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ON INEQUALITIES OF HERMITE-HADAMARD TYPE FOR STOCHASTIC PROCESSES WHOSE THIRD DERIVATIVE ABSOLUTE VALUES ARE QUASI-CONVEX

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Abstract. In this paper we give some estimates of the right-hand side inequality of Hermite-Hadamad type for stochastic processes whose third derivatives in absolute values are quasi-convex.

1. Introduction

In recent years, different inequalities have been established for convex functions and one of most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications (see [5], [13]). For Hermite-Hadamad's inequality, several authors have estimated the error in the approximation of its sides. The technique used consider derivatives of different orders and properties as convexity and quasi-convexity, (see [1], [2], [4], [7], [8], [12], [16]).

The stochastic processes study started from the endings of 30's, and it was not until 1980 when K. Nikodem established the notion of convexity for stochastic processes and some properties of this kind of processes in [10], based on the definition of additive stochastic processes introduced by B. Nagy in 1974, [9]. In the same year, K. Nikodem in [11] introduced some properties of quasi-convex stochastic processes.

Some inequalities for convex and quasi-convex stochastic processes have been established recently. In 2015, N. Merentes *et al.* in [3], started Jensen and Hermite-Hadamard type inequalities. Additionally, in [6] prove some error estimations of a Hermite-Hadamad type inequality for stochastic processes consider its first and second order derivatives convex and quasi-convex.

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In this paper, we present the counterpart of the research made by S. Qaisar *et al.* in [14] for stochastic processes to estimate different refinements of the right-hand side an inequality of Hermite-Hadamard type considering its third mean-square derivatives at certain powers quasi-convex.

2. Inequalities of Hermite-Hadamard type for quasi-convex stochastic processes

We will result some useful and important definitions for this research. Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is a *random variable* if it is $\mathscr{A}-$measurable. A stochastic processes is defined as a function $X : I \times \Omega \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

Consider a stochastic process $X(t,\cdot)$ such that the expectation squared is bounded, i.e. $\mathbb{E}[X(t)]^2 < \infty$ for all $t \in I$. The stochastic process X is defined:

(1) *Mean-square differentiable* in I, if there exists a stochastic process X' (the derivative of X) such that for all $t_0 \in I$ we have

$$\lim_{t \to t_0} E\left[\frac{X(t) - X(t_0)}{t - t_0} - X'(t_0)\right]^2 = 0,$$

(2) *Mean-square integrable* on $[a,b] \subseteq I$, if there exists a random variable Y such that for all normal sequence of partitions of the interval $[a,b]$ $a = t_0 < t_1 < \cdots < t_n = b$ and for all $\tau_k \in [t_{k-1}, t_k]$, $k = 1, \dots, n$, we have

$$\lim_{n \to \infty} \mathbb{E}\left[\left(\sum_{k=1}^{n} X(\tau_k, \cdot)(t_k - t_{k-1}) - Y(\cdot)\right)^2\right] = 0.$$

The random variable $Y : \Omega \to \mathbb{R}$ is called the mean-square integral of the process X on $[a,b]$. In such case, we write

$$Y(\cdot) = \int_a^b X(s,\cdot)ds, \quad \text{(a.e)}.$$

Definition and basic properties of the mean-square derivative and mean-square integral can be read in [15].

We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is a *quasi-convex stochastic process* if, for every $a, b \in I$, $\lambda \in (0,1)$, the following inequality is satisfied

$$X(\lambda t_1 + (1-\lambda)t_2, \cdot) \leq \max\{|X(a,\cdot)|, |X(b,\cdot)|\}, \quad \text{(a.e)}. \tag{2.1}$$

If in (2.1) the reversed inequality holds, the stochastic process is *quasi-concave*.

In order to prove some inequalities for quasi-convex differentiable stochastic processes which are connected with the right-hand side of Hermite-Hadamard's inequality, it is necessary to use the following lemma:

Lemma 2.1. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean-square differentiable stochastic process on I^0, $a, b \in I^0$ with $a < b$. If $X^{(3)}(t,\cdot)$ is mean-square integrable on $[a,b]$, then the following equality takes places almost everywhere:*

$$\begin{aligned}\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{b-a}\int_a^b X(t,\cdot)dt - \frac{b-a}{12}[X'(b,\cdot)-X'(a,\cdot)] \\ = \frac{(b-a)^3}{12}\int_0^1 \lambda(\lambda-1)(2\lambda-1)X^{(3)}(\lambda a+(1-\lambda)b,\cdot)d\lambda.\end{aligned} \tag{2.2}$$

Proof. Integrating by parts the right-hand side of equation (2.2) , we have

$$\begin{aligned}&\int_0^1 \lambda(\lambda-1)(2\lambda-1)X^{(3)}(\lambda a+(1-\lambda)b,\cdot)d\lambda \\ &= \frac{1}{a-b}\int_0^1 (6\lambda^2-6\lambda+1)X^{(2)}(\lambda a+(1-\lambda)b,\cdot))d\lambda \\ &= \frac{X'(a,\cdot)-X'(b,\cdot)}{(a-b)^2} - 6\left[\frac{X(a,\cdot)+X(b,\cdot)}{(a-b)^3}\right] + \frac{12}{(a-b)^3}\int_0^1 X(\lambda a+(1-\lambda)b,\cdot)d\lambda \\ &= \frac{X'(a,\cdot)-X'(b,\cdot)}{(a-b)^2} - 6\left[\frac{X(a,\cdot)+X(b,\cdot)}{(a-b)^3}\right] + \frac{12}{(a-b)^3}\int_0^1 X(\lambda a+(1-\lambda)b,\cdot)d\lambda.\end{aligned}$$

Multiplying by $\frac{(b-a)^3}{12}$:

$$\begin{aligned}&\frac{(b-a)^3}{12}\int_0^1 \lambda(\lambda-1)(2\lambda-1)X^{(3)}(\lambda a+(1-\lambda)b,\cdot)d\lambda \\ &= \frac{(b-a)}{12}[X'(a,\cdot)-X'(b,\cdot)] + \left[\frac{X(a,\cdot)+X(b,\cdot)}{2}\right] - \int_0^1 X(\lambda a+(1-\lambda)b,\cdot)d\lambda.\end{aligned}$$

Then, making the change of variable on the right hand side of the above equation $t = \lambda a + (1-\lambda)b$ and $dt = (a-b)d\lambda$, is obtained:

$$\begin{aligned}&\frac{(b-a)^3}{12}\int_0^1 \lambda(\lambda-1)(2\lambda-1)X^{(3)}(\lambda a+(1-\lambda)b,\cdot)d\lambda \\ &= \frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{b-a}\int_a^b X(t,\cdot)dt - \frac{(b-a)}{12}[X'(b,\cdot)-X'(a,\cdot)],\end{aligned}$$

Obtaining the desired result. □

Theorem 2.2. *Let $X : I \times \Omega \to \mathbb{R}$ be three times mean-square differentiable stochastic process on I^0 such that $a, b \in I^0$, $a < b$. If $X^{(3)}$ is mean-square integrable on $[a,b]$ and $|X^{(3)}|$ is quasi-convex on $[a,b]$, then we have the following inequality:*

$$\begin{aligned}\left|\frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{1}{b-a}\int_a^b X(t,\cdot)dt - \frac{b-a}{12}[X'(b,\cdot)-X'(a,\cdot)]\right| \\ \le \frac{(b-a)^3}{192}\max\{|X^{(3)}(a,\cdot)|, |X^{(3)}(b,\cdot)|\}.\end{aligned} \tag{2.3}$$

Proof. Using Lemma 2.1 and quasi-convexity of $|X^{(3)}|$, we get:

$$\begin{aligned}&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2}-\frac{1}{b-a}\int_a^b X(t,\cdot)dt-\frac{b-a}{12}[X'(b,\cdot)-X'(a,\cdot)]\right|\\&\quad\le\frac{(b-a)^3}{12}\int_0^1\lambda(1-\lambda)|2\lambda-1||X^{(3)}(\lambda a+(1-\lambda)b,\cdot)|d\lambda\\&\quad\le\frac{(b-a)^3}{12}\max\{|X^{(3)}(a,\cdot)|,|X^{(3)}(b,\cdot)|\}\int_0^1\lambda(1-\lambda)|2\lambda-1|d\lambda\\&\quad=\frac{(b-a)^3}{192}\max\{|X^{(3)}(a,\cdot)|,|X^{(3)}(b,\cdot)|\},\end{aligned}$$

where

$$\int_0^1\lambda(1-\lambda)|2\lambda-1|d\lambda=\int_0^{1/2}\lambda(1-\lambda)(1-2\lambda)d\lambda+\int_{1/2}^1\lambda(1-\lambda)(2\lambda-1)d\lambda=\frac{1}{16}. \tag{2.4}$$

□

In the following theorem, we establish the corresponding version for powers of the absolute value of the second derivative:

Theorem 2.3. *Let $X: I\times\Omega\to\mathbb{R}$ be three times mean-square differentiable stochastic process on I^0 such that $a,b\in I^0$, $a<b$. If $X^{(3)}$ is mean-square integrable on $[a,b]$ and $|X^{(3)}|^{p/(p-1)}$ is quasi-convex on $[a,b]$ and $p>1$, then we have the following inequality:*

$$\begin{aligned}&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2}-\frac{1}{b-a}\int_a^b X(t,\cdot)dt-\frac{b-a}{12}[X'(b,\cdot)-X'(a,\cdot)]\right|\\&\quad\le\frac{(b-a)^3}{96}\left(\frac{1}{p+1}\right)^{1/p}\max\{|X^{(3)}(a,\cdot)|^q,|X^{(3)}(b,\cdot)|^q\}^{1/q},\end{aligned} \tag{2.5}$$

where $q=p/(p-1)$.

Proof. By using Lemma 2.1 and the well know Hölder's integral inequality, we have:

$$\begin{aligned}&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2}-\frac{1}{b-a}\int_a^b X(t,\cdot)dt-\frac{b-a}{12}[X'(b,\cdot)-X'(a,\cdot)]\right|\\&\quad\le\frac{(b-a)^3}{12}\int_0^1\lambda(1-\lambda)|2\lambda-1||X^{(3)}(\lambda a+(1-\lambda)b,\cdot)|d\lambda\\&\quad\le\frac{(b-a)^3}{12}\left(\int_0^1\lambda^p(1-\lambda)^p|2\lambda-1|^p d\lambda\right)^{1/p}\left(\int_0^1|X^{(3)}(\lambda a+(1-\lambda b,\cdot))|^q d\lambda\right)^{1/q}.\end{aligned}$$

Since $0<2\lambda-1<1$ and $p>1$, we have:

$$\int_0^1\lambda^p(1-\lambda)^p|2\lambda-1|^p d\lambda\le\int_0^1\lambda^p(1-\lambda)^p|2\lambda-1|d\lambda.$$

Then,

$$\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2}-\frac{1}{b-a}\int_a^b X(t,\cdot)dt-\frac{b-a}{12}[X'(b,\cdot)-X'(a,\cdot)]\right|\\
&\le \frac{(b-a)^3}{12}\left(\int_0^1 \lambda^p(1-\lambda)^p|2\lambda-1|d\lambda\right)^{1/p}\left(\int_0^1 |X^{(3)}(\lambda a+(1-\lambda b,\cdot))|^q d\lambda\right)^{1/q}\\
&=\frac{(b-a)^3}{12}\left(\frac{1}{2^{2p+1}(1+p)}\right)^{1/p}\max\{|X^{(3)}(a,\cdot)|^q,|X^{(3)}(b,\cdot)|^q\}^{1/q},
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1(\lambda(1-\lambda))^p|2\lambda-1|d\lambda&=\int_0^{1/2}(\lambda(1-\lambda))^p(1-2\lambda)d\lambda+\int_{1/2}^1(\lambda(1-\lambda))^p(2\lambda-1)d\lambda\\
&=\left(\frac{1}{2^{2p+1}(1+p)}\right).
\end{aligned}$$

□

Theorem 2.4. *Let $X: I\times\Omega\to\mathbb{R}$ be three times mean-square differentiable stochastic process on I^0 such that $a,b\in I^0$, $a<b$. If $X^{(3)}$ is mean-square integrable on $[a,b]$ and $|X^{(3)}|^q$ is quasi-convex on $[a,b]$ and $p>1$, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2}-\frac{1}{b-a}\int_a^b X(t,\cdot)dt-\frac{b-a}{12}[X'(b,\cdot)-X'(a,\cdot)]\right|\\
&\le\frac{(b-a)^3}{192}\max\{|X^{(3)}(a,\cdot)|^q,|X^{(3)}(b,\cdot)|^q\}^{1/q}.
\end{aligned}\tag{2.6}$$

Proof. Using Lemma 2.1 and the well know power-mean inequality, we get:

$$\begin{aligned}
&\left|\frac{X(a,\cdot)+X(b,\cdot)}{2}-\frac{1}{b-a}\int_a^b X(t,\cdot)dt-\frac{b-a}{12}[X'(b,\cdot)-X'(a,\cdot)]\right|\\
&\le\frac{(b-a)^3}{12}\int_0^1\lambda(1-\lambda)|2\lambda-1||X^{(3)}(\lambda a+(1-\lambda)b,\cdot)|d\lambda\\
&\le\frac{(b-a)^3}{12}\left(\int_0^1\lambda(1-\lambda)|2\lambda-1|d\lambda\right)^{1-1/q}\left(\int_0^1\lambda(1-\lambda)|2\lambda-1||X^{(3)}(\lambda a+(1-\lambda b,\cdot))|^q d\lambda\right)^{1/q}\\
&=\frac{(b-a)^3}{12}\left(\frac{1}{16}\right)^{1-1/q}\left(\frac{1}{16}\max\{|X^{(3)}(a,\cdot)|^q,|X^{(3)}(b,\cdot)|^q\}\right)^{1/q}\\
&=\frac{(b-a)^3}{192}\max\{|X^{(3)}(a,\cdot)|^q,|X^{(3)}(b,\cdot)|^q\}^{1/q}
\end{aligned}$$

where we use the equality (2.4). □

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