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A DOUBLE INEQUALITY FOR REMAINDER OF POWER SERIES OF TANGENT FUNCTION

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Abstract. By mathematical induction, an identity and a double inequality for remainder of power series of tangent function are established.

1. Introduction

It is well known that Bernoulli numbers B_i are defined [11] by

$$\frac{x}{e^x-1}=1-\frac{1}{2}x+\sum_{i=1}^{\infty}(-1)^{i+1}\frac{B_i}{(2i)!}x^{2i},\quad |x|<2\pi. \tag{1}$$

About Bernoulli numbers, some new results can be found in [1, 3, 5].

The tangent and cotangent can be expanded into power series with coefficients involving Bernoulli numbers as follows [11, p.5]:

$$\tan x=\sum_{i=1}^{\infty}\frac{2^{2i}(2^{2i}-1)B_i}{(2i)!}x^{2i-1},\quad |x|<\frac{\pi}{2}; \tag{2}$$

$$\cot x=\frac{1}{x}-\sum_{i=1}^{\infty}\frac{2^{2i}B_i}{(2i)!}x^{2i-1},\qquad |x|<\pi. \tag{3}$$

Introduce two notations $S_n(x)$ and $r_n(x)$ by

$$S_n(x)=\sum_{i=1}^{n}\frac{2^{2i}(2^{2i}-1)B_i}{(2i)!}x^{2i-1}, \tag{4}$$

$$r_n(x)=\tan x-S_n(x) \tag{5}$$

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for $0 < x < \frac{\pi}{2}$. Then $\tan x = \lim_{n\to\infty} S_n(x)$. We call $r_n(x)$ the remainder of power series for tangent function.

For elementary functions $\sin x$, $\cos x$, and e^x, there are much literature on estimates of their remainder. For examples, see [6, 7, 9]. The methods used in [6, 7, 9] have been applied to construct inequalities of elliptic integrals. See [8, 10]. Some inequalities involving $\tan x$ were researched by the second author and others in [2].

In this article, we will establish a double inequality for remainder $r_n(x)$ of power series for $\tan x$. That is

Theorem 1. *For $x \in (0, \frac{\pi}{2})$ and $n \in \mathbb{N}$, we have*

$$\frac{2^{2(n+1)}(2^{2(n+1)}-1)B_{n+1}}{(2n+2)!}x^{2n}\tan x < \tan x - S_n(x) < \left(\frac{2}{\pi}\right)^{2n} x^{2n}\tan x. \tag{6}$$

Remark 1. If taking $n = 1$ in (6), we have for $x \in (0, 1)$

$$\frac{\pi}{2}\cdot\frac{x}{1-\frac{7\pi^2}{360}x^2} < \tan\frac{\pi x}{2} < \frac{\pi}{2}\cdot\frac{x}{1-x^2}. \tag{7}$$

For $0 < x < \frac{3}{\pi}\sqrt{\frac{5(\pi^2-8)}{38}}$, the left inequality in (7) is better than the left inequality in the following Becker-Stark inequality [4, p.351]:

$$\frac{4}{\pi}\cdot\frac{x}{1-x^2} < \tan\frac{\pi x}{2} < \frac{\pi}{2}\cdot\frac{x}{1-x^2}, \quad x \in (0, 1). \tag{8}$$

If taking $n = 2$ in (6), we obtain

$$x + \frac{1}{3}x^3 + \frac{2}{15}x^4\tan x < \tan x < x + \frac{1}{3}x^3 + \left(\frac{2}{\pi}\right)^4 x^4 \tan x, \quad x \in \left(0, \frac{\pi}{2}\right). \tag{9}$$

The constants $\frac{2}{15}$ and $(\frac{2}{\pi})^4$ in (9) are best possible.

For $x \in (0, \frac{\pi}{6})$, the Djokvie inequality states [4, p.350] that

$$x + \frac{1}{3}x^3 < \tan x < x + \frac{4}{9}x^3. \tag{10}$$

Since

$$\frac{1}{3} + \left(\frac{2}{\pi}\right)^4 x\tan x < \frac{1}{3} + \left(\frac{2}{\pi}\right)^4 \cdot \frac{\pi}{6}\cdot\frac{1}{\sqrt{3}} < \frac{4}{9},$$

thus, the inequality in (9) is better than those in (10).

2. Proof of Theorem

Let

$$h_n(x) = \frac{\tan x - S_n(x)}{x^{2n}\tan x} \tag{11}$$

for $n \in \mathbb{N}$. Then we have the following lemma.

Lemma 1. *For $x \in (0, \frac{\pi}{2})$ and $n \in \mathbb{N}$, we have*

$$h_n(x) = \sum_{j=1}^{n} \frac{2^{2(n-j+1)}[2^{2(n-j+1)}-1]B_{n-j+1}}{[2(n-j+1)]!} \sum_{k=j}^{\infty} \frac{2^{2k}B_k}{(2k)!} x^{2(k-j)}. \tag{12}$$

Proof. We shall prove this lemma by mathematical induction on n.

For $n = 1$, we have

$$\begin{aligned} h_1(x) &= \frac{\tan x - S_1(x)}{x^2 \tan x} \\ &= \frac{1}{x^2} - \frac{\cot x}{x} \\ &= \frac{1}{x^2} - \frac{1}{x}\left(\frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}B_k}{(2k)!} x^{2k-1}\right) \\ &= \sum_{k=1}^{\infty} \frac{2^{2k}B_k}{(2k)!} x^{2(k-1)}, \end{aligned}$$

the formula (12) holds for $n = 1$.

For $n = 2$, we have

$$\begin{aligned} h_2(x) &= \frac{\tan x - S_2(x)}{x^4 \tan x} \\ &= \frac{1}{x^4} - \frac{\cot x}{x^3} - \frac{\cot x}{3x} \\ &= \frac{1}{x^4} - \frac{1}{x^3}\left(\frac{1}{x} - \frac{1}{3}x - \sum_{k=2}^{\infty} \frac{2^{2k}B_k}{(2k)!} x^{2k-1}\right) - \frac{1}{3x}\left(\frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}B_k}{(2k)!} x^{2k-1}\right) \\ &= \sum_{k=2}^{\infty} \frac{2^{2k}B_k}{(2k)!} x^{2(k-1)} + \sum_{k=1}^{\infty} \frac{2^{2k}B_k}{3 \cdot (2k)!} x^{2(k-1)}, \end{aligned}$$

the formula (12) holds for $n = 2$.

Assume formula (12) holds for $n = m$. Then for $n = m + 1$, we have

$$\begin{aligned} h_{m+1} &= \frac{\tan x - S_{m+1}(x)}{x^{2(m+1)} \tan x} \\ &= \frac{\tan x - S_m(x) - \frac{2^{2(m+1)}(2^{2(m+1)}-1)B_{m+1}}{[2(m+1)]!} x^{2m+1}}{x^{2(m+1)} \tan x} \\ &= \frac{1}{x^2} \cdot \frac{\tan x - S_m(x)}{x^{2m} \tan x} - \frac{2^{2(m+1)}(2^{2(m+1)}-1)B_{m+1}}{[2(m+1)]!} \cdot \frac{\cot x}{x} \\ &= \frac{1}{x^2} \sum_{j=1}^{m} \frac{2^{2(m-j+1)}[2^{2(m-j+1)}-1]B_{m-j+1}}{[2(m-j+1)]!} \sum_{k=j}^{\infty} \frac{2^{2k}B_k}{(2k)!} x^{2(k-j)} \end{aligned}$$

$$
\begin{aligned}
&-\frac{2^{2(m+1)}(2^{2(m+1)}-1)B_{m+1}}{[2(m+1)]!}\cdot\frac{1}{x}\left(\frac{1}{x}-\sum_{k=1}^{\infty}\frac{2^{2k}B_k}{(2k)!}x^{2k-1}\right)\\
&=\frac{1}{x^2}\sum_{j=1}^{m}\frac{2^{2j}(2^{2j}-1)B_j}{(2j)!}\cdot\frac{2^{2(m-j+1)}B_{m-j+1}}{[2(m-j+1)]!} \qquad (13)\\
&+\sum_{j=1}^{m+1}\frac{2^{2(m-j+2)}(2^{2(m-j+2)}-1)B_{m-j+2}}{[2(m-j+2)]!}\sum_{k=j}^{\infty}\frac{2^{2k}B_k}{(2k)!}x^{2(k-j)}\\
&-\frac{2^{2(m+1)}[2^{2(m+1)}-1]B_{m+1}}{[2(m+1)]!}\cdot\frac{1}{x^2}+\frac{2^{2(m+1)}[2^{2(m+1)}-1]B_{m+1}}{[2(m+1)]!}\sum_{k=1}^{\infty}\frac{2^{2k}B_k}{(2k)!}x^{2(k-1)}\\
&=\sum_{j=1}^{m+1}\frac{2^{2(m-j+2)}[2^{2(m-j+2)}-1]B_{m-j+2}}{[2(m-j+2)]!}\sum_{k=j}^{\infty}\frac{2^{2k}B_k}{(2k)!}x^{2(k-j)}\\
&+\frac{1}{x^2}\sum_{j=1}^{m}\frac{2^{2j}(2^{2j}-1)B_j}{(2j)!}\cdot\frac{2^{2(m-j+1)}B_{m-j+1}}{[2(m-j+1)]!}-\frac{2^{2(m+1)}(2^{2(m+1)}-1)B_{m+1}}{[2(m+1)]!}\cdot\frac{1}{x^2}.
\end{aligned}
$$

Since $\tan x\cot x=1$, we have

$$\left(\sum_{i=1}^{\infty}\frac{2^{2i}(2^{2i}-1)B_i}{(2i)!}x^{2i-1}\right)\left(\frac{1}{x}-\sum_{i=1}^{\infty}\frac{2^{2i}B_i}{(2i)!}x^{2i-1}\right)=1,$$

which is equivalent to

$$\sum_{i=2}^{\infty}\frac{2^{2i}(2^{2i}-1)B_i}{(2i)!}x^{2i-2}=\left[\sum_{i=1}^{\infty}\frac{2^{2i}(2^{2i}-1)B_i}{(2i)!}x^{2i-1}\right]\sum_{i=1}^{\infty}\frac{2^{2i}B_i}{(2i)!}x^{2i-1}, \qquad (14)$$

equating coefficients of the term x^{2m} on both sides of (14) yields

$$\frac{2^{2(m+1)}(2^{2(m+1)}-1)B_{m+1}}{(2(m+1))!}=\sum_{j=1}^{m}\frac{2^{2j}(2^{2j}-1)B_j}{(2j)!}\cdot\frac{2^{2(m-j+1)}B_{m-j+1}}{[2(m-j+1)]!}. \qquad (15)$$

Substituting (15) into (13) and simplifying gives us

$$h_{m+1}(x)=\sum_{j=1}^{m+1}\frac{2^{2(m-j+2)}(2^{2(m-j+2)}-1)B_{m-j+2}}{[2(m-j+2)]!}\sum_{k=j}^{\infty}\frac{2^{2k}B_k}{(2k)!}x^{2(k-j)}. \qquad (16)$$

By induction, the proof of Lemma 1 is complete.

Now we give a proof of Theorem 1.

Proof of Theorem 1. From (12), it is deduced that $h_n'(x)>0$, and $h_n(x)$ is strictly increasing in $(0,\frac{\pi}{2})$, Easy computing yields

$$h_n(0+0)=\frac{2^{2n+2}(2^{2n+2)}-1)B_{n+1}}{(2n+2)!},$$

$$h\Big(\frac{\pi}{2}-0\Big)=\Big(\frac{2}{\pi}\Big)^{2n}.$$

Therefore, we have

$$\frac{2^{2n+2}(2^{2n+2}-1)B_{n+1}}{(2n+2)!}<h_n(x)<\Big(\frac{2}{\pi}\Big)^{2n}. \tag{17}$$

Inequalities in (17) are equivalent to the double inequality (6).

The proof of Theorem 1 is complete.

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