to 0 \text{ as } n \to \infty.$$

Combining these two, one can easily proved the assertion.

Now the following factorization of $\mathbf{x} = \{x_n\}$ is considered:

$$\mathbf{x} = \mathbf{y}.\mathbf{z} \qquad (x_n = y_n.z_n),$$

where

$$y_n = \Big(\frac{b_n|x_n|}{q_n^{p_n-1}}\Big)^{\frac{1}{p_n}} sign(x_n). \tag{9}$$

and

$$z_n = (q_n|x_n|)^{\frac{1}{p_n^*}} b_n^{-\frac{1}{p_n}}. \tag{10}$$

Therefore using equations (9) and (10), one obtains

$$\begin{aligned}
\sum_{n=0}^{\infty} (q_n|y_n|)^{p_n} &= \sum_{n=0}^{\infty} q_n|x_n|b_n \\
&= \sum_{n=0}^{\infty} q_n|x_n| \sum_{k=n}^{\infty} \frac{1}{Q_k^{p_k}} \Big(\sum_{i=0}^{k} q_i|x_i|\Big)^{p_k-1} \\
&= \sum_{k=0}^{\infty} \frac{1}{Q_k^{p_k}} \Big(\sum_{i=0}^{k} q_i|x_i|\Big)^{p_k-1} \Big(\sum_{n=0}^{k} q_n|x_n|\Big) \\
&= \sum_{k=0}^{\infty} \frac{1}{Q_k^{p_k}} \Big(\sum_{i=0}^{k} q_i|x_i|\Big)^{p_k}.
\end{aligned}$$

Hence $\mathbf{x} \in \text{ces}[\{p_n\},\{q_n\}]$ implies that $\mathbf{y} \in q_n^{-1} l(p_n)$.
Now using the Hölder's inequality, the following is obtained:

$$\begin{aligned}
\Big(\sum_{k=0}^{m} |z_k|^{p_k^*}\Big)^{p_m} &= \Big(\sum_{k=0}^{m} (q_k|x_k|)^{\frac{1}{p_m}+\frac{1}{p_m^*}} b_k^{-\frac{p_k^*}{p_k}}\Big)^{p_m} \\
&\le \Big(\sum_{k=0}^{m} q_k|x_k|\Big)^{\frac{p_m}{p_m^*}} \Big(\sum_{k=0}^{m} q_k|x_k|b_k^{-\frac{p_m p_k^*}{p_k}}\Big).
\end{aligned} \tag{11}$$

Therefore, for $m = 0, 1, 2, \ldots$, one gets

$$\sum_{n=m}^{\infty} \Big(\frac{1}{Q_n} \sum_{k=0}^{m} |z_k|^{p_k^*}\Big)^{p_m} \le \sum_{n=m}^{\infty} \frac{1}{Q_n^{p_m}} \Big(\sum_{k=0}^{m} q_k|x_k|\Big)^{p_m-1} \Big(\sum_{k=0}^{m} q_k|x_k|b_k^{-\frac{p_m p_k^*}{p_k}}\Big)$$

$$= \sum_{k=0}^{m} q_k |x_k| b_k^{-\frac{p_m p_k^*}{p_k}} \sum_{n=m}^{\infty} \frac{1}{Q_n^{p_m}} \Big(\sum_{k=0}^{m} q_k |x_k| \Big)^{p_m - 1}. \tag{12}$$

Since $\mathbf{x} \in \text{ces}[\{p_n\},\{q_n\}]$, so the sequence $\left\{ \frac{1}{Q_n} \sum_{k=0}^{n} q_k |x_k| \right\}$ is bounded, that is there exist constant $K_1(p)$ (infact, here and after we shall use $K_i(p)$ instead of $K_i(\mathbf{p})$ depending on the sequence $\mathbf{p}$ for each i), we have

$$\frac{1}{Q_n} \sum_{k=0}^{n} q_k |x_k| \le K_1(p) \quad \text{for each } n = 0,1,2,\dots.$$

Now, the following is established:

$$\begin{aligned}
\sum_{n=m}^{\infty} \frac{1}{Q_n^{p_m}} \Big(\sum_{k=0}^{m} q_k |x_k| \Big)^{p_m - 1} &= \sum_{n=m}^{\infty} \frac{1}{Q_n^{p_n}} \Big(\sum_{k=0}^{m} q_k |x_k| \Big)^{p_n - 1} Q_n^{p_n - p_m} \Big(\sum_{k=0}^{m} q_k |x_k| \Big)^{p_m - p_n} \\
&= \sum_{n=m}^{\infty} \frac{1}{Q_n^{p_n}} \Big(\sum_{k=0}^{m} q_k |x_k| \Big)^{p_n - 1} \Big(\frac{1}{Q_n} \sum_{k=0}^{m} q_k |x_k| \Big)^{p_m - p_n} \\
&\le K_2(p) \sum_{n=m}^{\infty} \frac{1}{Q_n^{p_n}} \Big(\sum_{k=0}^{m} q_k |x_k| \Big)^{p_n - 1} = K_2(p) b_m. \qquad (13)
\end{aligned}$$

Since the sequence $\mathbf{b}$ is bounded and $\mathbf{p}$ is non-increasing, so there exists a constant $K_3(p)$ and for each $k = 0,1,2,\dots$ with $k \le m$, one can assumed that

$$b_k^{\frac{p_k - p_m}{p_k - 1}} \le K_3(p).$$

Therefore, one gets

$$\begin{aligned}
\sum_{k=0}^{m} q_k |x_k| b_k^{-\frac{p_m p_k^*}{p_k}} &= \sum_{k=0}^{m} q_k |x_k| b_k^{-\frac{p_k^*}{p_k}} b_k^{\frac{p_k - p_m}{p_k - 1} - 1} \\
&\le K_3(p) \sum_{k=0}^{m} q_k |x_k| b_k^{-\frac{p_k^*}{p_k}} b_k^{-1} \\
&\le K_3(p) b_m^{-1} \sum_{k=0}^{m} z_k^{p_k^*}. \qquad (14)
\end{aligned}$$

Therefore, using inequalities (12), (13) and (14), there exists a constant $K_4(p)$, we have

$$\sum_{n=m}^{\infty} \frac{1}{Q_n^{p_m}} \Big(\sum_{k=0}^{m} |z_k|^{p_k^*} \Big)^{p_m - 1} \le K_4(p),$$

which implies that $\frac{1}{Q_m^{p_m}} \Big(\sum_{k=0}^{m} |z_k|^{p_k^*} \Big)^{p_m - 1} \le K_4(p)$. Hence an easy computation gives that

$$\sum_{k=0}^{m} |z_k|^{p_k^*} \le K(p)^{\frac{1}{p_m - 1}} Q_m = K(p)^{p_m^* - 1} Q_m.$$

This completes the part (**i**) of the theorem.

(**ii**) Since $\mathbf{z} \in g(p_n^*)$, we have

$$\sum_{k=0}^{n}|z_k|^{p_k^*}\le K(p)^{\frac{1}{p_n-1}}Q_n=K(p)^{\frac{1}{p_n-1}}\sum_{k=0}^{n}q_k.$$

Choose $w_k=Q_k^{-\frac{1}{2}}$, and applying the Lemma 1.1, one gets

$$\sum_{k=0}^{n}|z_k|^{p_k^*}Q_k^{-\frac{1}{2}}\le K(p)^{\frac{1}{p_n-1}}\sum_{k=0}^{n}q_kQ_k^{-\frac{1}{2}}. \tag{15}$$

Now applying the Hölder's inequality on the factorization $\mathbf{x}=\mathbf{y}.\mathbf{z}$, one obtains

$$\begin{aligned}\Big(\sum_{k=0}^{n}q_k|x_k|\Big)^{p_n}&=\Big(\sum_{k=0}^{n}q_k|y_k|Q_k^{\frac{1}{2p_n^*}}|z_k|Q_k^{-\frac{1}{2p_n^*}}\Big)^{p_n}\\&\le\Big(\sum_{k=0}^{n}(q_k|y_k|)^{p_n}Q_k^{\frac{p_n-1}{2}}\Big)\Big(\sum_{k=0}^{n}|z_k|^{p_n^*}Q_k^{-\frac{1}{2}}\Big)^{p_n-1}. \end{aligned}\tag{16}$$

Using inequalities (15) and (16), one gets

$$\begin{aligned}\sum_{n=0}^{\infty}\Big(\frac{1}{Q_n}\sum_{k=0}^{n}q_k|x_k|\Big)^{p_n}&\le\sum_{n=0}^{\infty}\frac{1}{Q_n^{p_n}}\Big(\sum_{k=0}^{n}(q_k|y_k|)^{p_n}Q_k^{\frac{p_n-1}{2}}\Big)\Big(\sum_{k=0}^{n}|z_k|^{p_n^*}Q_k^{-\frac{1}{2}}\Big)^{p_n-1}\\&\le K(p)\sum_{n=0}^{\infty}\frac{1}{Q_n^{p_n}}\Big(\sum_{k=0}^{n}(q_k|y_k|)^{p_n}Q_k^{\frac{p_n-1}{2}}\Big)\Big(\sum_{k=0}^{n}q_kQ_k^{-\frac{1}{2}}\Big)^{p_n-1}.\end{aligned}\tag{17}$$

Note that

$$\begin{aligned}\sum_{k=0}^{n}q_kQ_k^{-\frac{1}{2}}&=q_0Q_0^{-\frac{1}{2}}+\sum_{k=1}^{n}q_kQ_k^{-\frac{1}{2}}\\&\le q_0Q_0^{-\frac{1}{2}}+\sum_{k=1}^{n}\int_{Q_{k-1}}^{Q_k}x^{-\frac{1}{2}}dx\le 2Q_n^{\frac{1}{2}}.\end{aligned}$$

Using the above inequality, there exists a constant $K_5(p)=2^{p_n-1}K(p)$, from inequality (17) one can deduced that

$$\begin{aligned}\sum_{n=0}^{\infty}\Big(\frac{1}{Q_n}\sum_{k=0}^{n}q_k|x_k|\Big)^{p_n}&\le K_5(p)\sum_{n=0}^{\infty}\frac{1}{Q_n^{p_n}}\sum_{k=0}^{n}(q_k|y_k|)^{p_n}Q_k^{\frac{p_n-1}{2}}Q_n^{\frac{p_n-1}{2}}\\&=K_5(p)\sum_{k=0}^{\infty}(q_k|y_k|)^{p_n}\sum_{n=k}^{\infty}\Big(\frac{Q_k}{Q_n}\Big)^{\frac{p_n-1}{2}}.\frac{1}{Q_n}\\&=K_5(p)\sum_{k=0}^{\infty}(q_k|y_k|)^{p_k}(q_k|y_k|)^{p_n-p_k}\sum_{n=k}^{\infty}\Big(\frac{Q_k}{Q_n}\Big)^{\frac{p_k-1}{2}}\Big(\frac{Q_k}{Q_n}\Big)^{\frac{p_n-p_k}{2}}\frac{1}{Q_n}.\end{aligned}\tag{18}$$

Using the monotonicity, boundedness of the sequence $\mathbf{p}$ and since the sequence $\mathbf{qy}=\{q_ny_n\}$ is bounded as $\mathbf{y}\in {q_n}^{-1}l(p_n)$, so the sequences $\{(q_ky_k)^{p_n-p_k}\}$ and $\Big\{\Big(\frac{Q_k}{Q_n}\Big)^{\frac{p_n-p_k}{2}}\Big\}$ are also bounded and hence from inequality (18) there exists a constant $K_6(p)$, one gets

$$\sum_{n=0}^{\infty}\Big(\frac{1}{Q_n}\sum_{k=0}^{n}q_k|x_k|\Big)^{p_n}\le K_6(p)\sum_{k=0}^{\infty}(q_k|y_k|)^{p_k}\sum_{n=k}^{\infty}\Big(\frac{Q_k}{Q_n}\Big)^{\frac{p_k-1}{2}}.\frac{1}{Q_n}$$

$$= K_6(p)\sum_{k=0}^{\infty}(q_k|y_k|)^{p_k}Q_k^{\frac{p_k-1}{2}}\sum_{n=k}^{\infty}\frac{1}{Q_n^{1+\frac{1}{2}(p_k-1)}}. \qquad (19)$$

Since the sequence $\mathbf{q} = \{q_n\}$ is bounded away from zero, so by definition there exists a $c > 0$ such that $q_n \geq c$ for all $n \geq 0$. Then it is known by a simple calculation that

$$\begin{aligned}\sum_{n=k}^{\infty}\frac{1}{Q_n^{1+\frac{1}{2}(p_k-1)}} &\leq \frac{1}{c}\lim_{N\to\infty}\sum_{n=k}^{N} q_n Q_n^{-1-\frac{1}{2}(p_k-1)}\\ &\leq \frac{1}{c}\left\{q_k Q_k^{-1-\frac{1}{2}(p_k-1)} + \lim_{N\to\infty}\sum_{n=k+1}^{N}\int_{Q_{k-1}}^{Q_k} x^{-1-\frac{1}{2}(p_k-1)}dx\right\}\\ &\leq \frac{1}{c}\left\{Q_k^{-\frac{1}{2}(p_k-1)} + \frac{2}{p_k-1}Q_k^{-\frac{1}{2}(p_k-1)}\right\}\\ &= \frac{(p_k+1)}{c(p_k-1)}Q_k^{-\frac{1}{2}(p_k-1)}. \end{aligned} \qquad (20)$$

Choose a constant $K_7(p) = K_6(p)\frac{(p_k+1)}{c(p_k-1)}$, then by using inequality (20), from inequality (19) one obtains

$$\sum_{n=0}^{\infty}\left(\frac{1}{Q_n}\sum_{k=0}^{n} q_k|x_k|\right)^{p_n} \leq K_7(p)\sum_{k=0}^{\infty}(q_k|y_k|)^{p_k}. \qquad (21)$$

Inequality (21) clearly implies that $\mathbf{x} \in \text{ces}[\{p_n\},\{q_n\}]$, as the sequence $\mathbf{y} \in {q_n}^{-1}l(p_n)$. □

Remark 1. If $\mathbf{x} \in q_n^{-\frac{1}{p_n}}l(p_n)$, then $\mathbf{x}$ can be factorized as (3) with (6) by choosing $\mathbf{y} = \left\{x_n {q_n}^{-\frac{1}{p_n^*}}\right\} \in q_n^{-1}l(p_n)$ and $\mathbf{z} = \{q_n^{\frac{1}{p_n^*}}\}$. Therefore Part (**ii**) of our theorem clearly indicates that

$$q_n^{-\frac{1}{p_n}}l(p_n) \subseteq \text{ces}[\{p_n\},\{q_n\}],\ p_n > 1.$$

Using the hypothesis on $\mathbf{q}$, an easy computation gives

$$q_n^{-1}l(p_n) \subseteq q_n^{-\frac{1}{p_n}}l(p_n) \subseteq \text{ nor} - C_q^{-1}(\mathbf{q}^{-\frac{1}{p_n}}l(p_n)) \subseteq \text{ces}[\{p_n\},\{q_n\}],\ p_n > 1.$$

Remark 2. If $q_n = 1$ for each n, then Theorem 5 gives the result obtained by Leindler (see [10], Theorem **2.1**.).

Remark 3. If $p_n = p$ for each n, then Theorem 5 with Remark 1 gives the factorized enhancement of Copson's inequality.

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